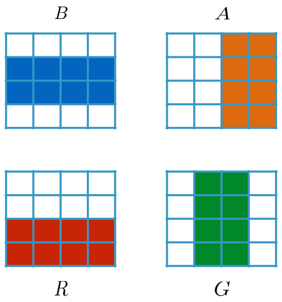
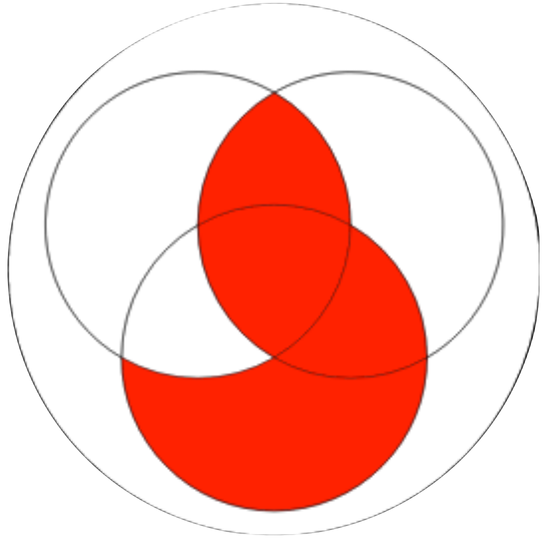


Informatics 1

Computation and Logic

CNF via Boolean Algebra

Michael Fourman



$x \vee (y \vee z) = (x \vee y) \vee z$	$x \wedge (y \wedge z) = (x \wedge y) \wedge z$	associative
$x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$	$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$	distributive
$x \vee y = y \vee x$	$x \wedge y = y \wedge x$	commutative
$x \vee 0 = x$	$x \wedge 1 = x$	identity
$x \vee 1 = 1$	$x \wedge 0 = 0$	annihilation
$x \vee x = x$	$x \wedge x = x$	idempotent
$x \vee \neg x = 1$	$\neg x \wedge x = 0$	complements
$x \vee (x \wedge y) = x$	$x \wedge (x \vee y) = x$	absorbtion
$\neg(x \vee y) = \neg x \wedge \neg y$	$\neg(x \wedge y) = \neg x \vee \neg y$	de Morgan
$\neg\neg x = x$	$x \to y = \neg x \leftarrow \neg y$	

$$\neg(a \to b) = a \wedge \neg b \qquad a \leftrightarrow b = (a \to b) \wedge (b \to a) \qquad a \to b = \neg a \vee b$$

$$\neg(a \vee b) = \neg a \wedge \neg b \qquad \neg(a \wedge b) = \neg a \vee \neg b$$

$$\neg 0 = 1 \qquad \neg\neg a = a \qquad \neg 1 = 0$$

$$a \vee 1 = 1 \qquad a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c) \qquad a \wedge 0 = 0$$

$$a \vee 0 = a \qquad a \vee \neg a = 1 \qquad a \wedge \neg a = 0 \qquad a \wedge 1 = a$$

2-SAT arrow rule

$\neg A \vee C$

$\neg B \vee D$

$\neg E \vee B$

$\neg E \vee A$

$A \vee E$

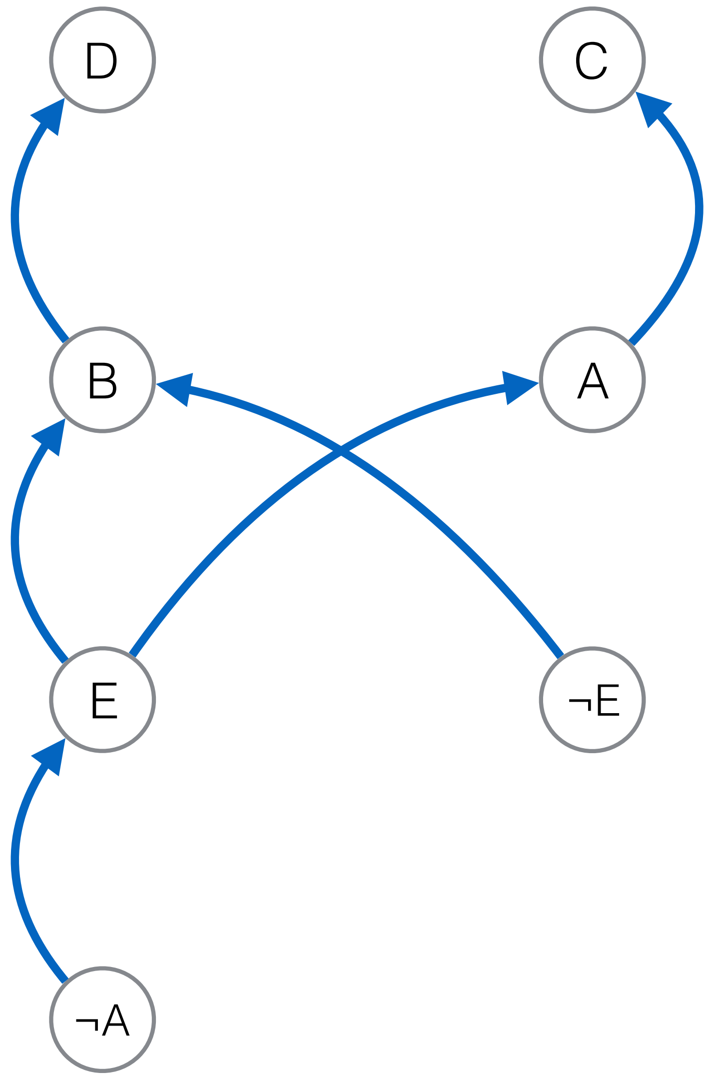
$E \vee B$

How many solutions are there to his set of constraints?

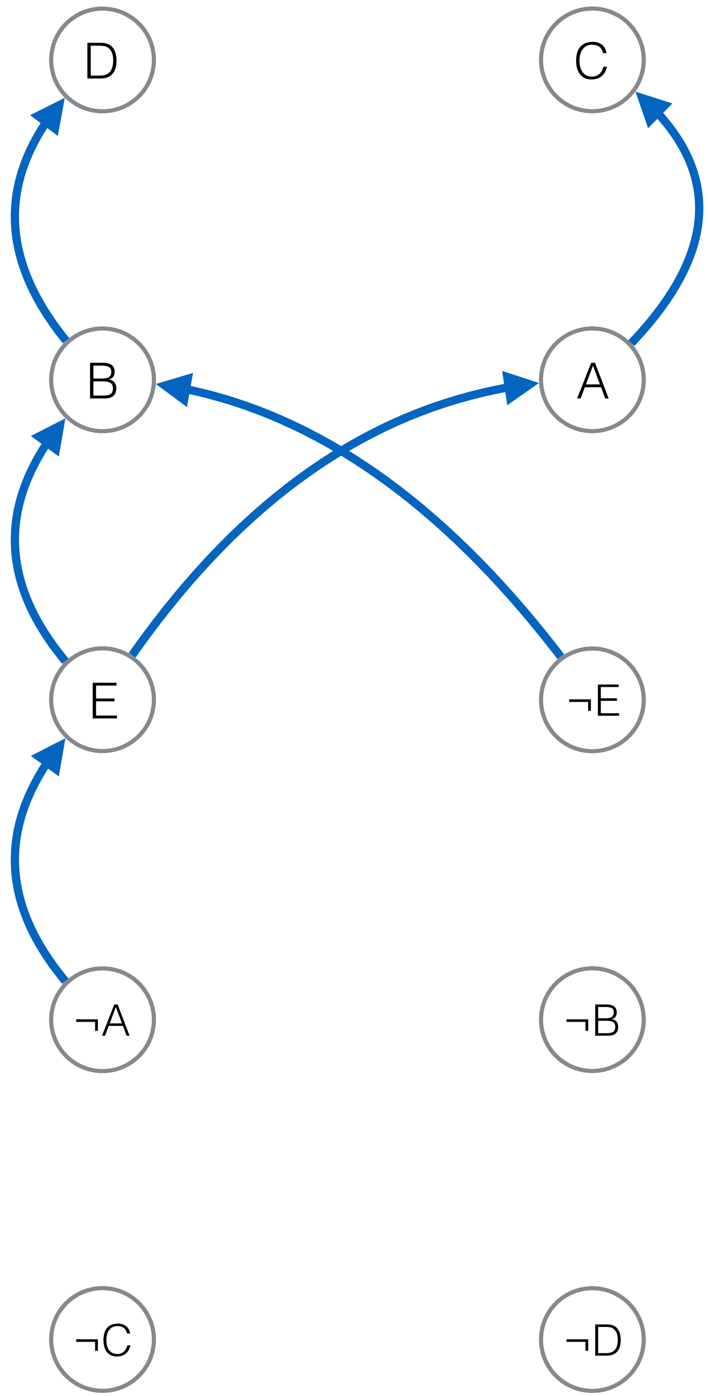
There are 32 states. Must we check them all?

For a 2-SAT problem we can use the arrow rule

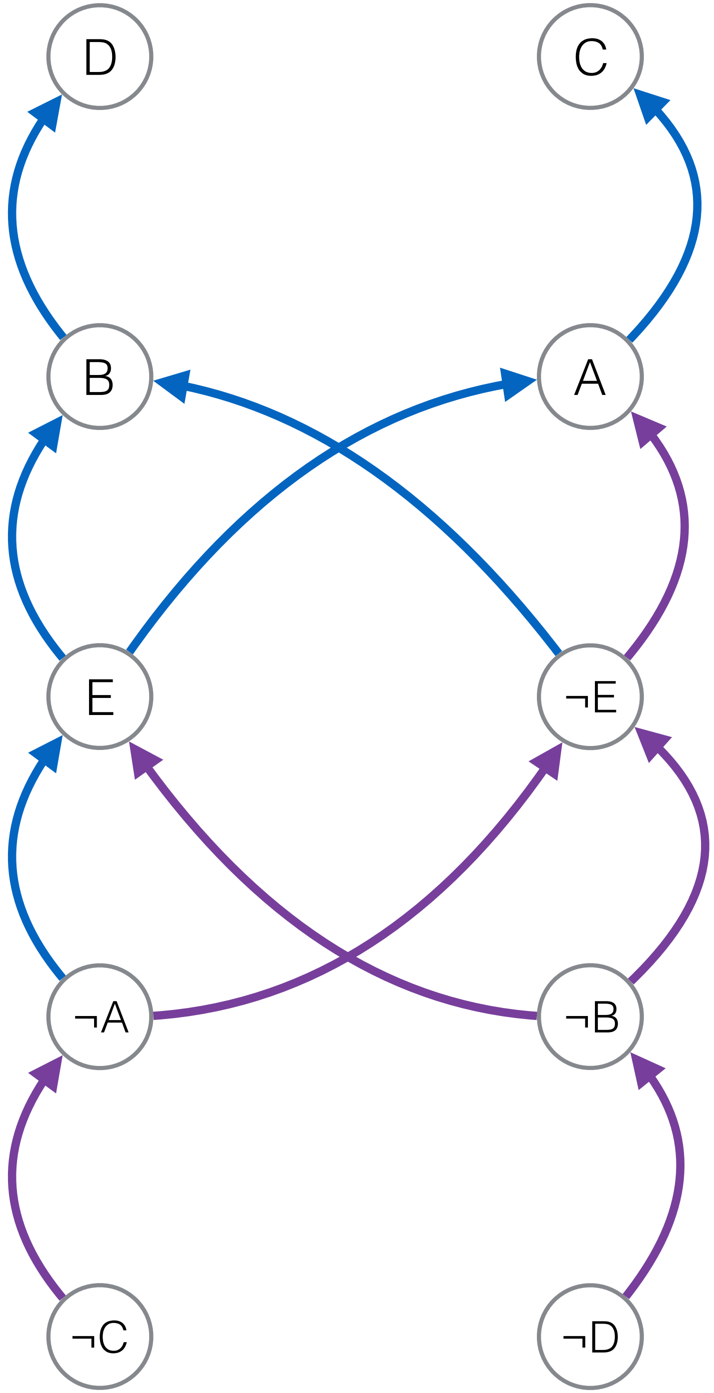
$\neg A \vee C$	$A \rightarrow C$
$\neg B \vee D$	$B \rightarrow D$
$\neg E \vee B$	$E \rightarrow B$
$\neg E \vee A$	$E \rightarrow A$
$A \vee E$	$\neg A \rightarrow E$
$E \vee B$	$\neg E \rightarrow B$



¬A ∨ C	A → C
¬B ∨ D	B → D
¬E ∨ B	E → B
¬E ∨ A	E → A
A ∨ E	¬A → E
E ∨ B	¬E → B



$\neg A \lor C$	$A \rightarrow C$	$\neg C \rightarrow \neg A$
$\neg B \lor D$	$B \rightarrow D$	$\neg D \rightarrow \neg B$
$\neg E \lor B$	$E \rightarrow B$	$\neg B \rightarrow \neg E$
$\neg E \lor A$	$E \rightarrow A$	$\neg A \rightarrow \neg E$
$A \lor E$	$\neg A \rightarrow E$	$\neg E \rightarrow A$
$E \lor B$	$\neg E \rightarrow B$	$\neg B \rightarrow E$



How many satisfying valuations?

¬A ∨ C

¬B ∨ D

¬E ∨ B

¬E ∨ A

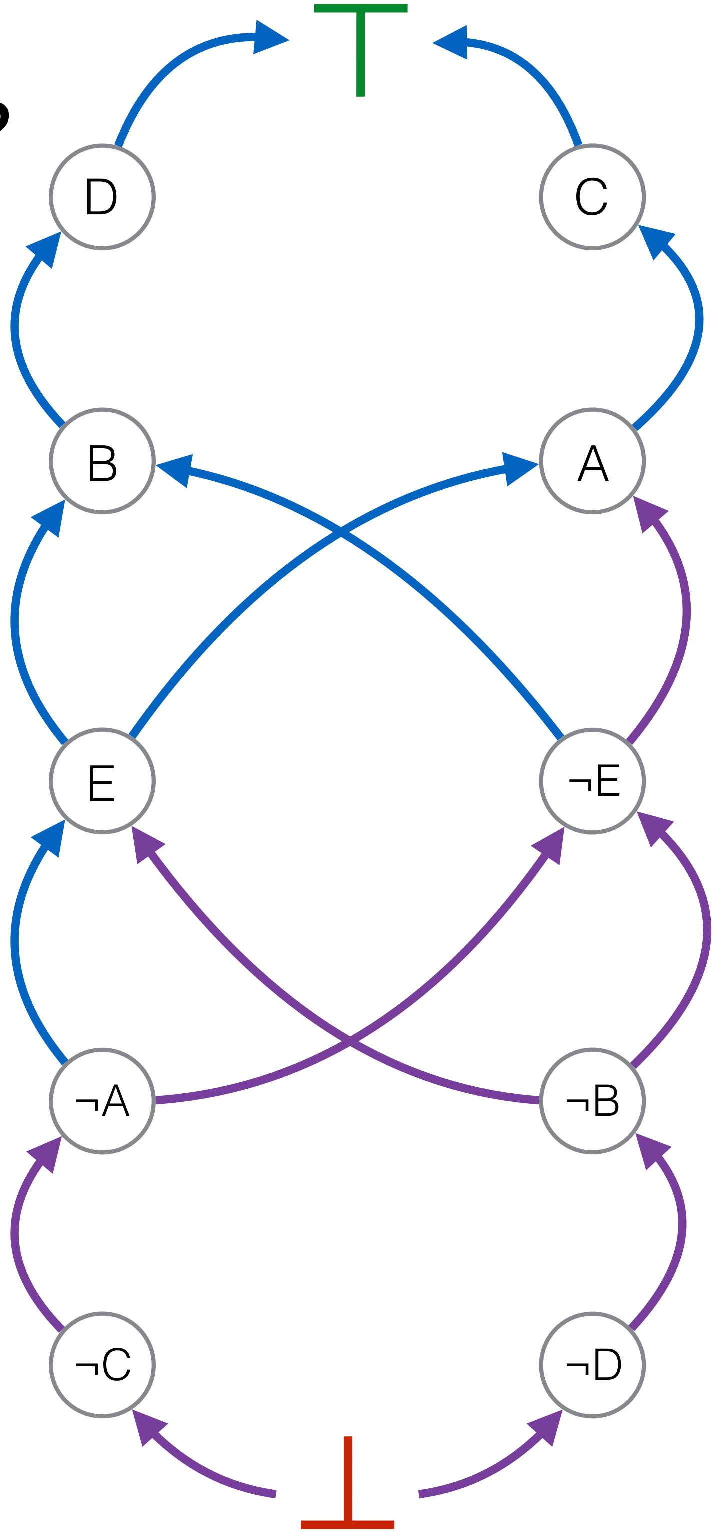
A ∨ E

E ∨ B

A satisfying valuation draws a line between false and true, such that

each atom is separated from its negation, and

no arrow leads from true to false.



How many satisfying valuations?

¬A ∨ C

¬B ∨ D

¬E ∨ B

¬E ∨ A

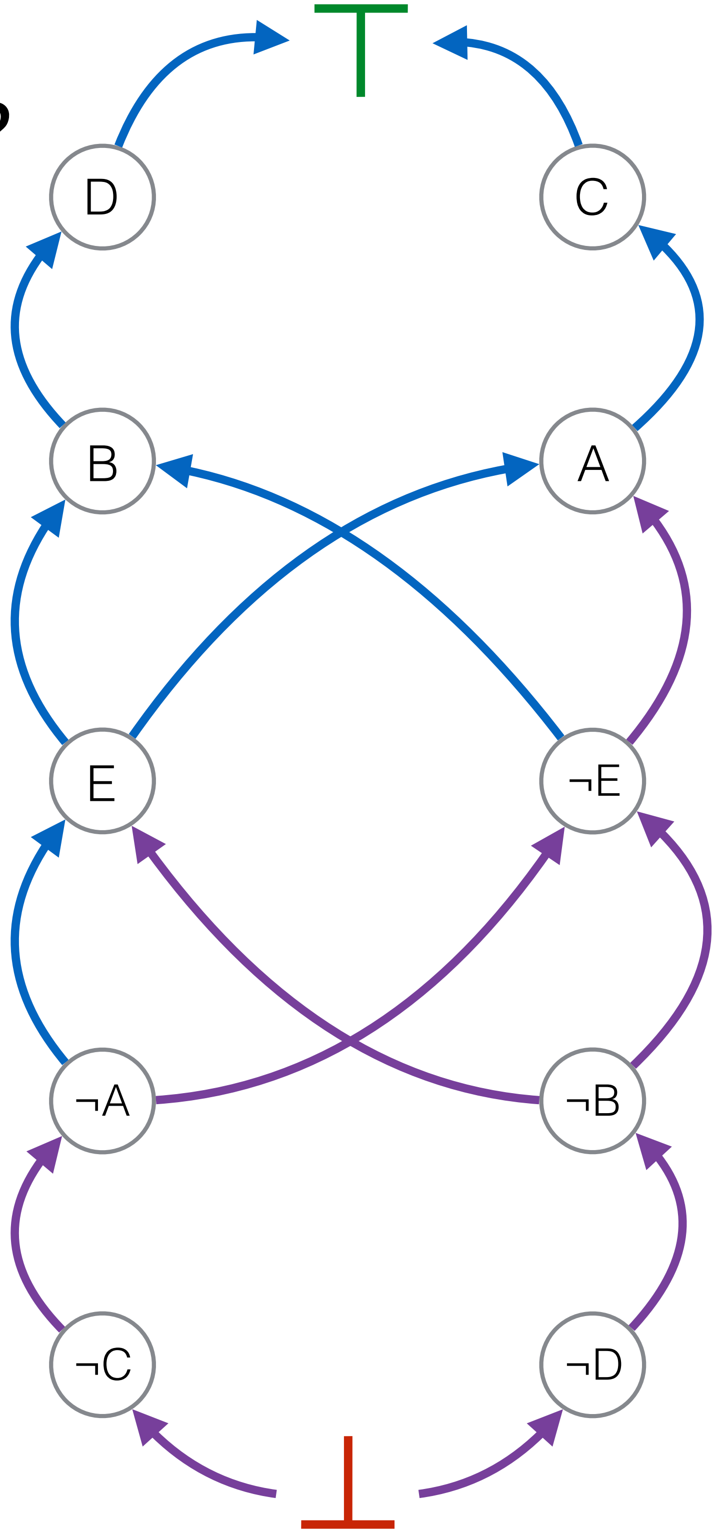
A ∨ E

E ∨ B

There is at least one satisfying valuation, **unless** there is some letter X with a cycle including both X and ¬X.

If there is a path ¬X→X then X must be true in every satisfying valuation.

If there is a path X→¬X then X must be false in every satisfying valuation.



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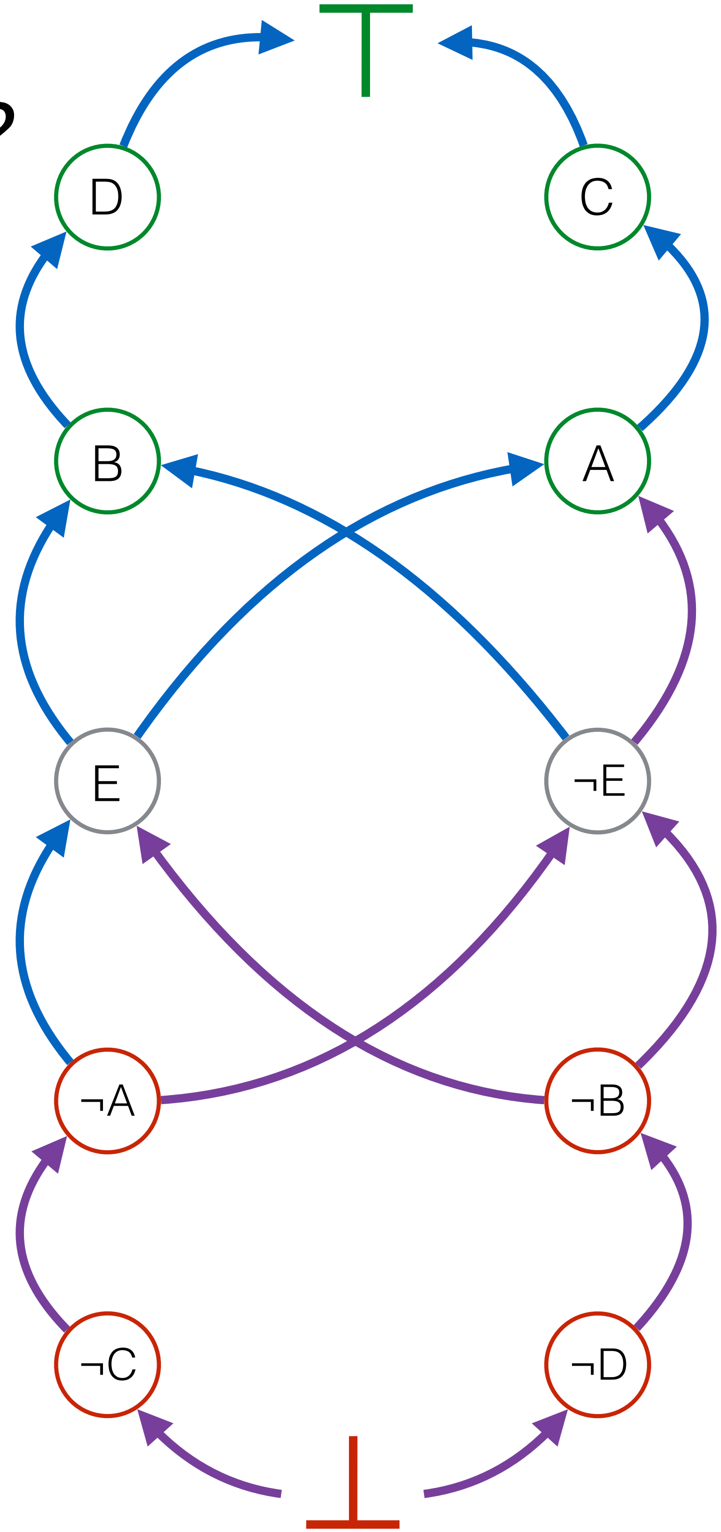

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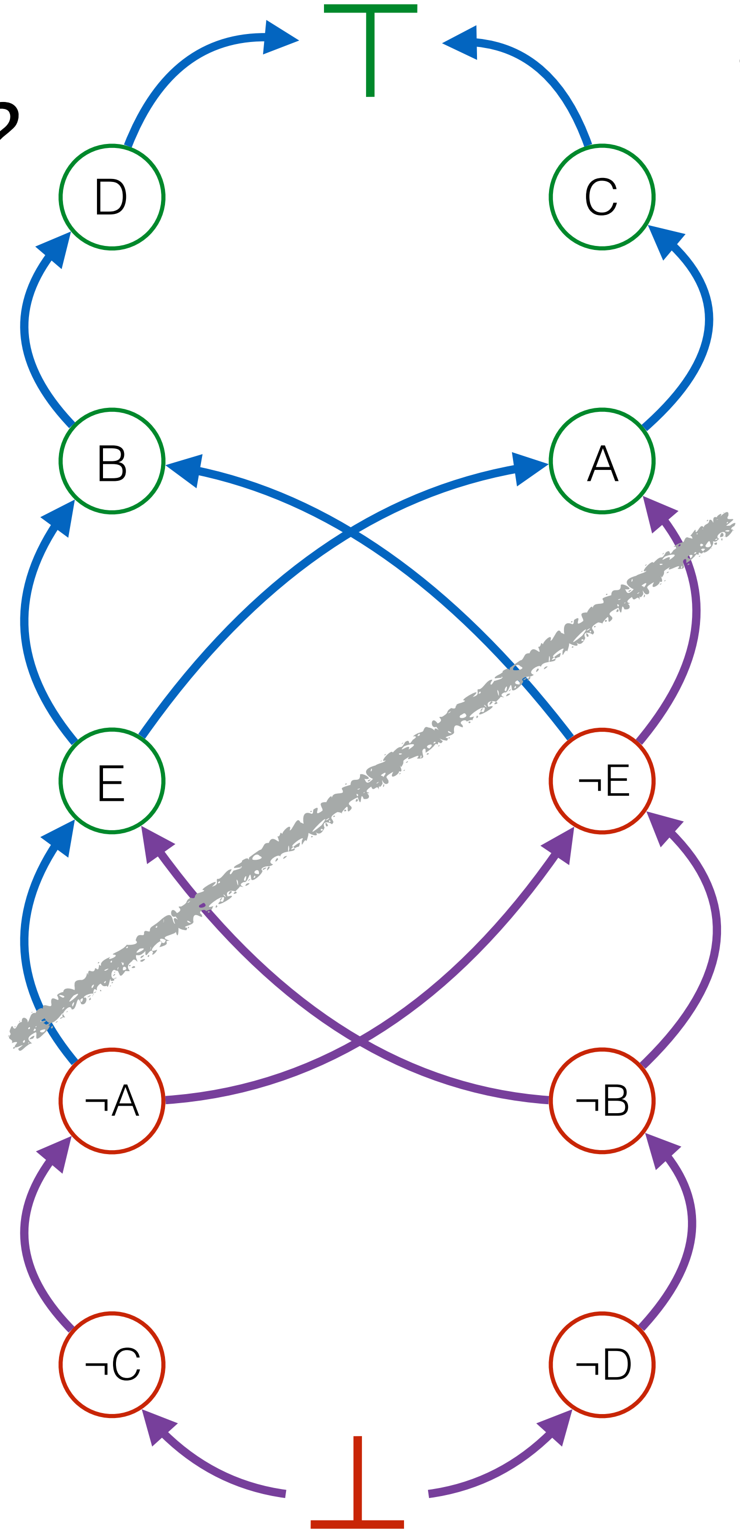


How many satisfying valuations?

$\neg A \lor C$

$\neg B \lor D$

$\neg E \lor B$

$\neg E \lor A$

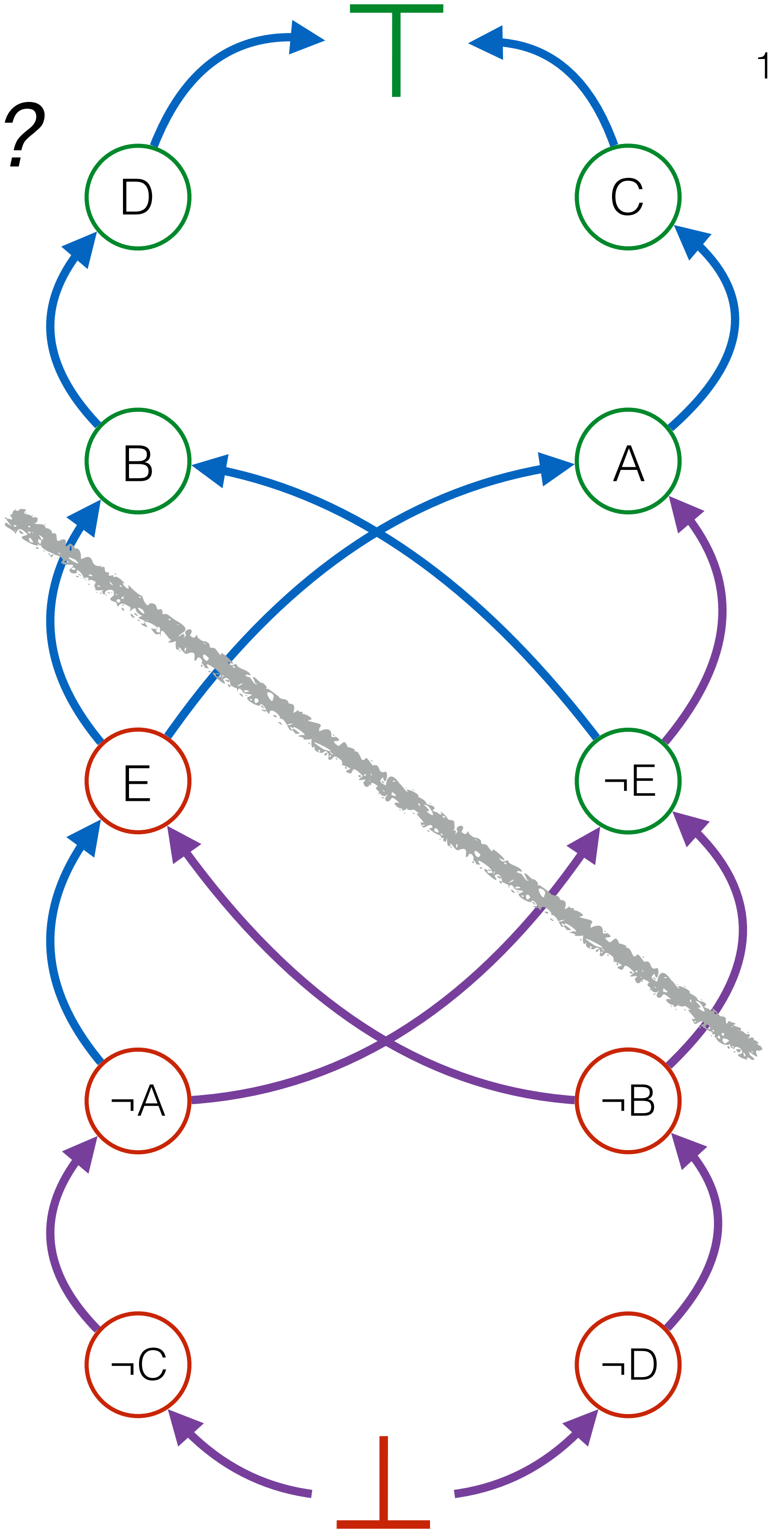
$A \lor E$

$E \lor B$

There is at least one satisfying valuation, unless there is some letter X with cycle including both X and ¬X.

If there is a path ¬X→X then X must be true in every satisfying valuation.

If there is a path X→¬X then X must be false in every satisfying valuation.



Boolean Algebra

$x \vee (y \vee z) = (x \vee y) \vee z$	$x \wedge (y \wedge z) = (x \wedge y) \wedge z$	associative
$x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$	$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$	distributive
$x \vee y = y \vee x$	$x \wedge y = y \wedge x$	commutative
$x \vee 0 = x$	$x \wedge 1 = x$	identity
$x \vee 1 = 1$	$x \wedge 0 = 0$	annihilation
$x \vee x = x$	$x \wedge x = x$	idempotent
$x \vee \neg x = 1$	$\neg x \wedge x = 0$	complements

$x \vee (x \wedge y) = x$	$x \wedge (x \vee y) = x$	absorbtion
$\neg(x \vee y) = \neg x \wedge \neg y$	$\neg(x \wedge y) = \neg x \vee \neg y$	de Morgan
$\neg\neg x = x$	$x \rightarrow y = \neg x \leftarrow \neg y$	

Derived Operations

Definitions:

$$x \rightarrow y \equiv \neg x \vee y \qquad \text{implication}$$

$$x \leftarrow y \equiv x \vee \neg y$$

$$x \leftrightarrow y \equiv (\neg x \wedge \neg y) \vee (x \wedge y) \qquad \text{equality (iff)}$$

$$x \oplus y \equiv (\neg x \wedge y) \vee (x \wedge \neg y) \qquad \text{inequality (xor)}$$

Some equations:

$$x \leftrightarrow y = (x \rightarrow y) \wedge (x \leftarrow y)$$

$$x \oplus y = \neg(x \leftrightarrow y)$$

$$x \oplus y = \neg x \oplus \neg y$$

$$x \leftrightarrow y = \neg(x \oplus y)$$

$$x \leftrightarrow y = \neg x \leftrightarrow \neg y$$

$$A \to B \wedge C$$

$$A \vee B \to C$$

$$
\begin{aligned}
& A \to B \wedge C \\
= \quad & \neg A \vee (B \wedge C) && \text{(implication)} \\
= \quad & (\neg A \vee B) \wedge (\neg A \vee C) && \text{(distributive)} \\
= \quad & (A \to B) \wedge (A \to C) && \text{(implication)}
\end{aligned}
$$

$$A \to B \wedge C = (A \to B) \wedge (A \to C)$$

$$A \vee B \to C = (A \to C) \wedge (B \to C)$$

CNF via Boolean Algebra

expand implications:

$$\neg(a \to b) = a \wedge \neg b \qquad a \leftrightarrow b = (a \to b) \wedge (b \to a) \qquad a \to b = \neg a \vee b$$

push negations down:

$$\neg(a \vee b) = \neg a \wedge \neg b \qquad \neg(a \wedge b) = \neg a \vee \neg b$$

$$\neg 0 = 1 \qquad \neg\neg a = a \qquad \neg 1 = 0$$

distribute disjunctions; absorb constants:

$$a \vee 1 = 1 \qquad a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c) \qquad a \wedge 0 = 0$$

$$a \vee 0 = a \qquad a \vee \neg a = 1 \qquad a \wedge \neg a = 0 \qquad a \wedge 1 = a$$

To produce conjunctive normal form (CNF)

eliminate $\leftrightarrow$ $\rightarrow$

push negations in

push $\vee$ inside $\wedge$

$\neg(a \rightarrow b) = a \wedge \neg b$ $\quad a \leftrightarrow b = (a \rightarrow b) \wedge (b \rightarrow a)$ $\quad a \rightarrow b = \neg a \vee b$

$\neg(a \vee b) = \neg a \wedge \neg b$ $\quad \neg(a \vee b) = \neg a \wedge \neg b$

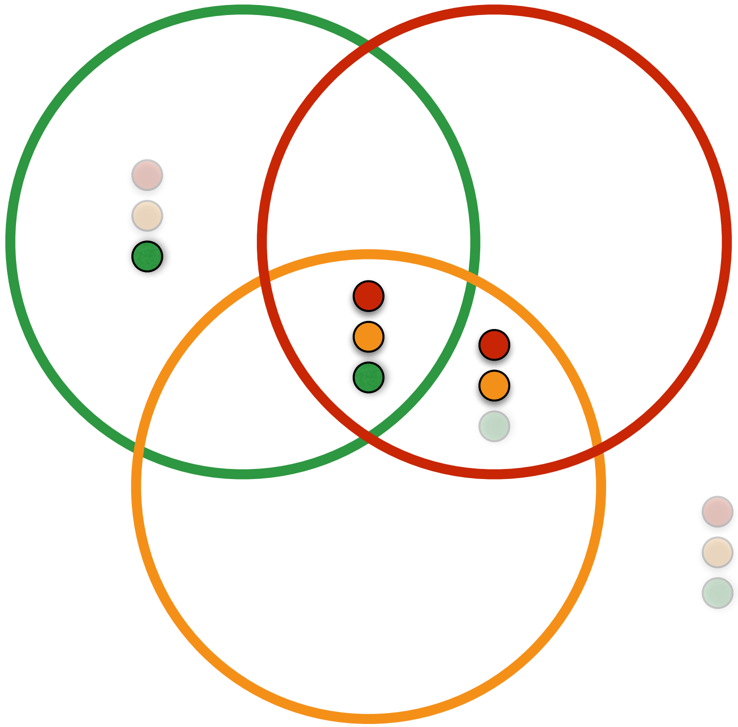
$\neg 0 = 1$ $\quad \neg\neg a = a$ $\quad \neg 1 = 0$

$a \vee 1 = 1$ $\quad a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$ $\quad a \wedge 0 = 0$

$a \vee 0 = a$ $\quad a \vee \neg a = 1$ $\quad a \wedge \neg a = 0$ $\quad a \wedge 1 = a$

eliminate $\leftrightarrow$ $\rightarrow$

$$\boxed{R \leftrightarrow A} = (R \rightarrow A) \wedge (A \rightarrow R)$$
$$= \boxed{(\neg R \vee A) \wedge (\neg A \vee R)}$$



eliminate $\leftrightarrow \quad \rightarrow$

$$\boxed{R \leftrightarrow A} = (R \rightarrow A) \wedge (A \rightarrow R)$$
$$= \boxed{(\neg R \vee A) \wedge (\neg A \vee R)}$$

$$G \leftrightarrow \boxed{(R \leftrightarrow A)}$$
$$= (\neg G \vee (\boxed{(\neg R \vee A) \wedge (\neg A \vee R)}))$$
$$\wedge$$
$$(\neg(\boxed{(\neg R \vee A) \wedge (\neg A \vee R)}) \vee G)$$

push negations in

$$
\begin{aligned}
G &\leftrightarrow (R \leftrightarrow A) \\
&= \big(\neg G \vee ((\neg R \vee A) \wedge (\neg A \vee R))\big) \\
&\quad \wedge \\
&\quad \big(\boxed{\neg((\neg R \vee A) \wedge (\neg A \vee R))} \vee G\big) \\
&= \big(\neg G \vee ((\neg R \vee A) \wedge (\neg A \vee R))\big) \\
&\quad \wedge \\
&\quad \big(\boxed{(\neg(\neg R \vee A) \vee \neg(\neg A \vee R))} \vee G\big)
\end{aligned}
$$

push negations in

$$
\begin{aligned}
G \leftrightarrow (R \leftrightarrow A)& \\
&= \big(\neg G \vee ((\neg R \vee A) \wedge (\neg A \vee R))\big) \\
&\quad \wedge \\
&\quad \big(\neg((\neg R \vee A) \wedge (\neg A \vee R)) \vee G\big) \\
&= \big(\neg G \vee ((\neg R \vee A) \wedge (\neg A \vee R))\big) \\
&\quad \wedge \\
&\quad \big(\boxed{(\neg(\neg R \vee A) \vee \neg(\neg A \vee R))} \vee G\big) \\
&= \big(\neg G \vee ((\neg R \vee A) \wedge (\neg A \vee R))\big) \\
&\quad \wedge \\
&\quad \big(\boxed{((R \wedge \neg A) \vee (A \wedge \neg R))} \vee G\big)
\end{aligned}
$$

push ∨ inside ∧

$$
\begin{aligned}
&G \leftrightarrow (R \leftrightarrow A) \\
&\quad = \left(\boxed{\neg G \vee ((\neg R \vee A) \wedge (\neg A \vee R))}\right) \\
&\qquad \wedge \\
&\qquad \left(((R \wedge \neg A) \vee (A \wedge \neg R)) \vee G\right) \\
&\quad = \left(\boxed{((\neg G \vee \neg R \vee A) \wedge (\neg G \vee \neg A \vee R))}\right) \\
&\qquad \wedge \\
&\qquad \left(((R \wedge \neg A) \vee (A \wedge \neg R)) \vee G\right)
\end{aligned}
$$

push $\vee$ inside $\wedge$

$$
\begin{aligned}
& G \leftrightarrow (R \leftrightarrow A) \\
& \quad = (\neg G \vee \neg R \vee A) \wedge (\neg G \vee \neg A \vee R) \\
& \qquad \wedge \\
& \qquad \left(\boxed{((R \wedge \neg A) \vee (A \wedge \neg R))} \vee G \right) \\
& \quad = (\neg G \vee \neg R \vee A) \wedge (\neg G \vee \neg A \vee R) \\
& \qquad \wedge \\
& \qquad \left(\boxed{((R \vee A) \wedge (\neg A \vee A) \wedge (R \vee \neg R) \wedge (\neg A \vee \neg R))} \vee G \right)
\end{aligned}
$$

simplify

$$\neg A \lor A = \top$$
$$R \lor \neg R = \top$$
$$x \land \top = x$$

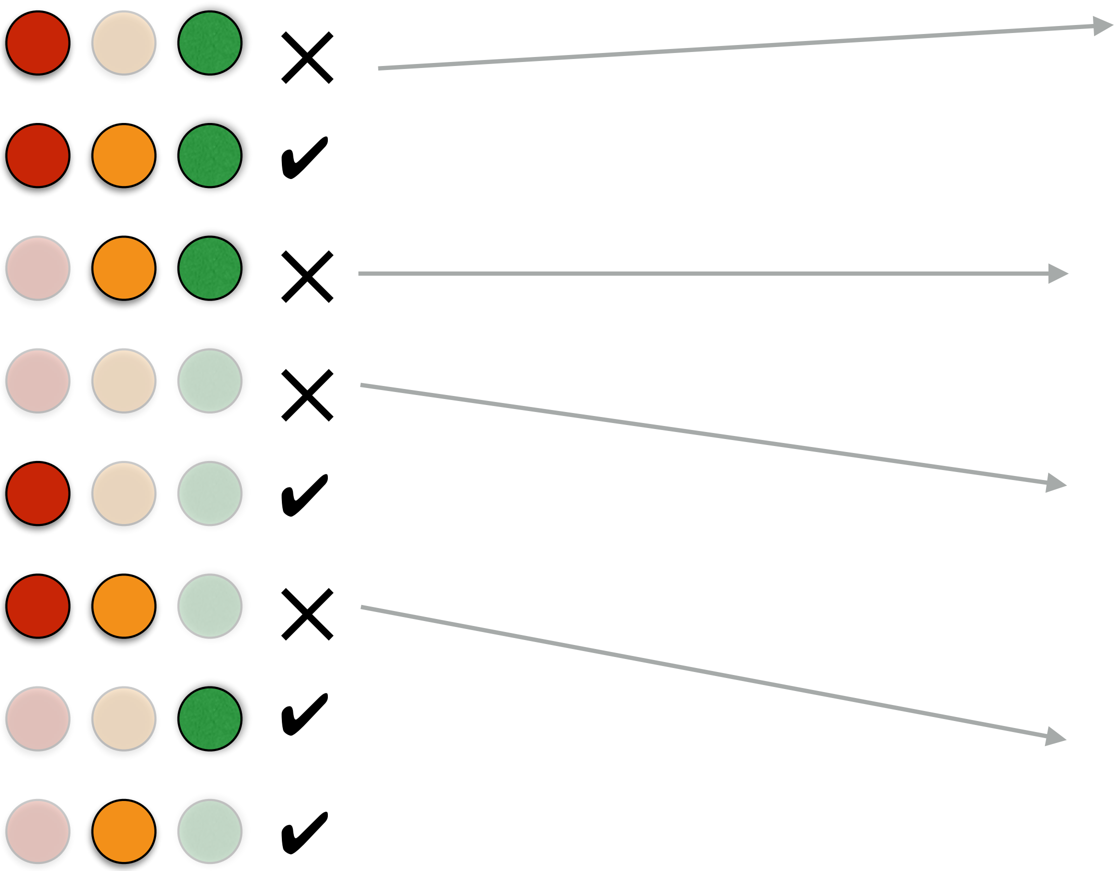
$$
\begin{aligned}
G &\leftrightarrow (R \leftrightarrow A) \\
&= (\neg G \lor \neg R \lor A) \land (\neg G \lor \neg A \lor R) \\
&\quad \land \\
&\quad \left(\boxed{((R \lor A) \land (\neg A \lor A) \land (R \lor \neg R) \land (\neg A \lor \neg R))} \lor G\right) \\
&= (\neg G \lor \neg R \lor A) \land (\neg G \lor \neg A \lor R) \\
&\quad \land \\
&\quad \left(\boxed{((R \lor A) \land (\neg A \lor \neg R))} \lor G\right)
\end{aligned}
$$

push ∨ inside ∧

$$
\begin{aligned}
G \leftrightarrow (R \leftrightarrow A) \\
&= (\neg G \vee \neg R \vee A) \wedge (\neg G \vee \neg A \vee R) \\
&\quad \wedge \\
&\quad \boxed{(((R \vee A) \wedge (\neg A \vee \neg R)) \vee G)} \\
&= (\neg G \vee \neg R \vee A) \wedge (\neg G \vee \neg A \vee R) \\
&\quad \wedge \\
&\quad \boxed{(R \vee A \vee G) \wedge (\neg A \vee \neg R \vee G)}
\end{aligned}
$$

check!

$$G \leftrightarrow (R \leftrightarrow A) =$$



$$(\neg G \vee \neg R \vee A)$$

$$\wedge$$

$$(\neg G \vee \neg A \vee R)$$

$$\wedge$$

$$(R \vee A \vee G)$$

$$\wedge$$

$$(\neg A \vee \neg R \vee G)$$

4-SAT ➥ 3-SAT

with an extra atom – and 3 extra clauses

replace $A \vee B \vee C \vee D$ by $(A \vee B \vee L) \wedge (L \leftrightarrow C \vee D)$

$$L \leftrightarrow C \vee D$$

$$= (L \rightarrow C \vee D) \wedge (C \vee D \rightarrow L)$$

$$= (\neg L \vee C \vee D) \wedge (\neg(C \vee D) \vee L)$$

$$= (\neg L \vee C \vee D) \wedge ((\neg C \wedge \neg D) \vee L)$$

$$= (\neg L \vee C \vee D) \wedge (\neg C \vee L) \wedge (\neg D \vee L)$$

$A \vee B \vee C \vee D \equiv$

$$(A \vee B \vee L) \wedge (\neg L \vee C \vee D) \wedge (\neg C \vee L) \wedge (\neg D \vee L)$$

4-SAT ➥ 3-SAT

with an extra atom – and 3 extra clauses

1: $A \vee B \vee C \vee D \equiv$

2: $(A \vee B \vee L) \wedge (\neg L \vee C \vee D) \wedge (\neg C \vee L) \wedge (\neg D \vee L)$

- Any state of ABCD in which (1) is true
  can be extended uniquely to a state in which (2) is true
  - just give L the value of $C \vee D$

- Any state of ABCDL in which (2) is true
  also makes (1) true

Satisfying a set of constraints including (1)
is equivalent to
satisfying the set given by replacing (1) by (2)

n-SAT ➥ 3-SAT

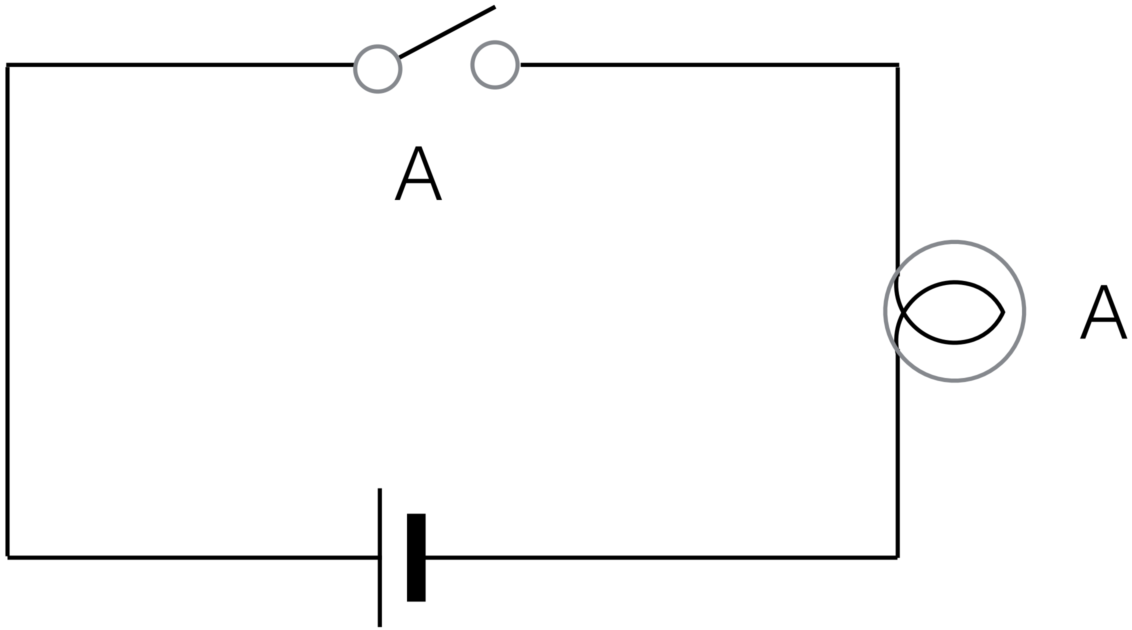
with extra atoms – and extra clauses

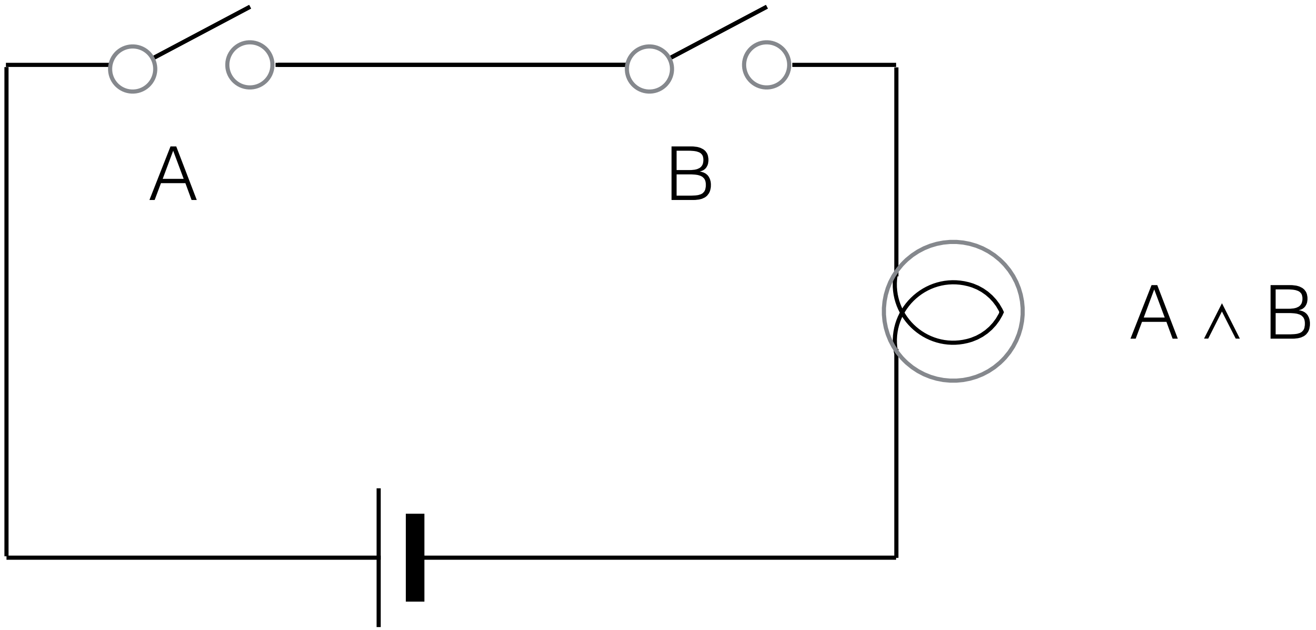
1: $A \lor B \lor C \lor D \equiv$

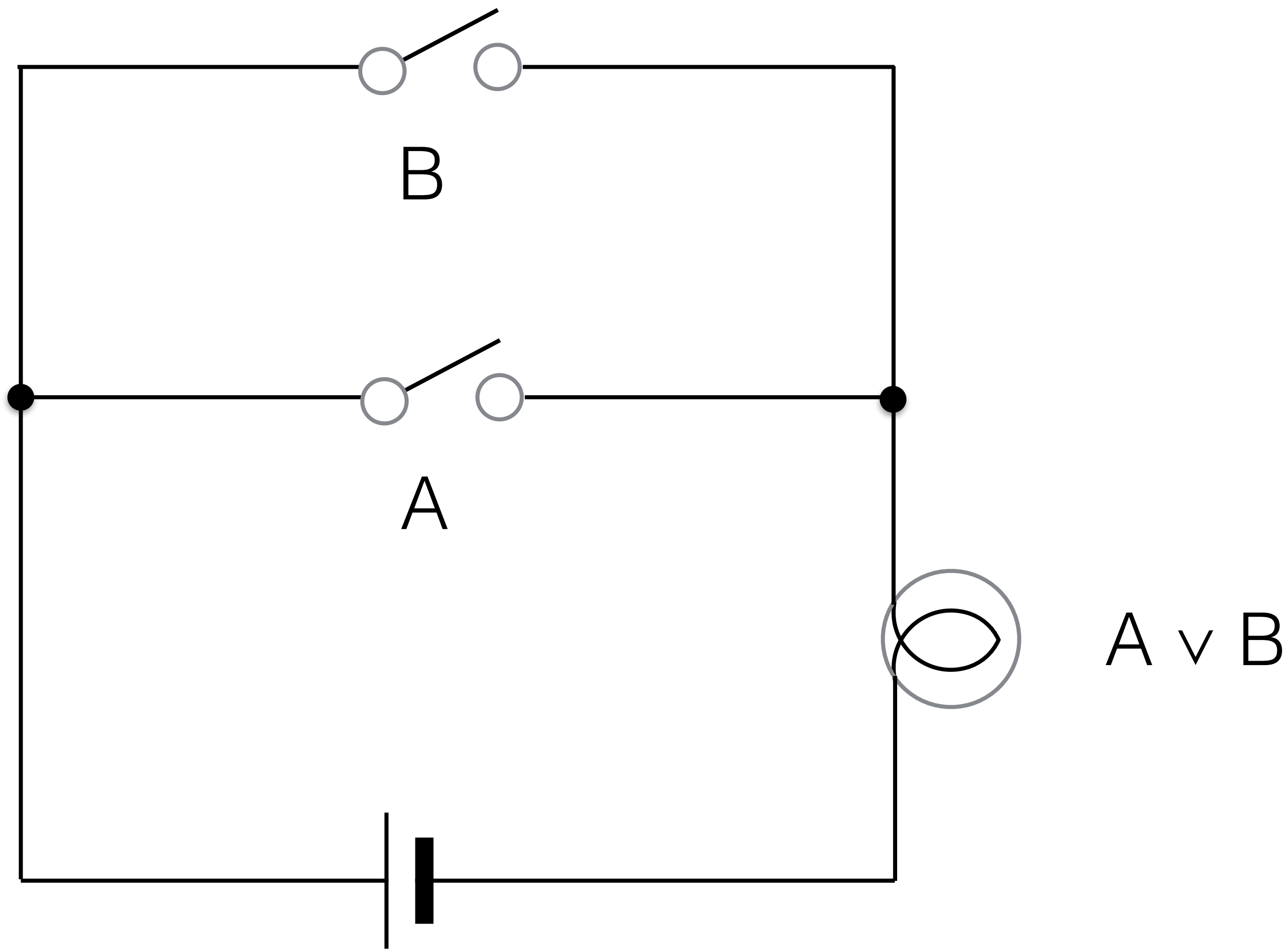
2: $(A \lor B \lor L) \land (\neg L \lor C \lor D) \land (\neg C \lor L) \land (\neg D \lor L)$

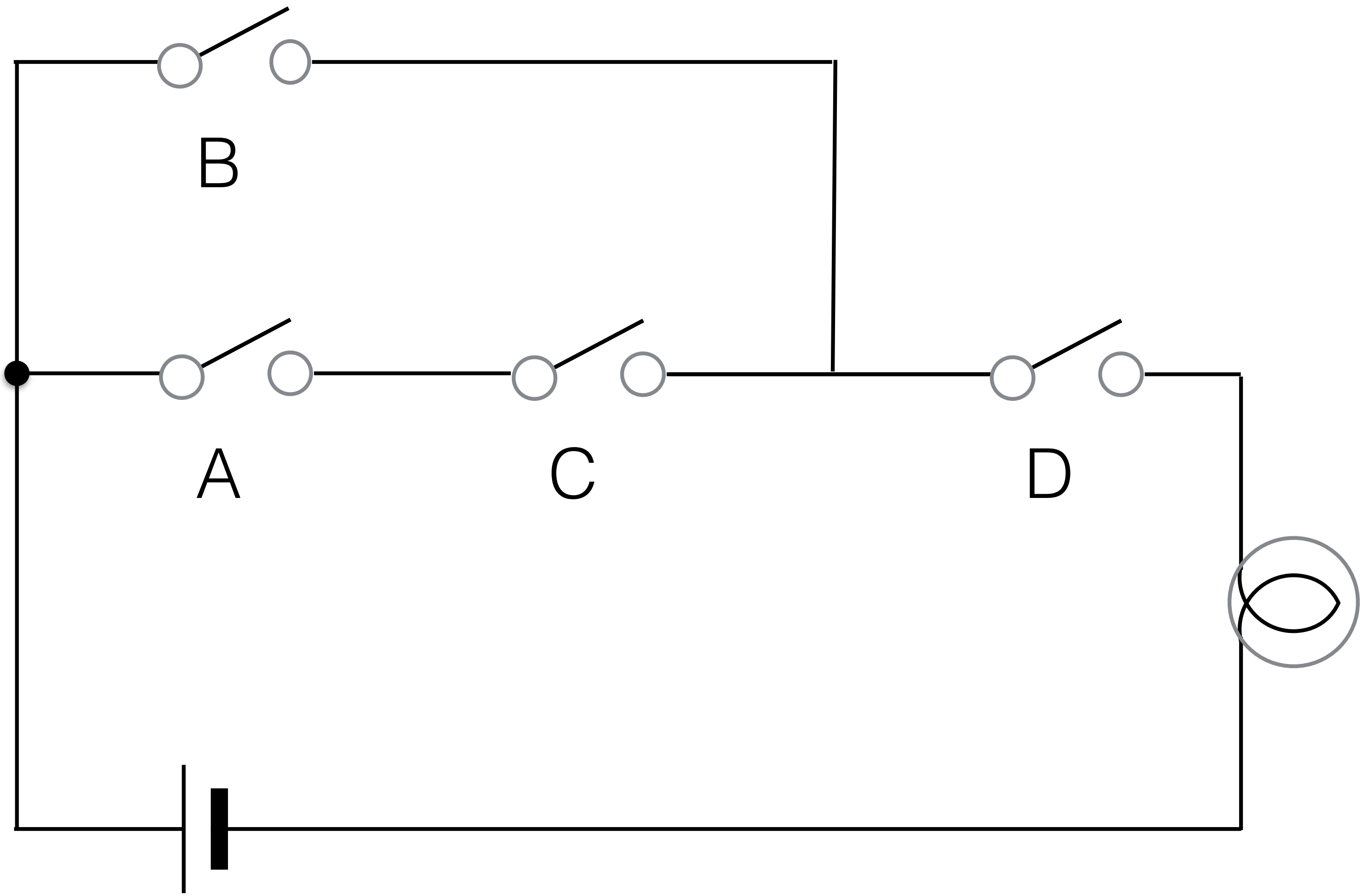
- We can use the same trick to reduce any (n+1)-SAT set of constraints to n-SAT (where n > 2)

If we can solve 3-SAT we can solve n-SAT

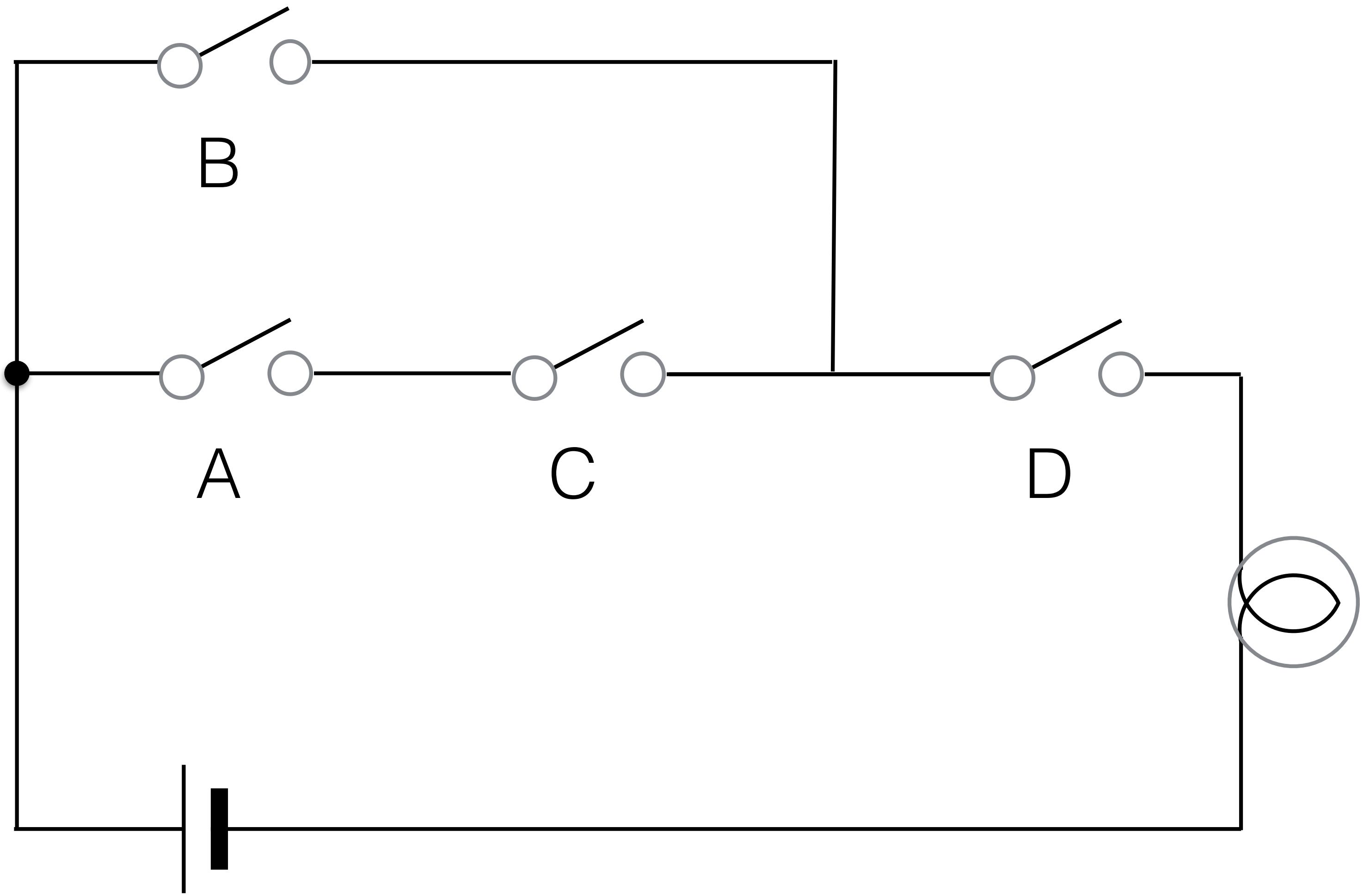

A
A

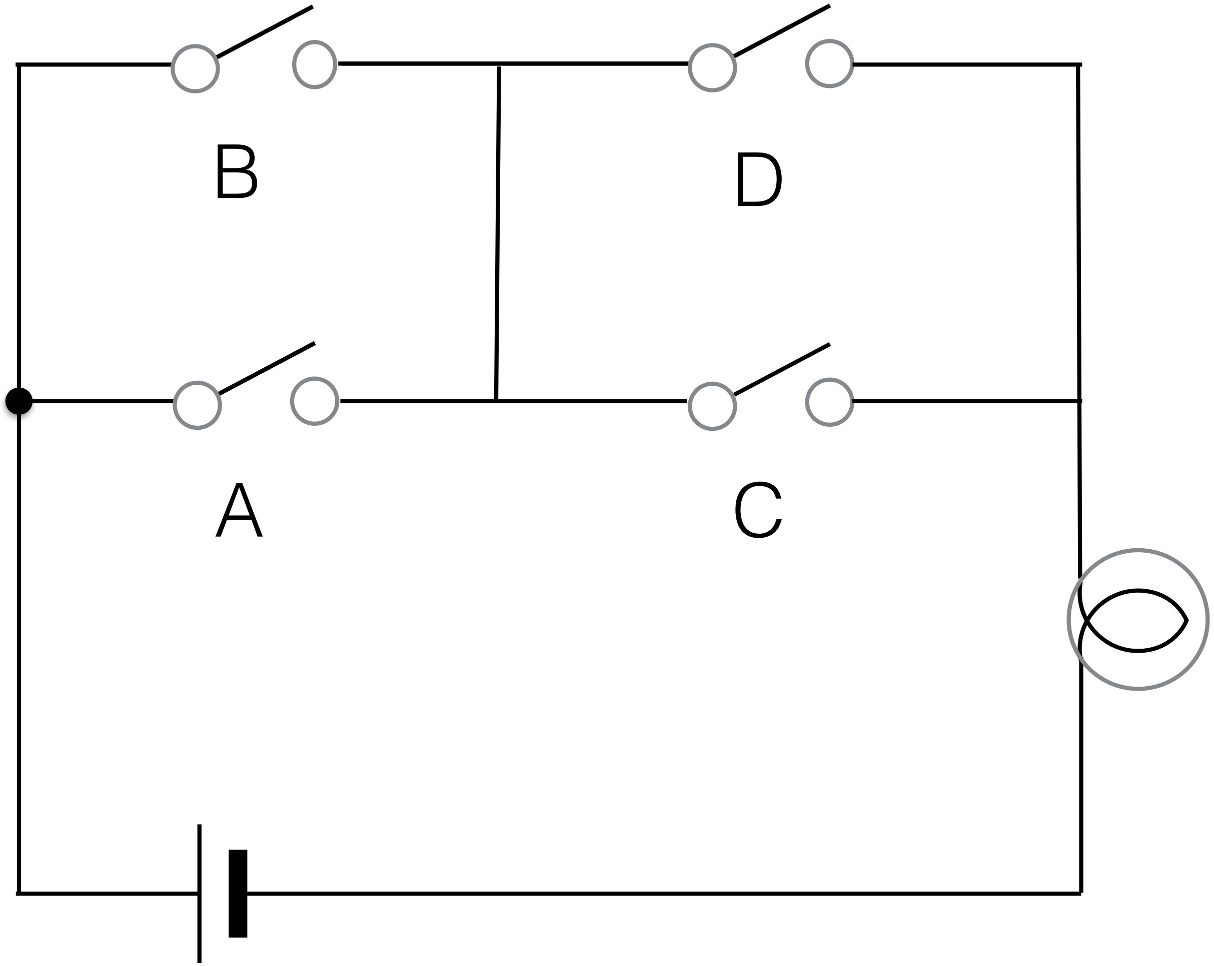

A
B
A ∧ B


B
A
A ∨ B


B
A
C
D

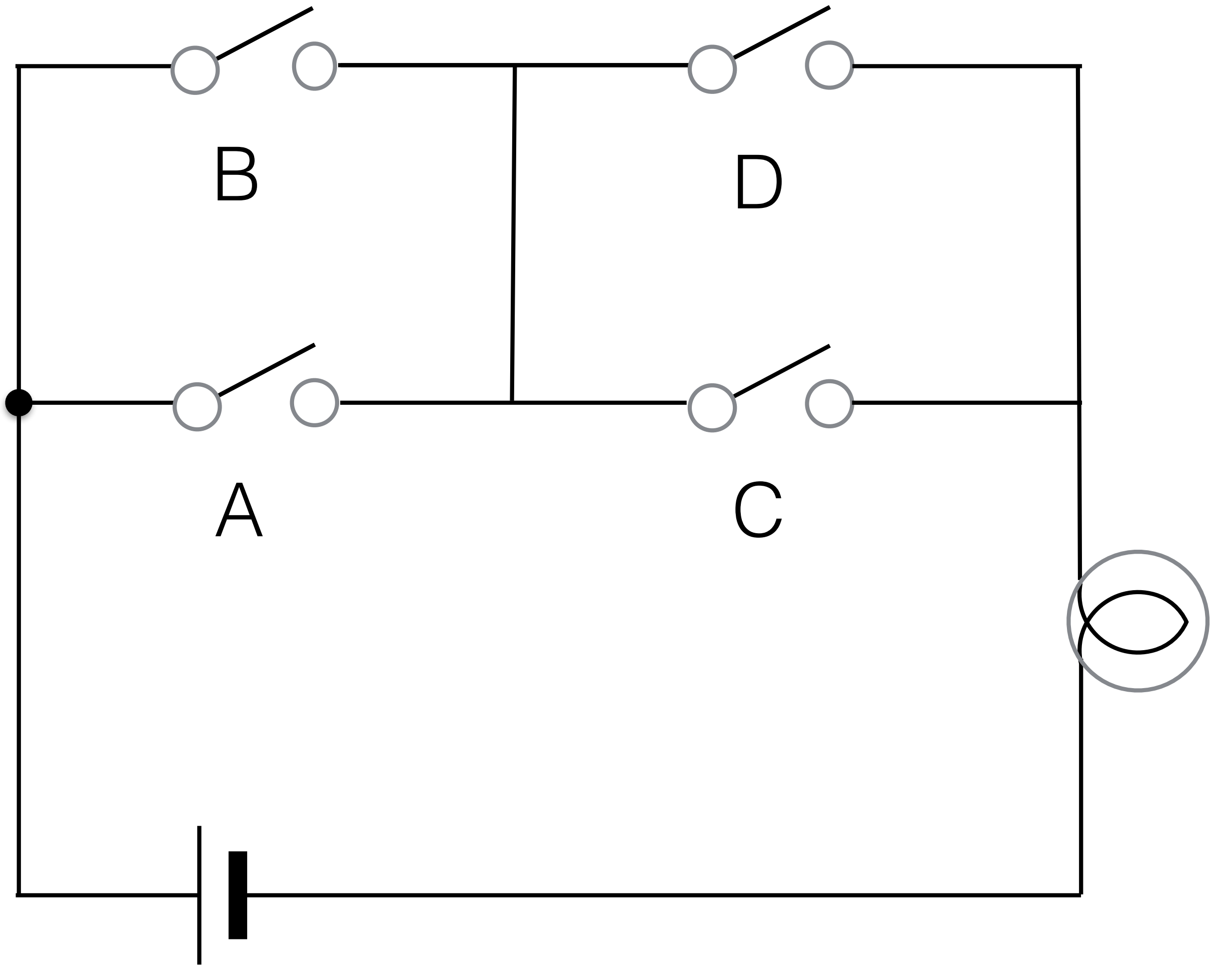
$(B \vee (A \wedge C)) \wedge D$




B
D
A
C

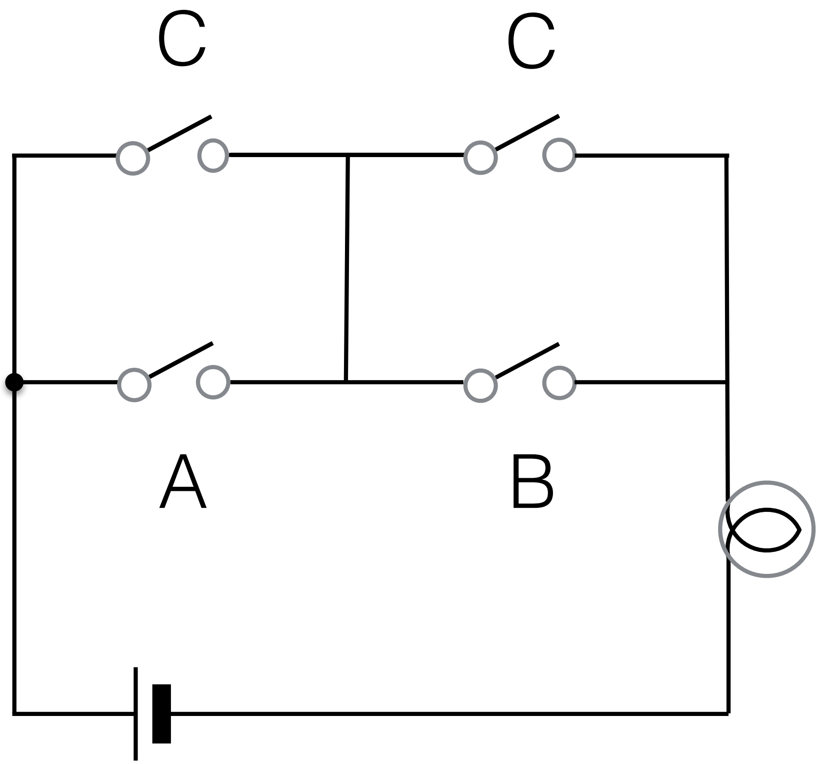
$$(A \wedge C) \vee (B \wedge D) \vee (B \wedge C) \vee (A \wedge D)$$

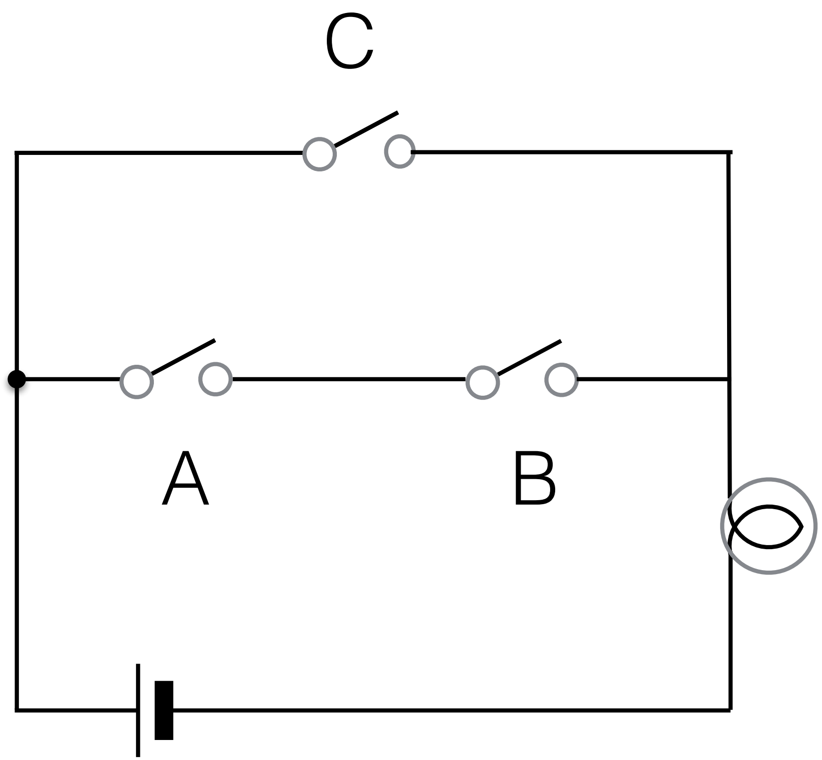
$$= (A \vee B) \wedge (C \vee D)$$



$$(A \vee C) \wedge (B \vee C)$$

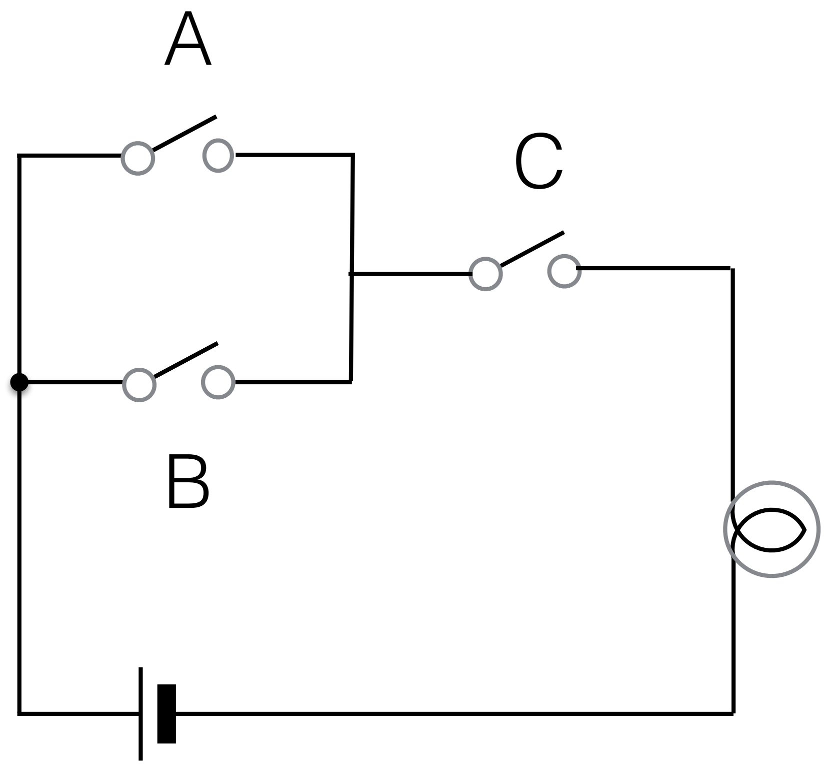
$$= (A \wedge B) \vee C$$

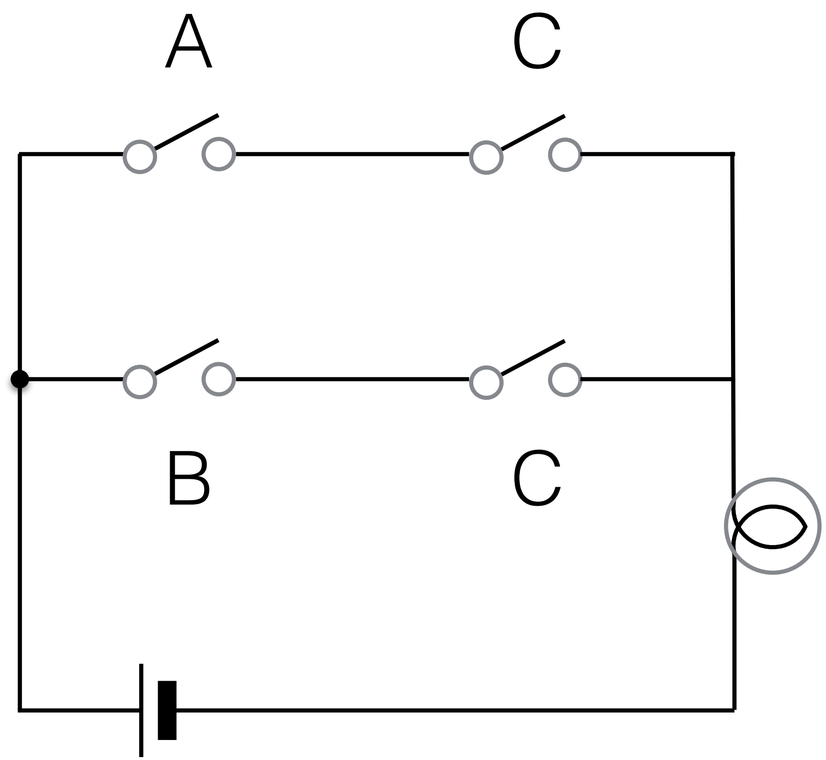




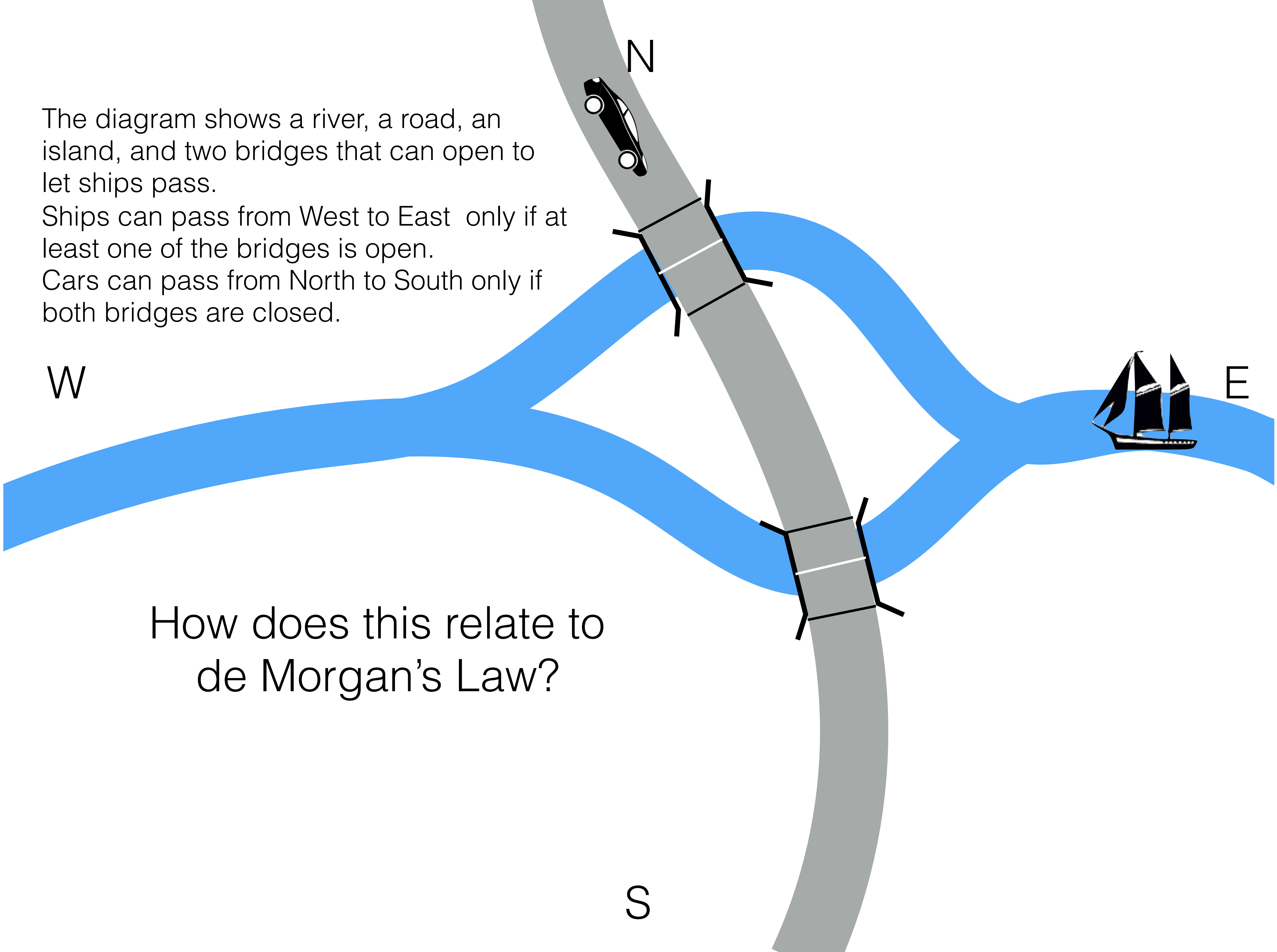
$$(A \vee B) \wedge C$$

$$= (A \wedge C) \vee (B \wedge C)$$





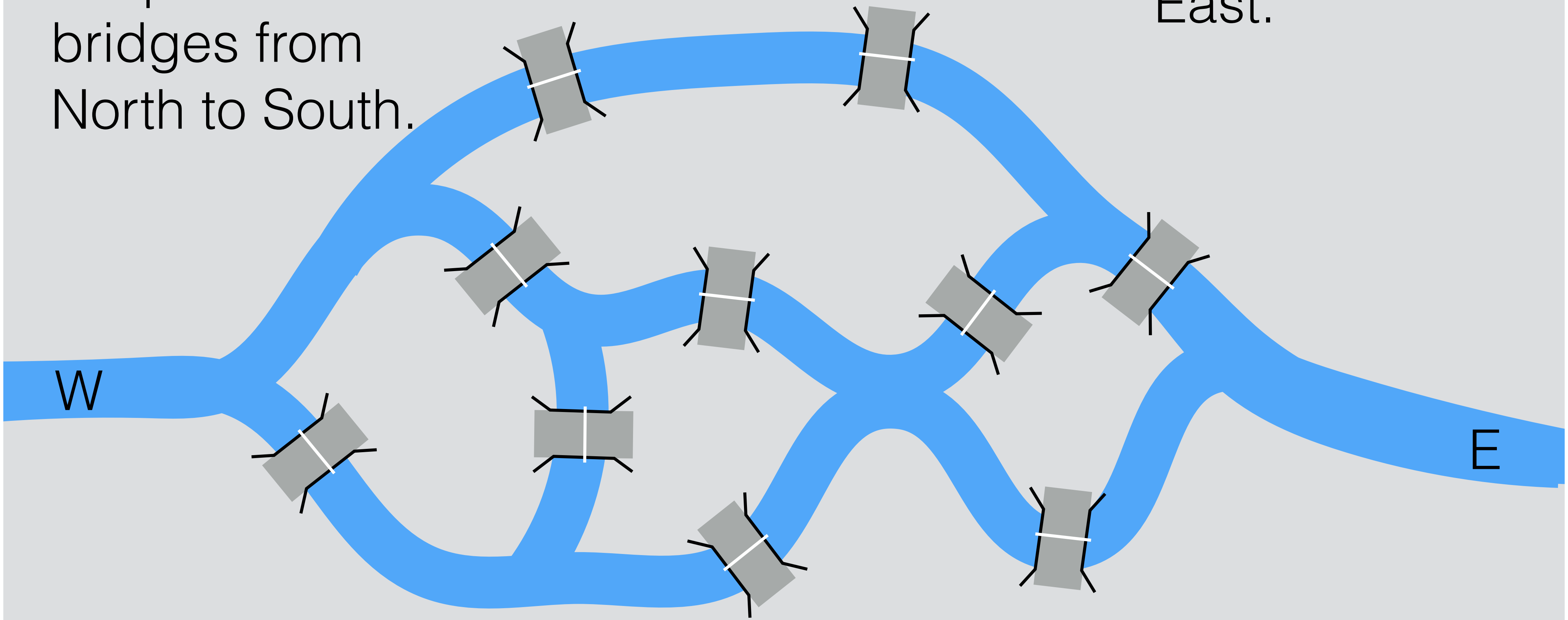
The diagram shows a river, a road, an island, and two bridges that can open to let ships pass.
Ships can pass from West to East only if at least one of the bridges is open.
Cars can pass from North to South only if both bridges are closed.



How does this relate to de Morgan's Law?

Draw a graph showing the paths across the bridges from North to South.

Draw a graph showing the paths under the bridges from West to East.



In each case, the bridges correspond to edges of the graph.

What is the logical relationship between the two graphs?