Informatics 1 - Computation & Logic: Tutorial 8

Propositional Logic: Sequent Calculus

Week 10: 23-27 November 2015

Please attempt the entire worksheet in advance of the tutorial, and bring with you all work, including (if a computer is involved) printouts of code and test results. Tutorials cannot function properly unless you do the work in advance.

You may work with others, but you must understand the work; you can't phone a friend during the exam.

Assessment is formative, meaning that marks from coursework do not contribute to the final mark. But coursework is not optional. If you do not do the coursework you are unlikely to pass the exams.

Attendance at tutorials is **obligatory**; please let your tutor know if you cannot attend.

In this tutorial we consider relations generated by rules. A rule of the form:

$$\frac{\beta_1 \quad \cdots \quad \beta_n}{\alpha}$$

allows us to derive the conclusion α from the premisses $\beta_1, \ldots, \beta_n$.

As a first example, consider defining the grammar of a language. We give the following rules:

$$\frac{}{\text{ideas} : \mathbf{N}} \quad \frac{}{\text{linguists} : \mathbf{N}} \quad \frac{}{\text{great} : \mathbf{A}} \quad \frac{}{\text{green} : \mathbf{A}} \quad \frac{}{\text{hate} : \mathbf{V}} \quad \frac{}{\text{generate} : \mathbf{V}}$$

$$\frac{X : \mathbf{V}}{X : \mathbf{VP}}\,(V) \qquad \frac{X : \mathbf{V} \quad Y : \mathbf{NP}}{XY : \mathbf{VP}}\,(VP) \qquad \frac{X : \mathbf{NP} \quad Y : \mathbf{VP}}{XY : \mathbf{S}}\,(S)$$

$$\frac{X : \mathbf{N}}{X : \mathbf{NP}}\,(N) \qquad \frac{X : \mathbf{A} \quad Y : \mathbf{NP}}{XY : \mathbf{NP}}\,(NP)$$

Here, "ideas:**N**" means that 'ideas' is a noun. Our rules allow us to infer that particular phrases belong to various grammatical categories: noun (**N**), adjective (**A**), verb (**V**), noun-phrase (**NP**), verb-phrase (**VP**), and sentence (**S**). The variables X, Y range over phrases, where phrases are non-empty lists of words. The rules are labelled, (V), (VP), etc., for ease of reference.

For example, we can show that, "great linguists generate green ideas" is a sentence. In symbols,

$$\text{great linguists generate green ideas} : \mathbf{S}$$

We do this by constructing a tree:

$$\dfrac{\dfrac{\dfrac{}{\text{great} : \mathbf{A}} \quad \dfrac{\dfrac{}{\text{linguists} : \mathbf{N}}}{\text{linguists} : \mathbf{NP}}\,(N)}{\text{great linguists} : \mathbf{NP}}\,(NP) \quad \dfrac{\dfrac{}{\text{generate} : \mathbf{V}} \quad \dfrac{\dfrac{}{\text{green} : \mathbf{A}} \quad \dfrac{\dfrac{}{\text{ideas} : \mathbf{N}}}{\text{ideas} : \mathbf{NP}}\,(N)}{\text{green ideas} : \mathbf{NP}}\,(NP)}{\text{generate green ideas} : \mathbf{VP}}\,(VP)}{\text{great linguists generate green ideas} : \mathbf{S}}\,(S)$$

1. Which of the following are sentences for this grammar?

   (a) green linguists hate great ideas   Yes

$$\dfrac{\dfrac{\dfrac{}{\text{green} : \mathbf{A}} \quad \dfrac{\dfrac{}{\text{linguists} : \mathbf{N}}}{\text{linguists} : \mathbf{NP}}\,(N)}{\text{green linguists} : \mathbf{NP}}\,(NP) \quad \dfrac{\dfrac{}{\text{hate} : \mathbf{V}} \quad \dfrac{\dfrac{}{\text{great} : \mathbf{A}} \quad \dfrac{\dfrac{}{\text{ideas} : \mathbf{N}}}{\text{ideas} : \mathbf{NP}}\,(N)}{\text{great ideas} : \mathbf{NP}}\,(NP)}{\text{hate great ideas} : \mathbf{VP}}\,(VP)}{\text{green linguists hate great ideas} : \mathbf{S}}\,(S)$$

   (b) green green green linguists hate   Yes

$$\dfrac{\dfrac{\dfrac{}{\text{green} : \mathbf{A}} \quad \dfrac{\dfrac{}{\text{green} : \mathbf{A}} \quad \dfrac{\dfrac{}{\text{green} : \mathbf{A}} \quad \dfrac{\dfrac{}{\text{linguists} : \mathbf{N}}}{\text{linguists} : \mathbf{NP}}\,(N)}{\text{green linguists} : \mathbf{NP}}\,(NP)}{\text{green green linguists} : \mathbf{NP}}\,(NP)}{\text{green green green linguists} : \mathbf{NP}}\,(NP) \quad \dfrac{\dfrac{}{\text{hate} : \mathbf{V}}}{\text{hate} : \mathbf{VP}}\,(V)}{\text{green green green linguists hate} : \mathbf{S}}\,(S)$$

   (c) generate ideas   No

   (d) green ideas generate hate   No

2. How might you extend the grammar to include the sentence, "colourless green ideas sleep furiously"?

$$\frac{}{\text{colourless} : \mathbf{A}} \quad \frac{}{\text{furiously} : \mathbf{AdV}} \quad \frac{X : \mathbf{VP} \quad Y : \mathbf{AdV}}{XY : \mathbf{VP}} \, (VP)$$

3. We say that a grammar is *sound* if it only generates grammatical sentences, and that it is *complete* if every grammatical sentence can be generated by the rules.

   (a) Is it is possible to give a sound grammar for a natural language?
   Yes, trivially; the empty grammar produces no ungrammatical sentences.

   (b) Is it possible to give a complete grammar for a natural language?
   Yes, trivially; one could produce grammar that generated all possible sequences of words for the given the vocabulary of the langauge. Of course, the real problem is, is it possible to produce a grammar that is both sound *and* complete for a natural language. This is very much a disputed issue; in the 1950's, Chomsky gave the production of complete, sound grammars of natural languages as a mission statement for the programme of Generative linguistics which has dominated the study of language from then until now, and so far, no-one has been able to convincingly do it.

   (c) Is every grammatical sentence true?
   No, nor even meaningful - witness "Colourless green ideas sleep furiously."

   (d) Is it possible to write a grammar that will only generate true sentences?
   Not if the truth conditions of some of its sentences refer to states of affairs in the world.

Note for tutors: it is probably worth noting at this point that this is a rather non-standard notation for Context-Free Grammars, and students should not be too surprised to see a different notation next year in INF2A.

For our second example we introduce some simple logical rules.

$$\frac{}{\mathcal{A}, X \vdash X} \, (I) \qquad\qquad \frac{\Gamma \vdash A \quad \Delta, A \vdash B}{\Gamma, \Delta \vdash B} \, Cut$$

$$\frac{\mathcal{A} \vdash X \quad \mathcal{A} \vdash Y}{\mathcal{A} \vdash X \wedge Y} \, (\wedge) \quad \frac{\mathcal{A}, X \vdash Z \quad \mathcal{A}, Y \vdash Z}{\mathcal{A}, X \vee Y \vdash Z} \, (\vee) \quad \frac{\mathcal{A}, X \vdash Y}{\mathcal{A} \vdash X \rightarrow Y} \, (\rightarrow)$$

Here, $\mathcal{A}$ is a variable over sets of expressions of propositional logic, and X, Y and Z are variables over expressions themselves. We read the 'turnstile' $\vdash$ symbol as *entails*.

The *immediate* rule (I) has no assumptions. The double line used for the other three rules means that the rule can be used in either direction. The entailment below the double line is valid iff *all* of the entailments above the line are valid. Read from top to bottom, they are called *introduction rules ($^+$)*, since they introduce a new connective into the argument. Read from bottom to top, they are *elimination rules ($^-$)* since a connective is eliminated.

These rules are designed to allow us to produce *valid* entailments. We say that a valuation makes $\mathcal{A} \vdash X$ true if it makes at least one of the assumptions $A \in \mathcal{A}$ *false* or it makes X *true*. The entailment is valid iff every valuation makes it true. So it is valid iff any valuation that makes all the premisses in $\mathcal{A}$ true also makes X true.[1]

4. We have claimed that these rules are sound. This exercise asks you to show something stronger. For each of the rules $(\wedge), (\vee), (\rightarrow)$ show that every valuation makes the entailment below the line true iff it makes all of the entailments above the line true.

   The truth of an entailment requires the falsity of its premises or the truth of its conclusions. Thus;

   - $\triangleright$ $(\wedge)$: The premises are identical, so a valuation falsifying $\mathcal{A}$ validates all the entailments. A valuation making $(X \wedge Y)$, true also makes both X and Y true, as by definition, a conjunction is true iff its conjuncts are true. Thus, any valuation validating the entailments above the line validates those below and vice versa.

   - $\triangleright$ $(\vee)$: Here, the conclusions are identical, so a valuation making Z true validates all the entailments. $\mathcal{A}$ is in all the premises of all the entailments, so any valuation falsifying $\mathcal{A}$ validates all the entailments. A valuation making $\mathcal{A}$ true and Z false, if the entailments are to be valid, must falsify X and Y (for above the line), and $X \vee Y$ (for below). Above the line, that is $\neg X$ and $\neg Y$, equivalent to $\neg X \wedge \neg Y$ by $\wedge^+$. Below the line, that is $\neg(X \vee Y)$, equivalent to $\neg X \wedge \neg Y$ by De Morgan's law. Thus, any valuation validating the entailments above the line validates those below and vice versa.

   - $\triangleright$ $(\rightarrow^-)$:Since $\mathcal{A}$ is in the premises of the entailments above and below the lines, a valuation falsifying it validates all the entailments. If $\mathcal{A}$ is true, the validity of $\mathcal{A}, X \vdash Y$ requires X to be false or Y to be true, or formally, $\neg X \vee Y$. If $\mathcal{A}$ is true, the validity of $\mathcal{A} \vdash X \rightarrow Y$ requires that $X \rightarrow Y$ is also true, which, by $\rightarrow$-equivalence, is equivalent to $\neg X \vee Y$. Thus, any valuation validating the entailments above the line validates those below and vice versa.

[1] Note that the rule (I) is certainly sound, since X occurs on both sides of the turnstile.

Using these rules we can prove validity. For example, the following proof tree:

$$\dfrac{\dfrac{\dfrac{\dfrac{}{A \to B, C \vdash A \to B}\,(I)}{A \to B, C, A \vdash B}\,(\to^-) \qquad \dfrac{}{A \to B, C, A \vdash C}\,(I)}{A \to B, C, A \vdash B \wedge C}\,(\wedge^+)}{A \to B, C \vdash A \to (B \wedge C)}\,(\to^+)$$

shows that $A \to B, C \vdash A \to (B \wedge C)$ is valid, using a mixture of introduction and elimination rules.

Here are some tips for proving the validity of arguments with the use of proof rules:

- A branch of the proof tree is considered to be proved when the *immediate* rule is applied. All the branches of the proof tree need to be proved.
- Each entailment introduced by a rule must be justified by another rule.
- Remember to include the name of the rule that you are applying at each point.

5. For each of the entailments listed either give a proof tree that shows it is valid, or give a valuation that shows it is invalid. Uses of (Cut) and (I) have been labelled. It is a good exercise to identify the other rules used. For these examples, the choice of cut formula is fairly easy — but in general it is open-ended.

   (a) $\mathsf{B \wedge C \vdash (A \to B) \wedge (A \to C)}$

$$\dfrac{B \wedge C \vdash A \to B \qquad B \wedge C \vdash A \to C}{B \wedge C \vdash (A \to B) \wedge (A \to C)}$$

We focus on one branch since they are similar:

$$\dfrac{\dfrac{\dfrac{}{B \wedge C \vdash B \wedge C}\,(I)}{B \wedge C \vdash B} \qquad \dfrac{\dfrac{}{A, B \vdash B}\,(I)}{B \vdash A \to B}}{B \wedge C \vdash A \to B}\,(\mathrm{Cut})$$

   (b) $\mathsf{A \wedge (B \wedge C) \vdash (A \vee B) \wedge C}$

$$\dfrac{A \wedge (B \wedge C) \vdash A \vee B \qquad \dfrac{\dfrac{\dfrac{}{A \wedge (B \wedge C) \vdash A \wedge (B \wedge C)}\,(I)}{A \wedge (B \wedge C) \vdash B \wedge C} \qquad \dfrac{\dfrac{}{B \wedge C \vdash B \wedge C}\,(I)}{B \wedge C \vdash C}}{A \wedge (B \wedge C) \vdash C}\,(\mathrm{Cut})}{A \wedge (B \wedge C) \vdash (A \vee B) \wedge C}$$

Focus on the remaining goal

$$\dfrac{\dfrac{\dfrac{}{A \wedge (B \wedge C) \vdash A \wedge (B \wedge C)}\,(I)}{A \wedge (B \wedge C) \vdash A} \qquad \dfrac{\dfrac{}{A \vee B \vdash A \vee B}\,(I)}{A \vdash A \vee B}}{A \wedge (B \wedge C) \vdash A \vee B}\,(\mathrm{Cut})$$

(c) $A \to B,\ A \wedge C \vdash B \wedge C$

$$
\dfrac{
  \dfrac{
    \dfrac{\dfrac{}{A \wedge C \vdash A \wedge C}\,(I)}{A \wedge C \vdash A}
    \qquad
    \dfrac{\dfrac{}{A \to B \vdash A \to B}\,(I)}{A,\ A \to B \vdash B}
  }{A \to B,\ A \wedge C \vdash B}\,(\mathrm{Cut})
  \qquad
  \dfrac{\dfrac{}{A \to B,\ A \wedge C \vdash A \wedge C}\,(I)}{A \to B,\ A \wedge C \vdash C}
}{A \to B,\ A \wedge C \vdash B \wedge C}
$$

(d) $A \vee B \to C,\ C \to A \vdash B \to C$

$$
\dfrac{
  \dfrac{
    \dfrac{\dfrac{}{C \to A, A \vee B \vdash A \vee B}\,(I)}{C \to A, B \vdash A \vee B}
    \qquad
    \dfrac{\dfrac{}{A \vee B \to C \vdash A \vee B \to C}\,(I)}{A \vee B, A \vee B \to C \vdash C}
  }{A \vee B \to C,\ C \to A, B \vdash C}\,(\mathrm{Cut})
}{A \vee B \to C,\ C \to A \vdash B \to C}
$$

(e) $A \to C \vdash A \to (B \vee C)$

$$
\dfrac{
  \dfrac{
    \dfrac{\dfrac{}{A \to C \vdash A \to C}\,(I)}{A \to C, A \vdash C}
    \qquad
    \dfrac{\dfrac{}{B \vee C \vdash B \vee C}\,(I)}{C \vdash B \vee C}
  }{A \to C, A \vdash B \vee C}\,(\mathrm{Cut})
}{A \to C \vdash A \to (B \vee C)}
$$

The proofs you have given will generally include a mixture of introduction and elimination rules—and you may have found that it is easy to go round in circles. This makes it tricky to find proofs, and tricky to show that this set of rules is complete.

For the remainder of this exercise sheet, we will use a different set of rules, designed to make it easy to find proofs, and to enable a simple completeness proof.

$$\frac{}{\Gamma, A \vdash \Delta, A}\ (I)$$

$$\frac{\Gamma, A, B \vdash \Delta}{\Gamma, A \wedge B \vdash \Delta}\ (\wedge L) \qquad \frac{\Gamma \vdash A, B, \Delta}{\Gamma \vdash A \vee B, \Delta}\ (\vee R)$$

$$\frac{\Gamma, A \vdash \Delta \quad \Gamma, B \vdash \Delta}{\Gamma, A \vee B \vdash \Delta}\ (\vee L) \qquad \frac{\Gamma \vdash A, \Delta \quad \Gamma \vdash B, \Delta}{\Gamma \vdash A \wedge B, \Delta}\ (\wedge R)$$

$$\frac{\Gamma \vdash A, \Delta \quad \Gamma, B \vdash \Delta}{\Gamma, A \to B \vdash \Delta}\ (\to L) \qquad \frac{\Gamma, A \vdash B, \Delta}{\Gamma \vdash A \to B, \Delta}\ (\to R)$$

$$\frac{\Gamma \vdash A, \Delta}{\Gamma, \neg A \vdash \Delta}\ (\neg L) \qquad \frac{\Gamma, A \vdash \Delta}{\Gamma \vdash \neg A, \Delta}\ (\neg R)$$

We now allow sequents that include multiple premisses *and* multiple conclusions: Γ, Δ vary over finite sets of expressions; A, B vary over expressions. The intended interpretation is that if *all* of the premises are true then at least one of the conclusions is true. This seemingly minor change allowed Gentzen to introduce this beautifully symmetric set of rules, which are all introduction rules. This means that a goal-directed proof will always produce simpler and simpler sequents (but maybe many many simpler sequents) as our trees grow upwards.

6. For each of the rules show that if some valuation makes at least one of the entailments above the line false, then it makes the statement below the line false.

   This is a straight-forward exercise in truth-table argumentation. A counter-example makes everything on the left true, and everything on the right false. So, for example, a counterexample to either of the premises of $(\to L)$ makes everything in Γ true, and everything in Δ false. A counter-example to the first premiss makes A false, while a counter-example to the second premiss makes B true; in either case, $A \to B$ is true and we have a counter-example to the conclusion.

7. For each of the entailments listed below, construct a proof tree, by applying these new rules until the leaves of your tree contain no connectives. Then say whether the entailment is valid. How can a proof attempt fail? How can you can construct a falsifying valuation from a failed proof attempt?

(a) $\mathsf{B} \wedge \mathsf{C} \vdash (\mathsf{A} \to \mathsf{B}) \wedge (\mathsf{A} \to \mathsf{C})$

$$\dfrac{\dfrac{\dfrac{\dfrac{}{A,B,C \vdash B}\,(I)}{B,C \vdash A \to B}\,(\to R) \qquad \dfrac{\dfrac{}{A,B,C \vdash C}\,(I)}{B,C \vdash A \to C}\,(\to R)}{B,C \vdash (A \to B) \wedge (A \to C)}\,(\wedge R)}{B \wedge C \vdash (A \to B) \wedge (A \to C)}\,(\wedge L)$$

(b) $\mathsf{A} \wedge (\mathsf{B} \wedge \mathsf{C}) \vdash (\mathsf{A} \wedge \mathsf{B}) \wedge \mathsf{C}$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{A,B,C \vdash A}\,(I) \qquad \dfrac{}{A,B,C \vdash B}\,(I)}{A,B,C \vdash A \wedge B}\,(\wedge R) \qquad \dfrac{}{A,B,C \vdash C}\,(I)}{A,B,C \vdash (A \wedge B) \wedge C}\,(\wedge R)}{A, B \wedge C \vdash (A \wedge B) \wedge C}\,(\wedge L)}{A \wedge (B \wedge C) \vdash (A \wedge B) \wedge C}\,(\wedge L)$$

(c) $\mathsf{A} \to \mathsf{B}, \mathsf{A} \wedge \mathsf{C} \vdash \mathsf{B} \wedge C$

$$\dfrac{\dfrac{\dfrac{}{A,C \vdash A, B \wedge C}\,(I) \qquad \dfrac{\dfrac{}{A,B,C \vdash B}\,(I) \qquad \dfrac{}{A,B,C \vdash C}\,(I)}{A,B,C \vdash B \wedge C}\,(\wedge R)}{A \to B, A, C \vdash B \wedge C}\,(\to L)}{A \to B, A \wedge C \vdash B \wedge C}\,(\wedge L)$$

(d) $\mathsf{A} \vee \mathsf{B} \to \mathsf{C},\ \mathsf{C} \to \mathsf{A} \vdash \mathsf{C} \to \mathsf{B}$

$$\dfrac{\dfrac{\dfrac{\dfrac{}{C \vdash B, A \vee B, C}\,(I) \qquad \dfrac{\dfrac{}{A,C \vdash B,A,B}\,(I)}{A,C \vdash B, A \vee B}\,(\vee R)}{C \to A, C \vdash B, A \vee B}\,(\to L) \qquad \dfrac{\dfrac{}{C \vdash B, C}\,(I) \qquad A, C \vdash B, \mathit{failure!}}{C \to A, C \vdash B}\,(\to L)}{A \vee B \to C, C \to A, C \vdash B}\,(\to L)}{A \vee B \to C, C \to A \vdash C \to B}\,(\to R)$$

$(A$	$\vee$	$B)$	$\to$	$C,$	C	$\to$	A	$\vdash$	C	$\to$	B
T	T	F	T	T	T	T	T		T	F	F

The valuation $A = \mathit{true}$, $B = \mathit{false}$, $C = \mathit{true}$ invalidates this inference.

(e) $\mathsf{A} \to \mathsf{C} \vdash \mathsf{A} \to (\mathsf{B} \vee \mathsf{C})$

$$\dfrac{\dfrac{\dfrac{\dfrac{}{A \vdash A,B,C}\,(I) \qquad \dfrac{}{C,A \vdash B,C}\,(I)}{A \to C, A \vdash B, C}\,(\to L)}{A \to C, A \vdash B \vee C}\,(\vee R)}{A \to C \vdash A \to (B \vee C)}\,(\to R)$$

Note that for these rules there are no cut formulae to find — and that the choice of rules to apply is limited by the context.

8. Explain why the set of rules on page 7 is complete, by arguing that applying them repeatedly, until there are no connectives in any leaf of the tree, will either produce a proof or a counterexample.

   Working in goal-directed mode, each rule except (I) removes a connective, so the depth of a proof attempt is bounded by $N + 1$ where N is the number of connectives in our original goal. (Note that, although there may be many paths, nothing is duplicated along any path up the tree.) So, every proof attempt must terminate. Any undischarged sub-goals have no connectives. Just lists of atoms either side of a turnstile, $\vdash$. For each subgoal, these lists are disjoint (otherwise we could apply (I) and discharge the goal) so making everything on the left true and all on the right false provides a counterexample.

   By 6 this is a counterexample to all succeeding steps in the proof attempt — in particular, to the conclusion.

This tutorial exercise sheet was written by Paolo Besana, and extended by Thomas French Areti Manataki, and Michael Fourman. Send comments to `Michael.Fourman@ed.ac.uk`