Informatics 1 - Computation & Logic: Tutorial 8

Propositional Logic: Sequent Calculus

Week 10: 23-27 November 2015

Please attempt the entire worksheet in advance of the tutorial, and bring with you all work, including (if a computer is involved) printouts of code and test results. Tutorials cannot function properly unless you do the work in advance.

You may work with others, but you must understand the work; you can't phone a friend during the exam.

Assessment is formative, meaning that marks from coursework do not contribute to the final mark. But coursework is not optional. If you do not do the coursework you are unlikely to pass the exams.

Attendance at tutorials is **obligatory**; please let your tutor know if you cannot attend.

In this tutorial we consider relations generated by rules. A rule of the form:

$$\frac{\beta_1 \quad \cdots \quad \beta_n}{\alpha}$$

allows us to derive the conclusion α from the premisses $\beta_1, \ldots, \beta_n$.

As a first example, consider defining the grammar of a language. We give the following rules:

$$\frac{}{\text{ideas} : \mathbf{N}} \quad \frac{}{\text{linguists} : \mathbf{N}} \quad \frac{}{\text{great} : \mathbf{A}} \quad \frac{}{\text{green} : \mathbf{A}} \quad \frac{}{\text{hate} : \mathbf{V}} \quad \frac{}{\text{generate} : \mathbf{V}}$$

$$\frac{X : \mathbf{V}}{X : \mathbf{VP}}\,(V) \qquad \frac{X : \mathbf{V} \quad Y : \mathbf{NP}}{XY : \mathbf{VP}}\,(VP) \qquad \frac{X : \mathbf{NP} \quad Y : \mathbf{VP}}{XY : \mathbf{S}}\,(S)$$

$$\frac{X : \mathbf{N}}{X : \mathbf{NP}}\,(N) \qquad \frac{X : \mathbf{A} \quad Y : \mathbf{NP}}{XY : \mathbf{NP}}\,(NP)$$

Here, "ideas:**N**" means that 'ideas' is a noun. Our rules allow us to infer that particular phrases belong to various grammatical categories: noun (**N**), adjective (**A**), verb (**V**), noun-phrase (**NP**), verb-phrase (**VP**), and sentence (**S**). The variables X, Y range over phrases, where phrases are non-empty lists of words. The rules are labelled, (V), (VP), etc., for ease of reference.

For example, we can show that, "great linguists generate green ideas" is a sentence. In symbols,

$$\text{great linguists generate green ideas} : \mathbf{S}$$

We do this by constructing a tree:

$$\dfrac{\dfrac{\dfrac{}{\text{great} : \mathbf{A}} \quad \dfrac{\dfrac{}{\text{linguists} : \mathbf{N}}}{\text{linguists} : \mathbf{NP}}\,(N)}{\text{great linguists} : \mathbf{NP}}\,(NP) \quad \dfrac{\dfrac{}{\text{generate} : \mathbf{V}} \,(VP)\quad \dfrac{\dfrac{}{\text{green} : \mathbf{A}} \quad \dfrac{\dfrac{}{\text{ideas} : \mathbf{N}}}{\text{ideas} : \mathbf{NP}}\,(N)}{\text{green ideas} : \mathbf{NP}}\,(NP)}{\text{generate green ideas} : \mathbf{VP}}}{\text{great linguists generate green ideas} : \mathbf{S}}\,(S)$$

1. Which of the following are sentences for this grammar?
   (a) green linguists hate great ideas
   (b) green green green linguists hate
   (c) generate ideas
   (d) green ideas generate hate
2. How might you extend the grammar to include the sentence, "colourless green ideas sleep furiously"?
3. We say that a grammar is *sound* if it only generates grammatical sentences, and that it is *complete* if every grammatical sentence can be generated by the rules.
   (a) Is it is possible to give a sound grammar for a natural language?

   (b) Is it possible to give a complete grammar for a natural language?

(c) Is every grammatical sentence true?

(d) Is it possible to write a grammar that will only generate true sentences?

For our second example we introduce some simple logical rules.

$$\frac{}{\mathcal{A}, X \vdash X}\ (I) \qquad\qquad \frac{\Gamma \vdash A \quad \Delta, A \vdash B}{\Gamma, \Delta \vdash B}\ \mathit{Cut}$$

$$\begin{array}{c} \mathcal{A} \vdash X \quad \mathcal{A} \vdash Y \\ \hline\hline \mathcal{A} \vdash X \wedge Y \end{array}\ (\wedge) \qquad \begin{array}{c} \mathcal{A}, X \vdash Z \quad \mathcal{A}, Y \vdash Z \\ \hline\hline \mathcal{A}, X \vee Y \vdash Z \end{array}\ (\vee) \qquad \begin{array}{c} \mathcal{A}, X \vdash Y \\ \hline\hline \mathcal{A} \vdash X \to Y \end{array}\ (\to)$$

Here, $\mathcal{A}$ is a variable over sets of expressions of propositional logic, and X, Y and Z are variables over expressions themselves. We read the 'turnstile' $\vdash$ symbol as *entails*.

The *immediate* rule (I) has no assumptions. The double line used for the other three rules means that the rule can be used in either direction. The entailment below the double line is valid iff *all* of the entailments above the line are valid. Read from top to bottom, they are called *introduction rules* $(^+)$, since they introduce a new connective into the argument. Read from bottom to top, they are *elimination rules* $(^-)$ since a connective is eliminated.

These rules are designed to allow us to produce *valid* entailments. We say that a valuation makes $\mathcal{A} \vdash X$ true if it makes at least one of the assumptions $A \in \mathcal{A}$ *false* or it makes X *true*. The entailment is valid iff every valuation makes it true. So it is valid iff any valuation that makes all the premisses in $\mathcal{A}$ true also makes X true.[1]

4. We have claimed that these rules are sound. This exercise asks you to show something stronger. For each of the rules $(\wedge), (\vee), (\to)$ show that every valuation makes the entailment below the line true iff it makes all of the entailments above the line true.

Using these rules we can prove validity. For example, the following proof tree:

$$\dfrac{\dfrac{\dfrac{\dfrac{}{A \to B, C \vdash A \to B}\,(I)}{A \to B, C, A \vdash B}\,(\to^-) \qquad \dfrac{}{A \to B, C, A \vdash C}\,(I)}{A \to B, C, A \vdash B \wedge C}\,(\wedge^+)}{A \to B, C \vdash A \to (B \wedge C)}\,(\to^+)$$

shows that $A \to B, C \vdash A \to (B \wedge C)$ is valid, using a mixture of introduction and elimination rules.

Here are some tips for proving the validity of arguments with the use of proof rules:

[1]Note that the rule (I) is certainly sound, since X occurs on both sides of the turnstile.

- A branch of the proof tree is considered to be proved when the *immediate* rule is applied. All the branches of the proof tree need to be proved.
- Each entailment introduced by a rule must be justified by another rule.
- Remember to include the name of the rule that you are applying at each point.

5. For each of the entailments listed either give a proof tree that shows it is valid, or give a valuation that shows it is invalid.

   (a) $\mathsf{B} \wedge \mathsf{C} \vdash (\mathsf{A} \to \mathsf{B}) \wedge (\mathsf{A} \to \mathsf{C})$

   (b) $\mathsf{A} \wedge (\mathsf{B} \wedge \mathsf{C}) \vdash (\mathsf{A} \vee \mathsf{B}) \wedge \mathsf{C}$

   (c) $\mathsf{A} \to \mathsf{B}, \mathsf{A} \wedge \mathsf{C} \vdash \mathsf{B} \wedge \mathsf{C}$

   (d) $\mathsf{A} \vee \mathsf{B} \to \mathsf{C}, \mathsf{C} \to \mathsf{A} \vdash \mathsf{B} \to \mathsf{C}$

(e) $\mathsf{A} \to \mathsf{C} \vdash \mathsf{A} \to (\mathsf{B} \lor \mathsf{C})$

The proofs you have given will generally include a mixture of introduction and elimination rules—and you may have found that it is easy to go round in circles. This makes it tricky to find proofs, and tricky to show that this set of rules is complete.

For the remainder of this exercise sheet, we will use a different set of rules, designed to make it easy to find proofs, and to enable a simple completeness proof.

$$\frac{}{\Gamma, A \vdash \Delta, A}\ (I)$$

$$\frac{\Gamma, A, B \vdash \Delta}{\Gamma, A \wedge B \vdash \Delta}\ (\wedge L) \qquad \frac{\Gamma \vdash A, B, \Delta}{\Gamma \vdash A \vee B, \Delta}\ (\vee R)$$

$$\frac{\Gamma, A \vdash \Delta \quad \Gamma, B \vdash \Delta}{\Gamma, A \vee B \vdash \Delta}\ (\vee L) \qquad \frac{\Gamma \vdash A, \Delta \quad \Gamma \vdash B, \Delta}{\Gamma \vdash A \wedge B, \Delta}\ (\wedge R)$$

$$\frac{\Gamma \vdash A, \Delta \quad \Gamma, B \vdash \Delta}{\Gamma, A \to B \vdash \Delta}\ (\to L) \qquad \frac{\Gamma, A \vdash B, \Delta}{\Gamma \vdash A \to B, \Delta}\ (\to R)$$

$$\frac{\Gamma \vdash A, \Delta}{\Gamma, \neg A \vdash \Delta}\ (\neg L) \qquad \frac{\Gamma, A \vdash \Delta}{\Gamma \vdash \neg A, \Delta}\ (\neg R)$$

We now allow sequents that include multiple premisses *and* multiple conclusions: Γ, Δ vary over finite sets of expressions; A, B vary over expressions. The intended interpretation is that if *all* of the premises are true then at least one of the conclusions is true. This seemingly minor change allowed Gentzen to introduce this beautifully symmetric set of rules, which are all introduction rules. This means that a goal-directed proof will always produce simpler and simpler sequents (but maybe many many simpler sequents) as our trees grow upwards.

6. For each of the rules show that if some valuation makes at least one of the entailments above the line false, then it makes the statement below the line false.

7. For each of the entailments listed below, construct a proof tree, by applying these new rules until the leaves of your tree contain no connectives. Then say whether the entailment is valid. How can a proof attempt fail? How can you can construct a falsifying valuation from a failed proof attempt?

   (a) $\mathsf{B} \wedge \mathsf{C} \vdash (\mathsf{A} \rightarrow \mathsf{B}) \wedge (\mathsf{A} \rightarrow \mathsf{C})$

   (b) $\mathsf{A} \wedge (\mathsf{B} \wedge \mathsf{C}) \vdash (\mathsf{A} \wedge \mathsf{B}) \wedge \mathsf{C}$

   (c) $\mathsf{A} \rightarrow \mathsf{B}, \mathsf{A} \wedge \mathsf{C} \vdash \mathsf{B} \wedge C$

   (d) $\mathsf{A} \vee \mathsf{B} \rightarrow \mathsf{C},\ \mathsf{C} \rightarrow \mathsf{A} \vdash \mathsf{C} \rightarrow \mathsf{B}$

   (e) $\mathsf{A} \rightarrow \mathsf{C} \vdash \mathsf{A} \rightarrow (\mathsf{B} \vee \mathsf{C})$

8. Explain why the set of rules on page 6 is complete, by arguing that applying them repeatedly, until there are no connectives in any leaf of the tree, will either produce a proof or a counterexample.

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