Informatics 1 - Computation & Logic: Tutorial 2

Propositional Logic: Truth Tables

Week 4: 12-18 October 2014

Please attempt the entire worksheet in advance of the tutorial, and bring with you all work, including (if a computer is involved) printouts of code and test results. Tutorials cannot function properly unless you do the work in advance.

You may work with others, but you must understand the work; you can't phone a friend during the exam.

Assessment is formative, meaning that marks from coursework do not contribute to the final mark. But coursework is not optional. If you do not do the coursework you are unlikely to pass the exams.

Attendance at tutorials is **obligatory**; please let your tutor know if you cannot attend.

1. Look up the following terms, on the internet or elsewhere, and describe, in words, when an expression in propositional logic is:

   (a) Satisfiable:

Of an expression in propositional logic, that it is true for at least one valuation of its atoms

   (b) Tautologous:

Of an expression in propositional logic, that it is true for at all valuations of its atoms.

(c) Contingent:

Of an expression in propositional logic, that it is true for at least one valuation of its atoms, and false for at least one valuation

(d) Inconsistent:

Of an expression in propositional logic, that it is false for at all valuations of its atoms.

2. Which combinations of these four properties, SATISFIABLE/TAUTOLOGOUS/CONTINGENT/INCONSISTENT, are possible?

   Your answer should be a boolean formula, using the propositional letters S, T, C, I, that characterises the possible combinations of these properties.

   $(S \wedge T \wedge \neg C \wedge \neg I) \vee (S \wedge \neg T \wedge C \wedge \neg I) \vee (\neg S \wedge \neg T \wedge \neg C \wedge I)$

3. Construct truth tables for the following expressions of propositional logic, and use these to decide whether the expressions are satisfiable, tautologous, contingent, or inconsistent:

   (a) $(A \rightarrow B) \vee (\neg A \vee \neg B)$

   Draw the truth table here:

(A	→	B)	∨	(¬	A	∨	¬	B)
T	T	T	**T**	F	T	F	F	T
T	F	F	**T**	F	T	T	T	F
F	T	T	**T**	T	F	T	F	T
F	T	F	**T**	T	F	T	T	F

   This expression is SATISFIABLE/TAUTOLOGOUS

   (b) $\neg(A \wedge \neg B) \leftrightarrow \neg(\neg A \vee B)$

   Draw the truth table here:

¬	(A	∧	¬	B)	↔	¬	(¬	A	∨	B)
T	T	F	F	T	**F**	F	F	T	T	T
F	T	T	T	F	**F**	T	F	T	F	F
T	F	F	F	T	**F**	F	T	F	T	T
T	F	F	T	F	**F**	F	T	F	T	F

This expression is INCONSISTENT

(c) $A \to (B \wedge (A \vee B))$

Draw the truth table here:

A	→	(B	∧	(A	∨	B))
T	**T**	T	T	T	T	T
T	**F**	F	F	T	T	F
F	**T**	T	T	F	T	T
F	**T**	F	F	F	F	F

This expression is SATISFIABLE/CONTINGENT

(d) $(\neg A \wedge B) \vee C \leftrightarrow ((A \vee \neg B) \to C)$

Draw the truth table here:

(¬	A	∧	B)	∨	C	↔	((A	∨	¬	B)	→	C)
F	T	F	T	T	T	**T**	T	T	F	T	T	T
F	T	F	T	F	F	**T**	T	T	F	T	F	F
F	T	F	F	T	T	**T**	T	T	T	F	T	T
F	T	F	F	F	F	**T**	T	T	T	F	F	F
T	F	T	T	T	T	**T**	F	F	F	T	T	T
T	F	T	T	T	F	**T**	F	F	F	T	T	F
T	F	F	F	T	T	**T**	F	T	T	F	T	T
T	F	F	F	F	F	**T**	F	T	T	F	F	F

This expression is SATISFIABLE/TAUTOLOGOUS

4. This question concens the same expressions of propositional logic as the previous exercise. In each case, use the laws of Boolean Algebra to derive an equivalent CNF for the given expression. Eliminate arrows; push negations down; push disjunctions down; and simplify, using commutativity and associativity of $\vee, \wedge$, together with the following rules:

$$\neg(a \to b) = a \wedge \neg b \qquad a \leftrightarrow b = (a \to b) \wedge (b \to a) \qquad a \to b = \neg a \vee b$$

$$\neg(a \vee b) = \neg a \wedge \neg b \qquad \neg(a \wedge b) = \neg a \vee \neg b$$

$$\neg 0 = 1 \qquad \neg\neg a = a \qquad \neg 1 = 0$$

$$a \vee 1 = 1 \qquad a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c) \qquad a \wedge 0 = 0$$

$$a \vee 0 = a \qquad a \vee \neg a = 1 \qquad a \wedge \neg a = 0 \qquad a \wedge 1 = a$$

(a) $(A \to B) \vee (\neg A \vee \neg B)$

Show your working here:

$(A \to B) \vee (\neg A \vee \neg B)$
$(\neg A \vee B) \vee (\neg A \vee \neg B)$ - by arrow elimination
$\neg A \vee B \vee \neg A \vee \neg B$ - by associativity. Note, this is CNF.
$\neg A \vee B \vee \neg B$
$\neg A \vee 1$
1

This expression is SATISFIABLE/TAUTOLOGOUS

(b) $\neg(A \wedge \neg B) \leftrightarrow \neg(\neg A \vee B)$

Show your working here:

$\neg(A \wedge \neg B) \leftrightarrow \neg(\neg A \vee B)$
$\neg(A \wedge \neg B) \leftrightarrow (\neg\neg A \wedge \neg B)$ - by De Morgan's Law
$(\neg A \vee \neg\neg B) \leftrightarrow (\neg\neg A \wedge \neg B)$ - De Morgan
$(\neg A \vee B) \leftrightarrow (\neg\neg A \wedge \neg B)$ - double negation elimination
$(\neg A \vee B) \leftrightarrow (A \wedge \neg B)$ - $\neg\neg$ elim
$((\neg A \vee B) \rightarrow (A \wedge \neg B)) \wedge ((A \wedge \neg B) \rightarrow (\neg A \vee B))$ - biconditional equivalence
$(\neg(\neg A \vee B) \vee (A \wedge \neg B)) \wedge ((A \wedge \neg B) \rightarrow (\neg A \vee B))$ - arrow elim
$(\neg(\neg A \vee B) \vee (A \wedge \neg B)) \wedge (\neg(A \wedge \neg B) \vee (\neg A \vee B))$ - arrow elim
$((\neg\neg A \wedge \neg B) \vee (A \wedge \neg B)) \wedge (\neg(A \wedge \neg B) \vee (\neg A \vee B))$ - De Morgan
$((\neg\neg A \wedge \neg B) \vee (A \wedge \neg B)) \wedge ((\neg A \vee \neg\neg B) \vee (\neg A \vee B))$ - De Morgan
$((\neg\neg A \wedge \neg B) \vee (A \wedge \neg B)) \wedge ((\neg A \vee B) \vee (\neg A \vee B))$ - $\neg\neg$ elim
$((A \wedge \neg B) \vee (A \wedge \neg B)) \wedge ((\neg A \vee B) \vee (\neg A \vee B))$ - $\neg\neg$ elim
$((A \wedge \neg B) \vee (A \wedge \neg B)) \wedge (\neg A \vee B \vee \neg A \vee B)$ - associativity
$((A \wedge \neg B) \vee (A \wedge \neg B)) \wedge (B \vee \neg A \vee B)$
$((A \wedge \neg B) \vee (A \wedge \neg B)) \wedge (\neg A \vee B)$
$(A \wedge \neg B) \wedge (\neg A \vee B)$
$A \wedge \neg B \wedge (\neg A \vee B)$ - associativity. Note, this is CNF
$A \wedge \neg B \wedge \neg\neg(\neg A \vee B)$ - double negation introduction
$A \wedge \neg B \wedge \neg(\neg\neg A \wedge \neg B)$ - De Morgan
$A \wedge \neg B \wedge \neg(A \wedge \neg B)$ - $\neg\neg$ elim
$(A \wedge \neg B) \wedge \neg(A \wedge \neg B)$ - associativity
0

This expression is INCONSISTENT

(c) $A \rightarrow (B \wedge (A \vee B))$

Show your working here:

$A \rightarrow (B \wedge (A \vee B))$
$\neg A \vee (B \wedge (A \vee B))$ - arrow elim
$\neg A \vee ((B \wedge A) \vee (B \wedge B))$ - distributivity
$\neg A \vee ((B \wedge A) \vee B)$
$\neg A \vee (B \wedge A) \vee B$ - associativity
$((\neg A \vee B) \wedge (\neg A \vee A)) \vee B$ - distributivity
$((\neg A \vee B) \wedge 1) \vee B$
$(\neg A \vee B) \vee B$
$\neg A \vee B \vee B$ - associativity
$\neg A \vee B$ - associativity

This expression is SATISFIABLE/CONTINGENT

(d) $(\neg A \wedge B) \vee C \leftrightarrow ((A \vee \neg B) \rightarrow C)$

Show your working here:

$(\neg A \wedge B) \vee C \leftrightarrow ((A \vee \neg B) \rightarrow C)$
$(\neg A \wedge B) \vee C \leftrightarrow (\neg(A \vee \neg B) \vee C)$ - arrow elim
$(\neg A \wedge B) \vee C \leftrightarrow ((\neg A \wedge \neg\neg B) \vee C)$ - De Morgan
$(\neg A \wedge B) \vee C \leftrightarrow ((\neg A \wedge B) \vee C)$ - $\neg\neg$ elim
$((\neg A \wedge B) \vee C) \leftrightarrow ((\neg A \wedge B) \vee C)$ - associativity
$\phi \leftrightarrow \phi$ - For clarity, we stipulate $\phi = (\neg A \wedge B) \vee C$
$(\phi \rightarrow \phi) \wedge (\phi \rightarrow \phi)$ - biconditional elim
$(\phi \rightarrow \phi)$
$\neg\phi \vee \phi$ - arrow elim
1

This expression is SATISFIABLE/TAUTOLOGOUS

5. (a) How many rows will a truth table for an expression in propositional logic with n atomic propositions have? Why?

   2^n - because there are two possible valuations for each atom, and each must be combined with every possible valuation of all other atoms.

   (b) In general, is this a limitation? Why?

   Yes, in relation to the computational resources required to determine the satisfiability of an expression in propositional logic; number of lines in the truth table for an expression doubles for every additional atom - and so the problem becomes unmanageably complex for even a fairly modest number of atoms. The truth table for a 10-atom expression will have 1,024 lines; for a 20 atoms, 1,048,576; for 100, 1,267,650,600,228,229,401,496,703,205,376!

6. An *entailment* of propositional logic is of the form

$$\phi_1, \ldots, \phi_n \models \psi$$

where ϕ_i, ψ are all expressions of propositional logic. The ϕ_i expressions are the *premises* of the entailment and ψ is the *conclusion*. An entailment is *valid* if and only if there is no possible assignment of truth values to atomic propositional symbols such that the premises are all true and the conclusion false.

Using a truth table, determine whether the following entailments are valid or invalid:

   (a) $(A \wedge B) \rightarrow A,\ B \vee \neg A\ \models\ A \vee B$

(A	∧	B)	→	A	B	∨	¬	A	⊨	A	∨	B
T	T	T	**T**	T	T	**T**	F	T		T	**T**	T
T	F	F	**T**	T	F	**F**	F	T		T	**T**	F
F	F	T	**T**	F	T	**T**	T	F		F	**T**	T
F	F	F	**T**	F	F	**T**	T	F	counterexample	F	**F**	F

   This entailment is INVALID

(b) $\neg A \vee (B \rightarrow C),\ B \wedge C,\ C \rightarrow A \ \models\ A$

¬	A	∨	(B	→	C)	B	∧	C	C	→	A	⊨	A
F	T	**T**	T	T	T	T	**T**	T	T	**T**	T		**T**
F	T	**F**	T	F	F	T	**F**	F	F	**T**	T		**T**
F	T	**T**	F	T	T	F	**F**	T	T	**T**	T		**T**
F	T	**T**	F	T	F	F	**F**	F	F	**T**	T		**T**
T	F	**T**	T	T	T	T	**T**	T	T	**F**	F		**F**
T	F	**T**	T	F	F	T	**F**	F	F	**T**	F		**F**
T	F	**T**	F	T	T	F	**F**	T	T	**F**	F		**F**
T	F	**T**	F	T	F	F	**F**	F	F	**T**	F		**F**

This entailment is VALID

This tutorial exercise sheet was written by Mark McConville, revised by Paolo Besana, Thomas French, Areti Manataki, and Michael Fourman. Solutions by Dave Cochran. Send comments to `michael.fourman@ed.ac.uk` and `dcochra2@inf.ed.ac.uk`

Summary of useful symbols

Capital	Lowercase	Name
A	α	alpha
B	β	beta
Γ	γ	gamma
Δ	δ	delta
E	ϵ	epsilon
Z	ζ	zeta
H	η	eta
Θ	θ	theta
I	ι	iota
K	κ	kappa
Λ	λ	lambda
M	μ	mu
N	ν	nu
Ξ	ξ	xi
Π	π	pi
P	ρ	rho
Σ	σ	sigma
T	τ	tau
Υ	υ	upsilon
Φ	ϕ	phi
X	χ	chi
Ψ	ψ	psi
Ω	ω	omega

Symbol	Meaning	Example
$\neg$	not	$\neg A$
$\wedge$	and	$A \wedge B$
$\vee$	or	$A \vee B$
$\rightarrow$	implies	$A \rightarrow B$
$\models$	entails	$[\beta_1, ..., \beta_n] \models \alpha$
$\leftrightarrow$	equivalent	$A \leftrightarrow B$
$\vdash$	can be proved	$[\beta_1, ..., \beta_n] \vdash \alpha$