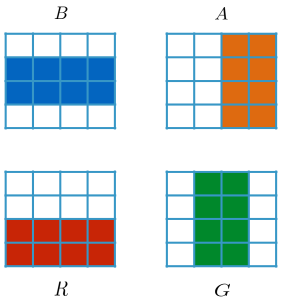


Informatics 1

Computation and Logic

2-SAT

Michael Fourman

SATisfiability

$$\neg R \vee \neg A \vee \neg G$$
$$\wedge$$
$$B \vee A \vee \neg G$$
$$\wedge$$
$$R \vee \neg B \vee G$$
$$\wedge$$
$$R \vee A \vee G$$
$$\wedge$$
$$\neg R \vee B \vee \neg A$$

Can we find a state that satisfies a given set of constraints?

		AG			
		00	01	11	10
RB	00				
	01				
	11				
	10				

SATisfiability

$$\neg R \vee \neg A \vee \neg G$$
$$\wedge$$
$$B \vee A \vee \neg G$$
$$\wedge$$
$$R \vee \neg B \vee G$$
$$\wedge$$
$$R \vee A \vee G$$
$$\wedge$$
$$\neg R \vee B \vee \neg A$$

Can we find a state that satisfies a given set of constraints?

Is there a state that satisfies a given set of constraints?

How many states are there?

How many states are eliminated by each constraint?

2-SAT (KISS) binary constraints

$$\neg R \vee \neg A$$

$$A \vee \neg G$$

$$R \vee A$$

$$\neg R \vee B$$

each binary constraint is equivalent to an arrow

$$\neg R \vee \neg A \quad \equiv \quad R \to \neg A$$

$$A \vee \neg G \quad \equiv \quad \neg A \to \neg G$$

$$R \vee A \quad \equiv \quad \neg R \to A$$

$$\neg R \vee B \quad \equiv \quad R \to B$$

each binary constraint is equivalent to two arrows

$$\neg R \vee \neg A \quad \equiv \quad R \to \neg A \quad \equiv \quad A \to \neg R$$

$$A \vee \neg G \quad \equiv \quad \neg A \to \neg G \quad \equiv \quad G \to A$$

$$R \vee A \quad \equiv \quad \neg R \to A \quad \equiv \quad \neg A \to R$$

$$\neg R \vee B \quad \equiv \quad R \to B \quad \equiv \quad \neg B \to \neg R$$

each binary constraint is equivalent to two arrows

$$\neg R \vee \neg A \quad \equiv \quad R \rightarrow \neg A \quad \equiv \quad A \rightarrow \neg R \quad \equiv \quad \neg A \vee \neg R$$

$$A \vee \neg G \quad \equiv \quad \neg A \rightarrow \neg G \quad \equiv \quad G \rightarrow A \quad \equiv \quad \neg G \vee A$$

$$R \vee A \quad \equiv \quad \neg R \rightarrow A \quad \equiv \quad \neg A \rightarrow R \quad \equiv \quad A \vee R$$

$$\neg R \vee B \quad \equiv \quad R \rightarrow B \quad \equiv \quad \neg B \rightarrow \neg R \quad \equiv \quad B \vee \neg R$$

→	0	1
0	1	1
1	0	1

$$A \to B = \top$$

iff

$$A \leq B$$

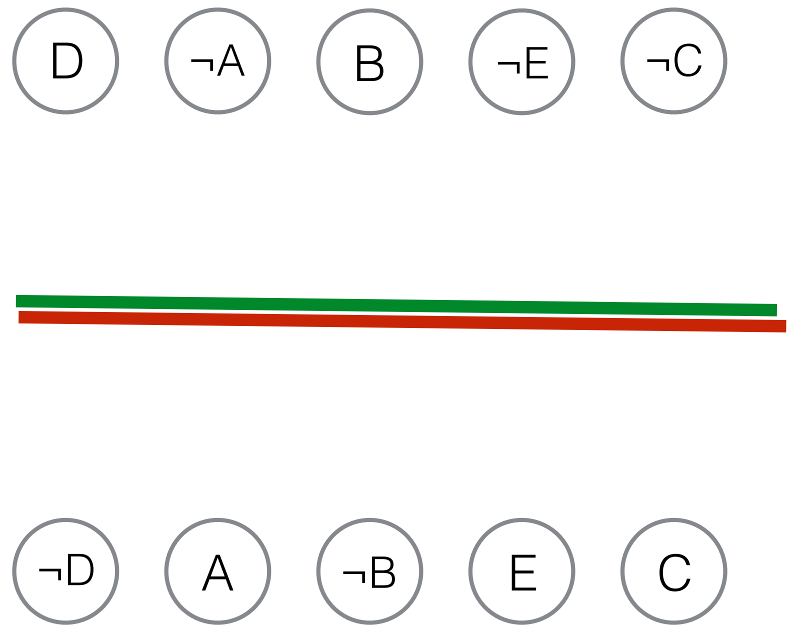
$$A \to B = \top$$

iff

$$A \leq B$$

$$\begin{array}{cccc}
\top & \top & \top & \top \\
\uparrow & \uparrow & \uparrow & \cancel{\uparrow} \\
B & \top & \top & \bot \\
\uparrow & \uparrow & \cancel{\uparrow} & \uparrow \\
A & \top & \bot & \bot \\
\uparrow & \cancel{\uparrow} & \uparrow & \uparrow \\
\bot & \bot & \bot & \bot
\end{array}$$

A **valuation,** or **state,** makes some atoms true and the rest false. Once we have a valuation, for each atom, we can compute the truth value of every expression.
If an atom is true its negation is false, and vice versa.



We draw a line to visualise a valuation, placing the true literals above the line, and the false literals below it.

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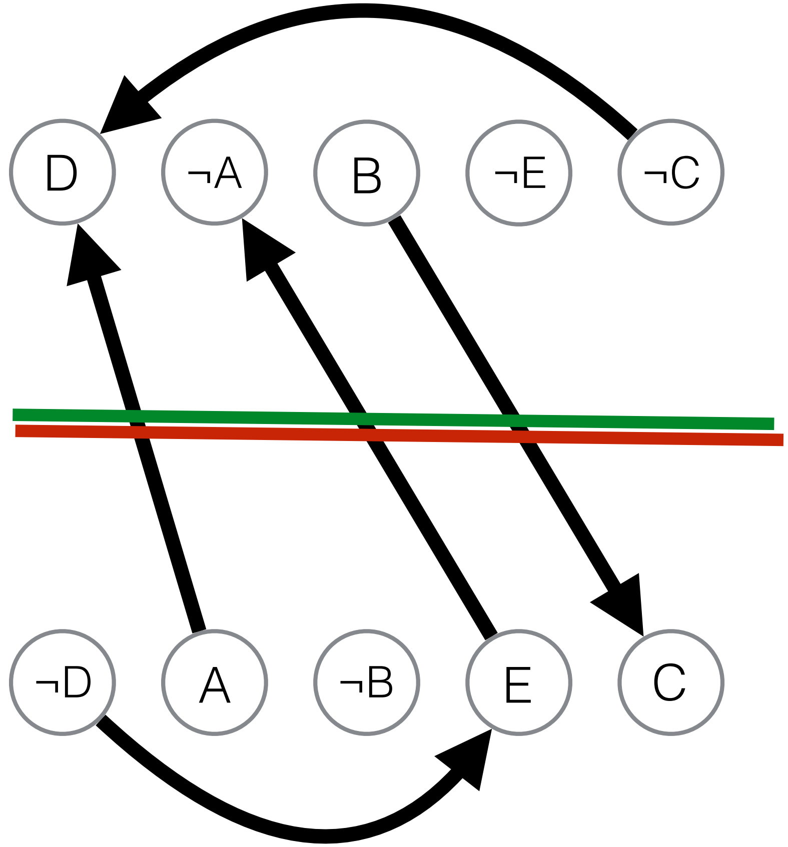
An implication between literals is represented by an arrow.



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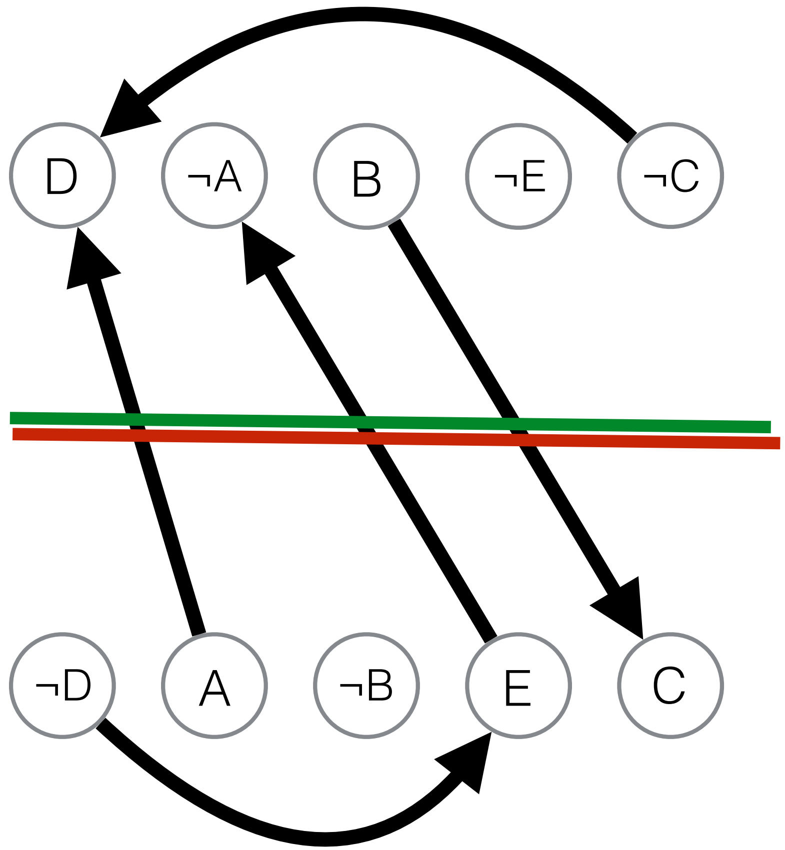

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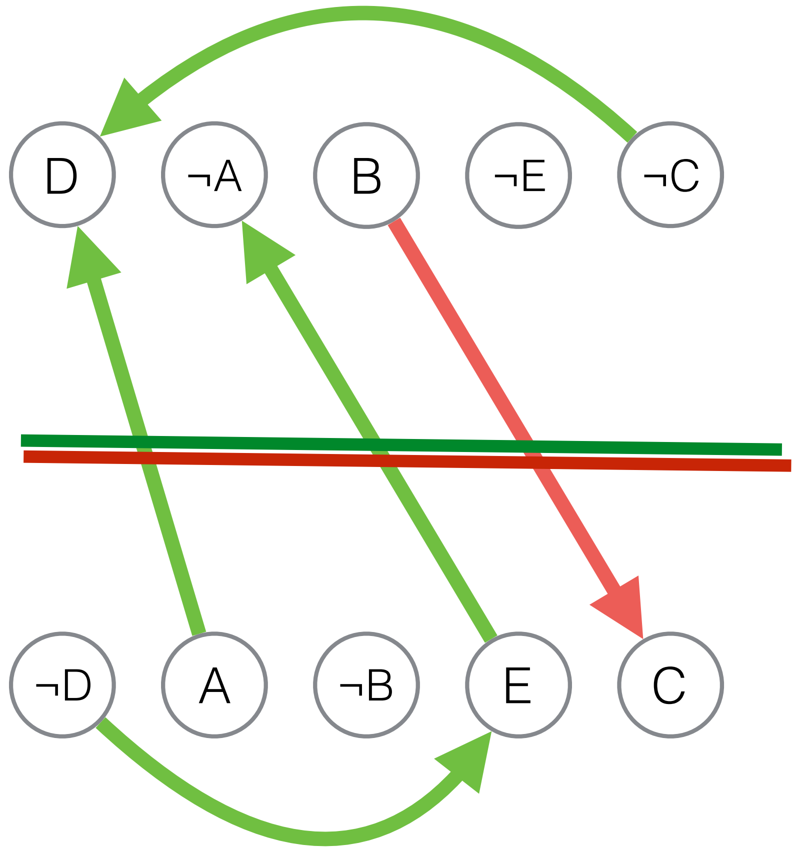
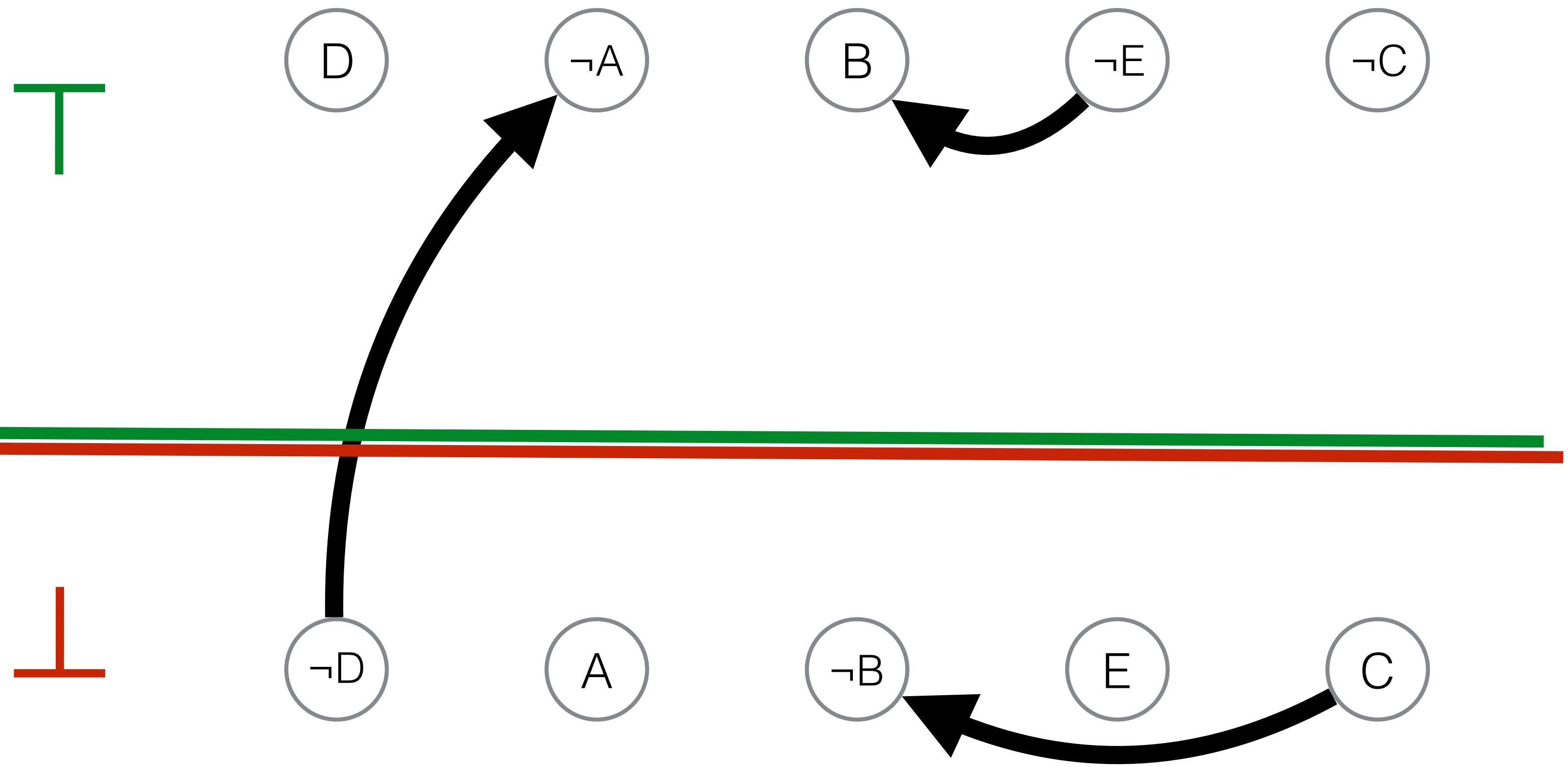


An implication between literals is represented by an arrow.

The valuation makes the implication true, unless the arrow goes from true to false.

$X \rightarrow Y$

if X is $\top$ then Y is $\top$

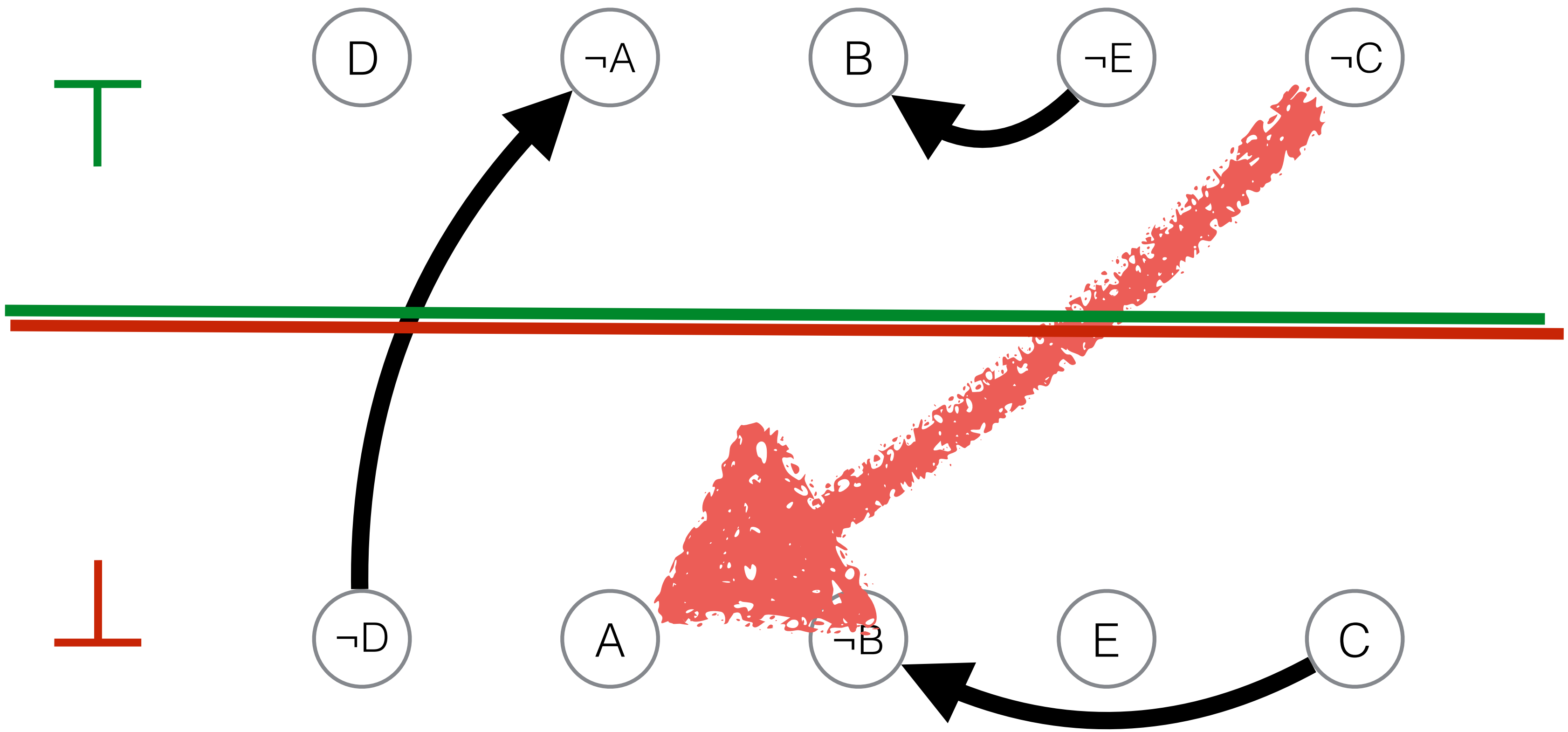


$X \rightarrow Y$

if Y is $\perp$ then X is $\perp$

$X \to Y$

if X is $\top$ then Y is $\top$



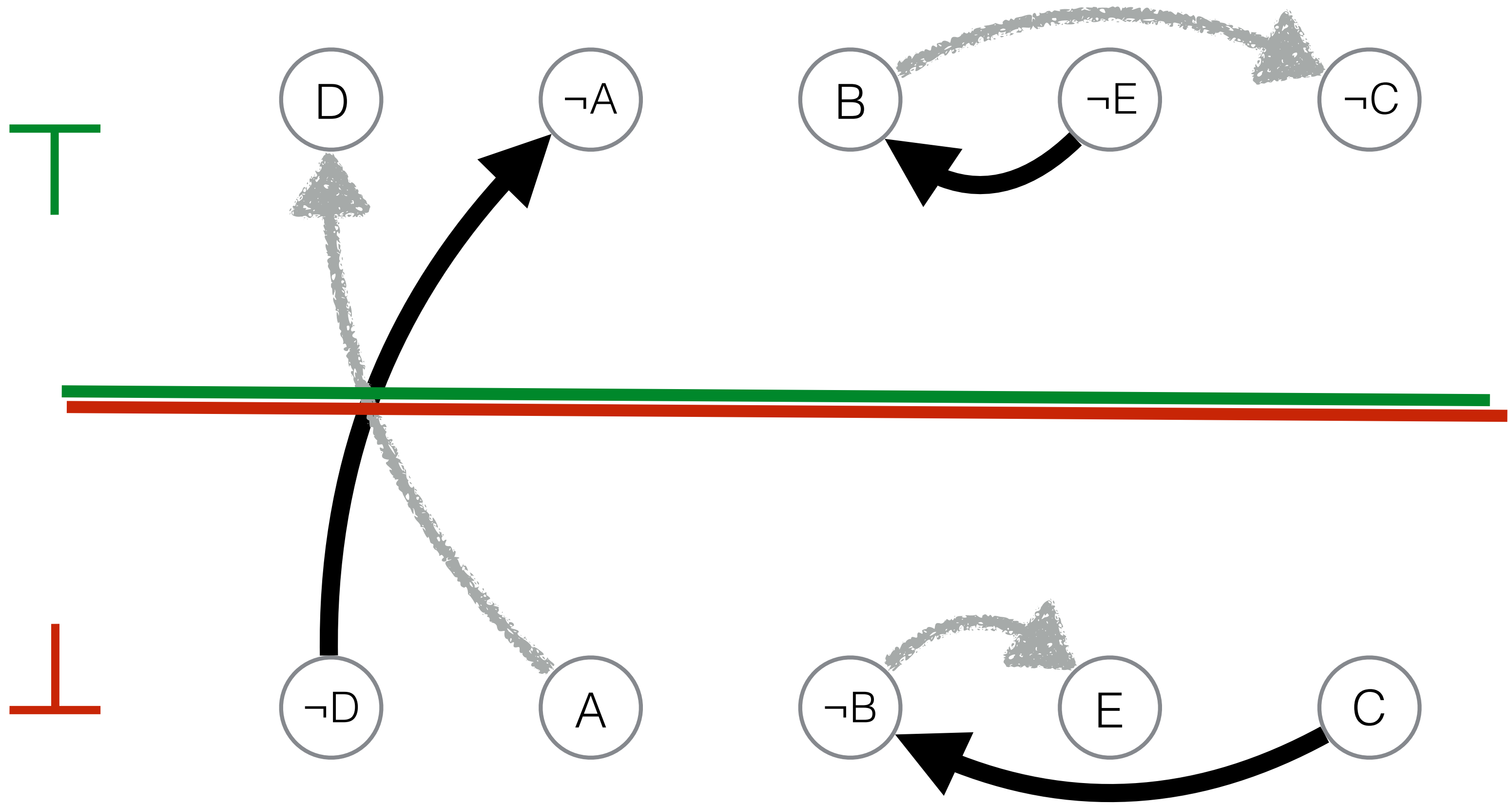
$X \to Y$

if Y is $\perp$ then X is $\perp$

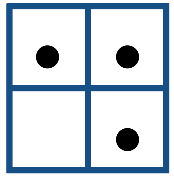
$$\neg X \vee Y \equiv X \rightarrow Y \equiv \neg Y \rightarrow \neg X$$

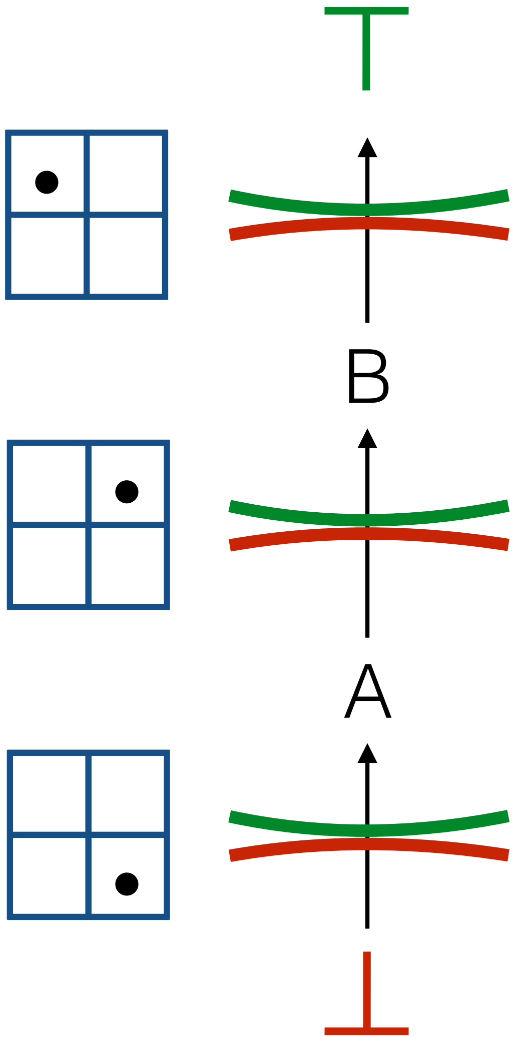
$$X \vee Y \equiv \neg X \rightarrow Y \equiv \neg Y \rightarrow X$$

$$\neg X \vee \neg Y \equiv X \rightarrow \neg Y \equiv Y \rightarrow \neg X$$



$A \to B$





Suppose $A \to B$
there are three possible
truth valuations for A and B
(we exclude only ($A = \top$, $B = \bot$))

Propositions are ordered
by $x \leq y$ iff $x \to y = \top$
Any valid truth assignment
must draw a line
between $\bot$ and $\top$

⊤

↑

Z

↑

Y

↑

X

↑

◉

◉

◉

↑

C

↑

B

↑

A

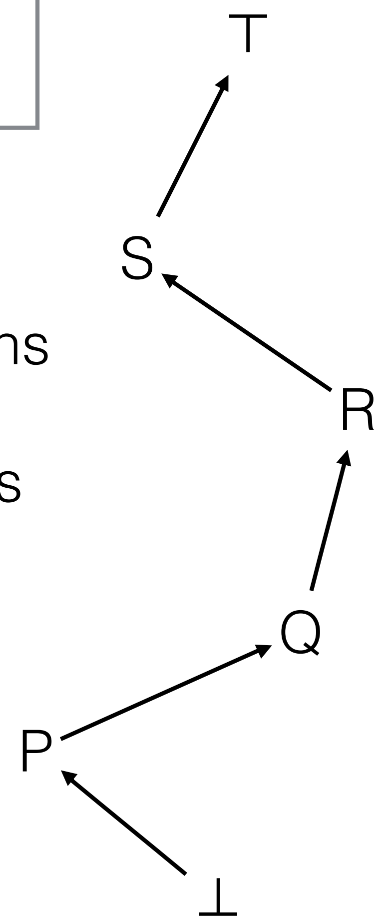
↑

⊥

$\top$
$\uparrow$
Z
$\uparrow$
Y
$\uparrow$
X
$\uparrow$
◉
◉
◉
$\uparrow$
C
$\uparrow$
B
$\uparrow$
A
$\uparrow$
$\bot$

$$\begin{array}{c} P \rightarrow Q \\ \wedge \\ Q \rightarrow R \\ \wedge \\ R \rightarrow S \end{array}$$

$$\begin{array}{c} \neg P \vee Q \\ \wedge \\ \neg Q \vee R \\ \wedge \\ \neg R \vee S \end{array}$$



If we have a chain of n-1 implications
between n variables
we can draw the line in n+1 places
making any number, from 0 to n,
of these variables true.

We can draw the line
before each letter, or after them all.

$$\begin{array}{c} \neg P \to Q \\ \wedge \\ Q \to \neg R \\ \wedge \\ \neg R \to S \end{array} \qquad \qquad \begin{array}{c} P \vee Q \\ \wedge \\ \neg Q \vee \neg R \\ \wedge \\ R \vee S \end{array}$$

If some of the variables are negated we can do the same (but making the negated variables false when they fall above the line and true when they fall below)

$$\bot \to \neg P \to Q \to \neg R \to S \to \top$$

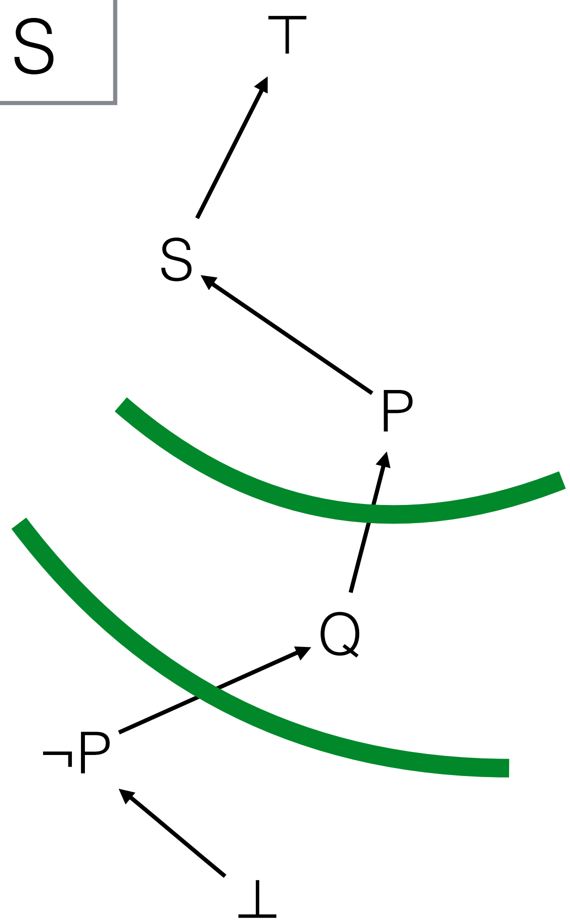
$$
\begin{gathered}
\neg P \to Q \\
\wedge \\
Q \to P \\
\wedge \\
P \to S
\end{gathered}
$$

$$
\begin{gathered}
P \vee Q \\
\wedge \\
\neg Q \vee P \\
\wedge \\
\neg P \vee S
\end{gathered}
$$

If a variable appears together with its negation, we have to draw the line between them.

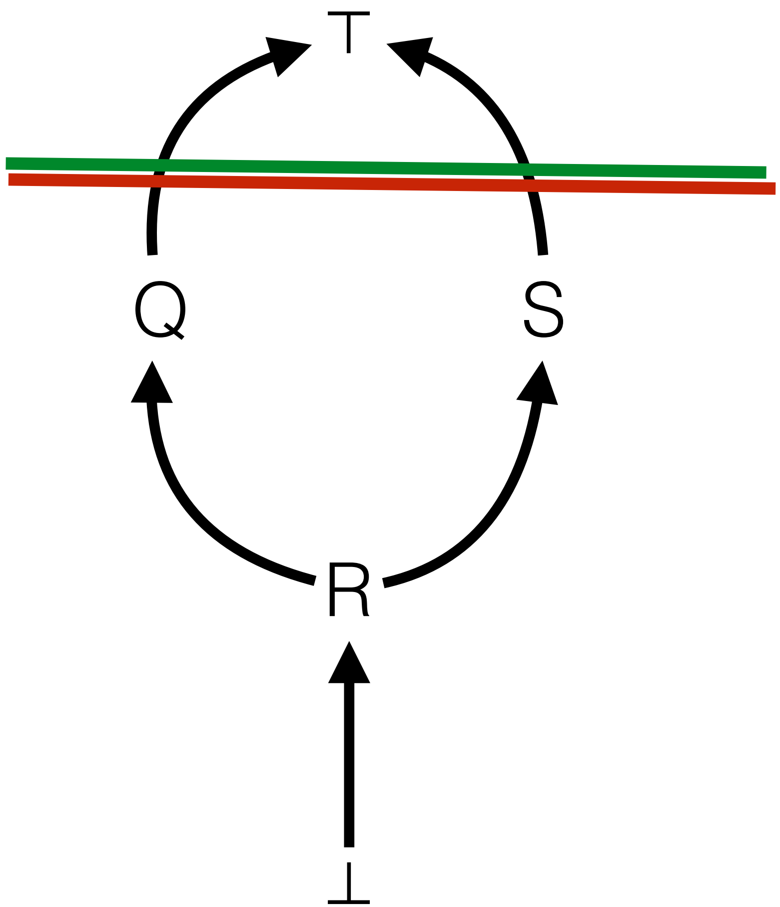
Here, P must be true.

$(\neg P \to P) \to P$
is a tautology



$$R \to Q$$
$$\wedge$$
$$R \to S$$

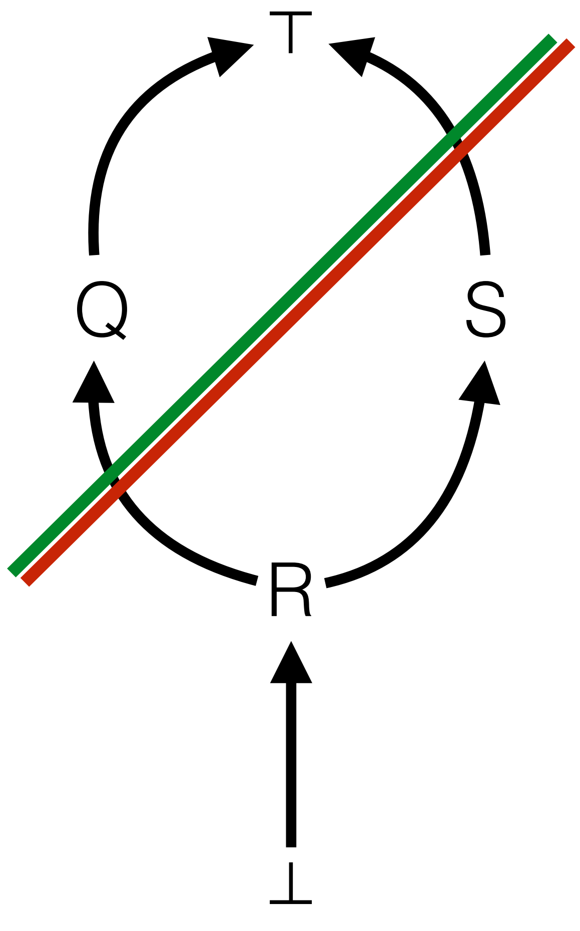
$$\neg R \vee Q$$
$$\wedge$$
$$\neg R \vee S$$



$\neg R$ $\neg Q$ $\neg S$

$$R \rightarrow Q \;\wedge\; R \rightarrow S$$

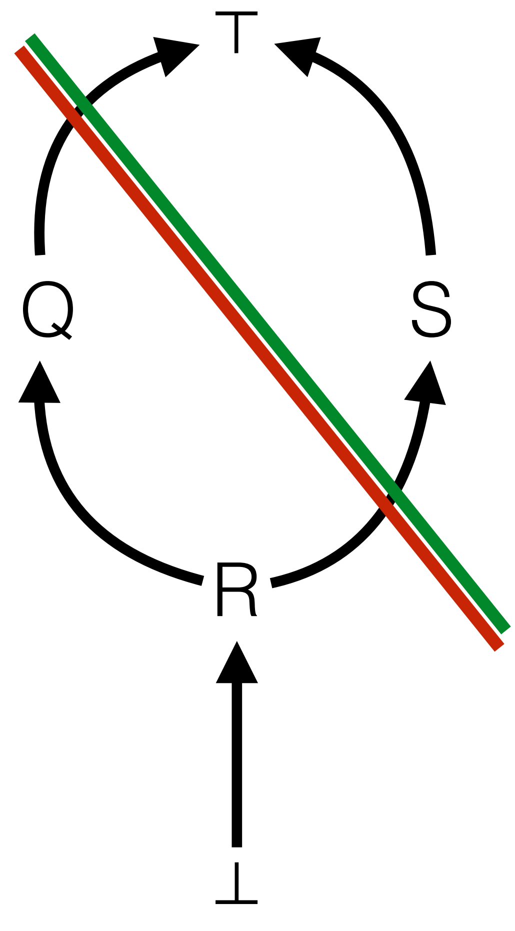
$$\neg R \vee Q \;\wedge\; \neg R \vee S$$



$\neg R$ $\quad Q \quad$ $\neg S$

$$R \rightarrow Q \wedge R \rightarrow S$$

$$\neg R \vee Q \wedge \neg R \vee S$$

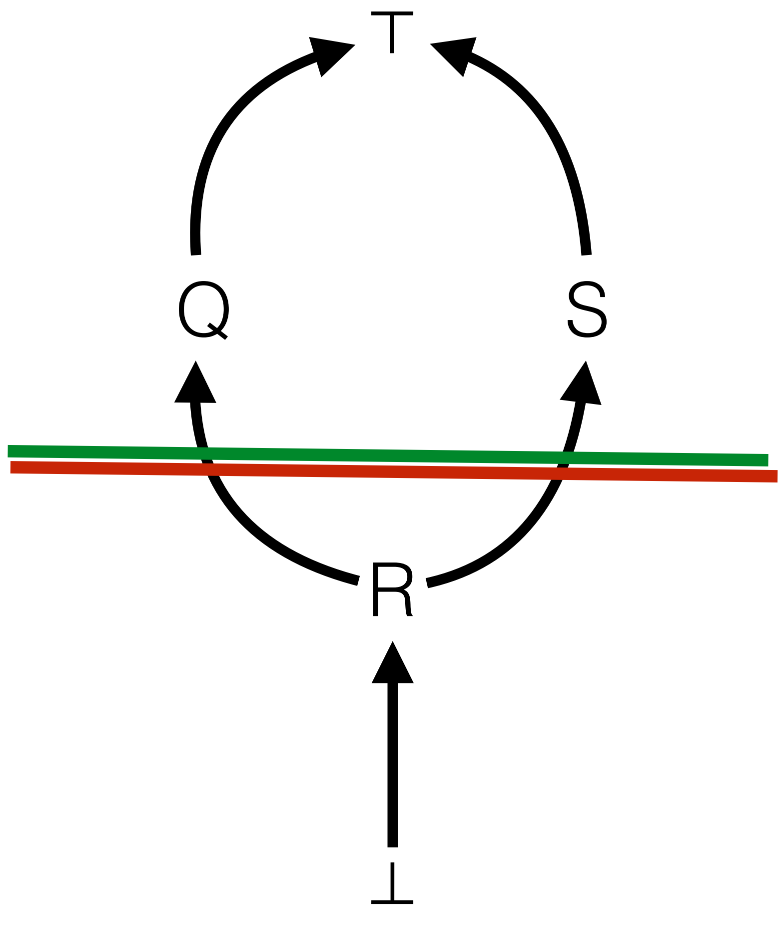


$\neg R$ $\neg Q$ S

$$R \rightarrow Q$$
$$\wedge$$
$$R \rightarrow S$$

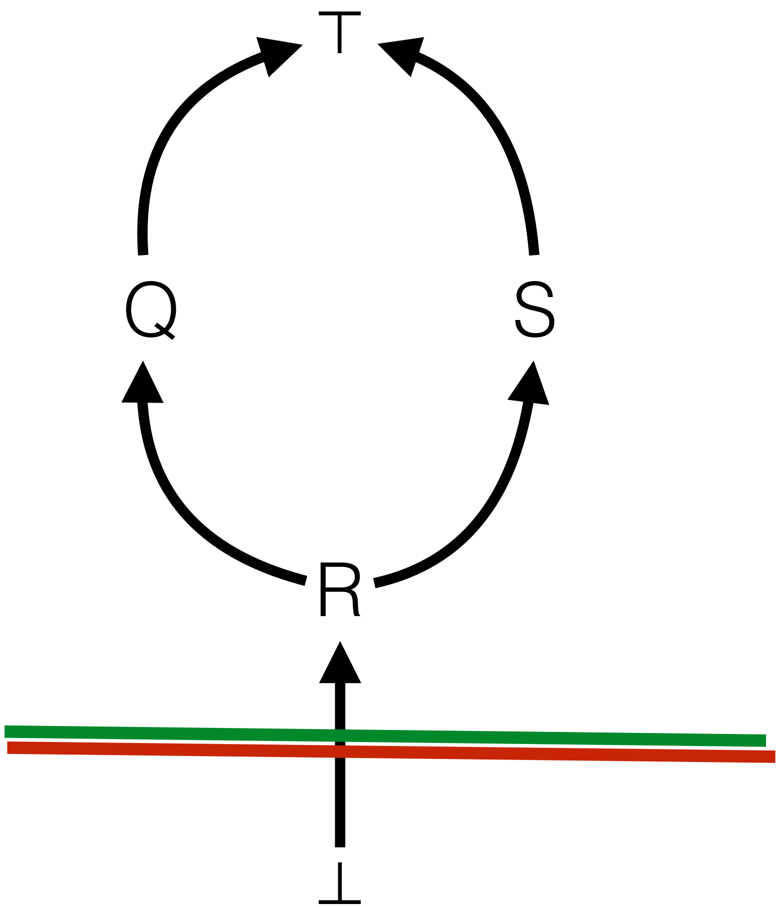
$$\neg R \vee Q$$
$$\wedge$$
$$\neg R \vee S$$

$\neg R$ Q S



$$(R \rightarrow Q) \land (R \rightarrow S)$$

$$(\neg R \lor Q) \land (\neg R \lor S)$$

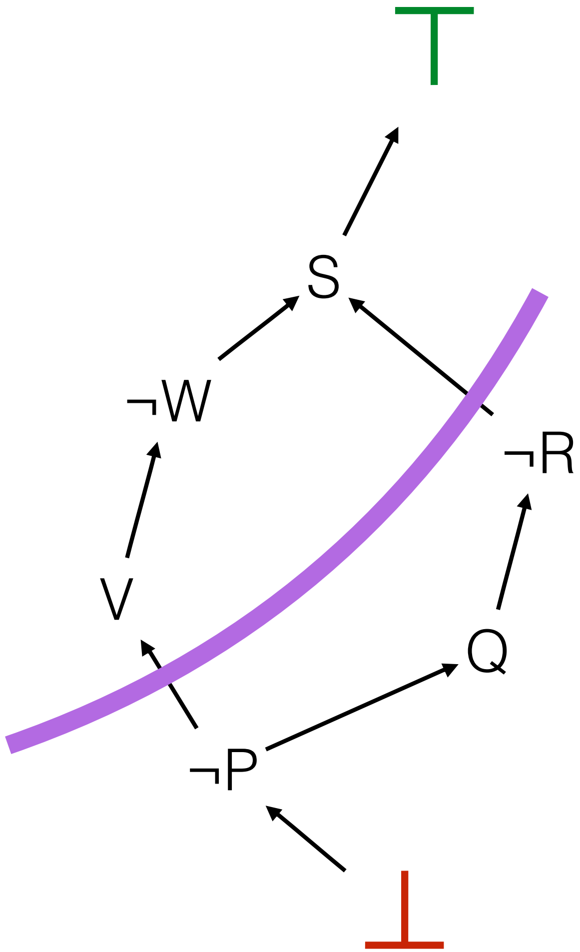


R Q S

$$\begin{array}{c} \neg P \to V \\ \wedge \\ V \to \neg W \\ \wedge \\ \neg W \to S \end{array} \quad \wedge \quad \begin{array}{c} \neg P \to Q \\ \wedge \\ Q \to \neg R \\ \wedge \\ \neg R \to S \end{array}$$

$$\begin{array}{c} P \vee V \\ \wedge \\ \neg V \vee \neg W \\ \wedge \\ W \vee S \end{array} \quad \wedge \quad \begin{array}{c} P \vee Q \\ \wedge \\ \neg Q \vee \neg R \\ \wedge \\ R \vee S \end{array}$$

The same trick works if our implications form a partial order.
But we have more options since we can draw a wavy line.

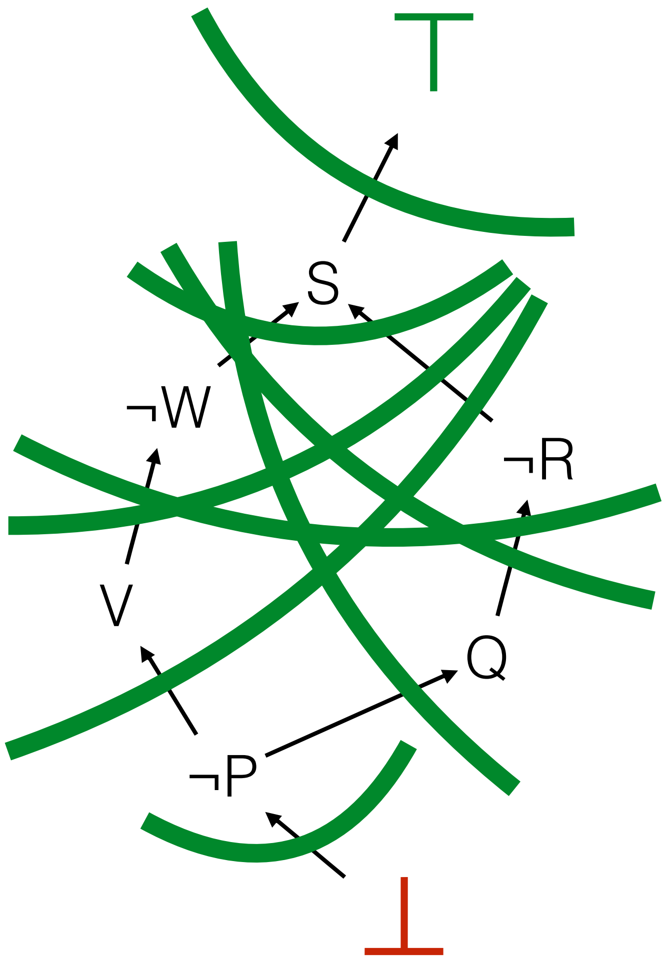


The **arrow rule** says that, whenever our line cuts an arrow, then the head must be on the side of true and the tail on the side of false.

The same trick works if our implications form a partial order.
But we have more options since we can draw a wavy line.

Not all of the valid truth assignments are represented in this diagram.

How many are missing?



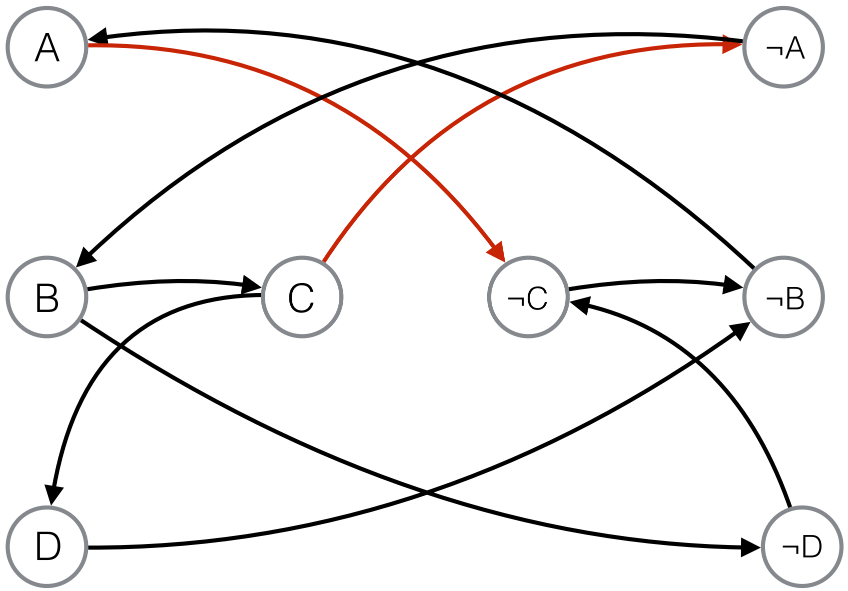
Binary constraints

You may not take both Archeology and Chemistry
If you take Biology you must take Chemistry
You must take Biology or Archeology
If you take Chemistry you must take Divinity
You may not take both Divinity and Biology

$$(\neg A \vee \neg C) \wedge (\neg B \vee C) \wedge (B \vee A) \wedge (\neg C \vee D) \wedge (\neg D \vee \neg B)$$

$$(A \rightarrow \neg C) \wedge (B \rightarrow C) \wedge (\neg B \rightarrow A) \wedge (C \rightarrow D) \wedge (D \rightarrow \neg B)$$

$$(\neg A \lor \neg C) \land (\neg B \lor C) \land (B \lor A) \land (\neg C \lor D) \land (\neg D \lor \neg B)$$

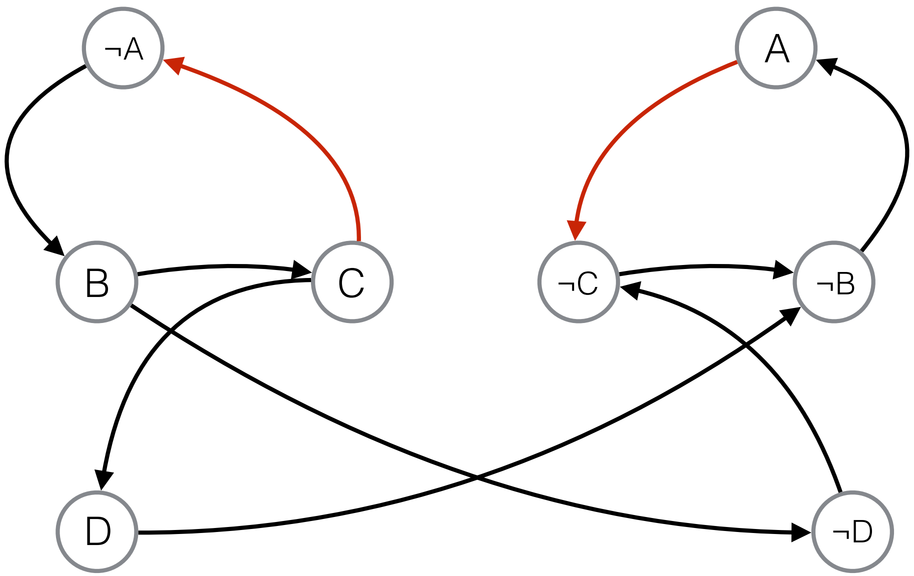


$$(\neg A \lor \neg C) \land (\neg B \lor C) \land (B \lor A) \land (\neg C \lor D) \land (\neg D \lor \neg B)$$

$$\equiv$$

$$(A \rightarrow \neg C) \land (B \rightarrow C) \land (\neg B \rightarrow A) \land (C \rightarrow D) \land (D \rightarrow \neg B)$$

$(\neg A \vee \neg C) \wedge (\neg B \vee C) \wedge (B \vee A) \wedge (\neg C \vee D) \wedge (\neg D \vee \neg B)$

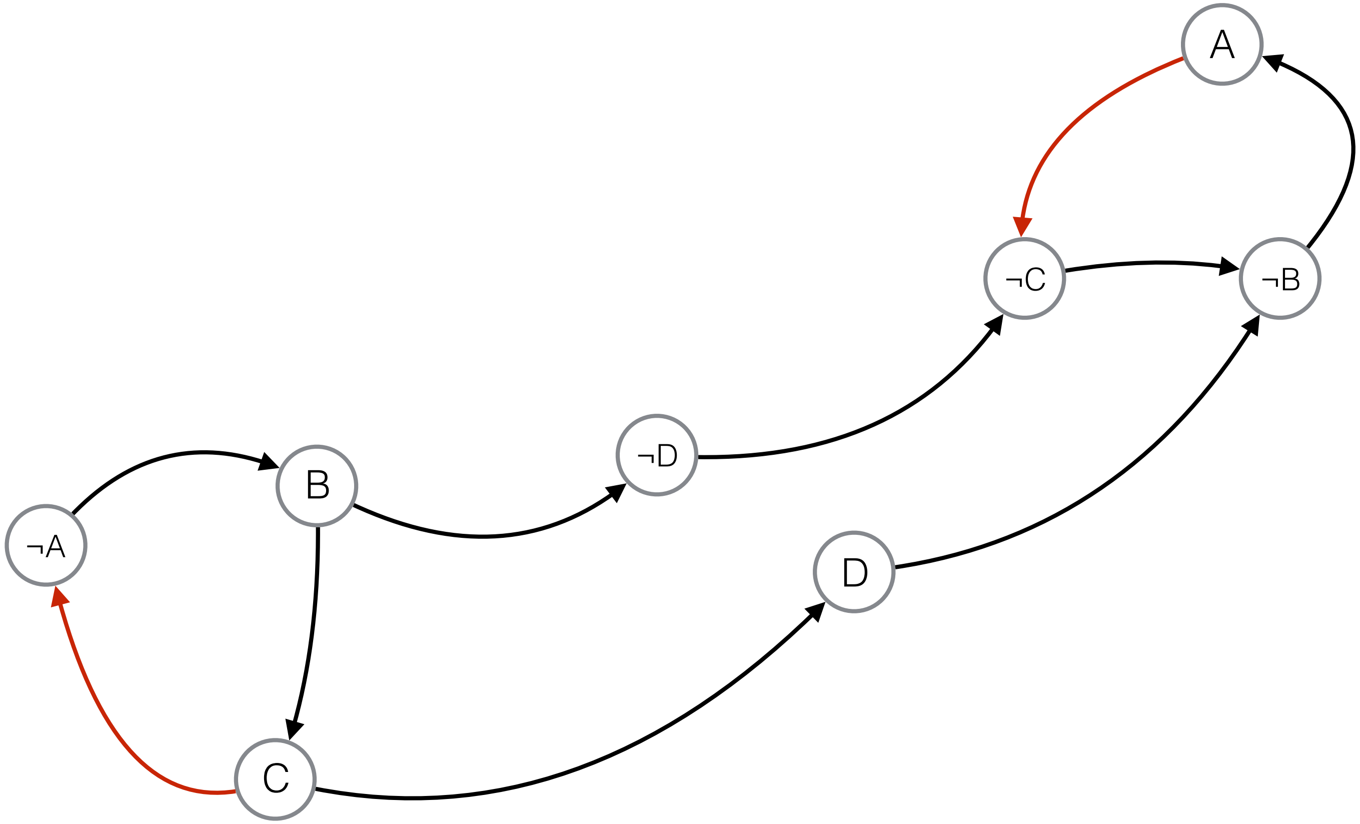


$(\neg A \vee \neg C) \wedge (\neg B \vee C) \wedge (B \vee A) \wedge (\neg C \vee D) \wedge (\neg D \vee \neg B)$

$\equiv$

$(A \to \neg C) \wedge (B \to C) \wedge (\neg B \to A) \wedge (C \to D) \wedge (D \to \neg B)$

$(\neg A \lor \neg C) \land (\neg B \lor C) \land (B \lor A) \land (\neg C \lor D) \land (\neg D \lor \neg B)$



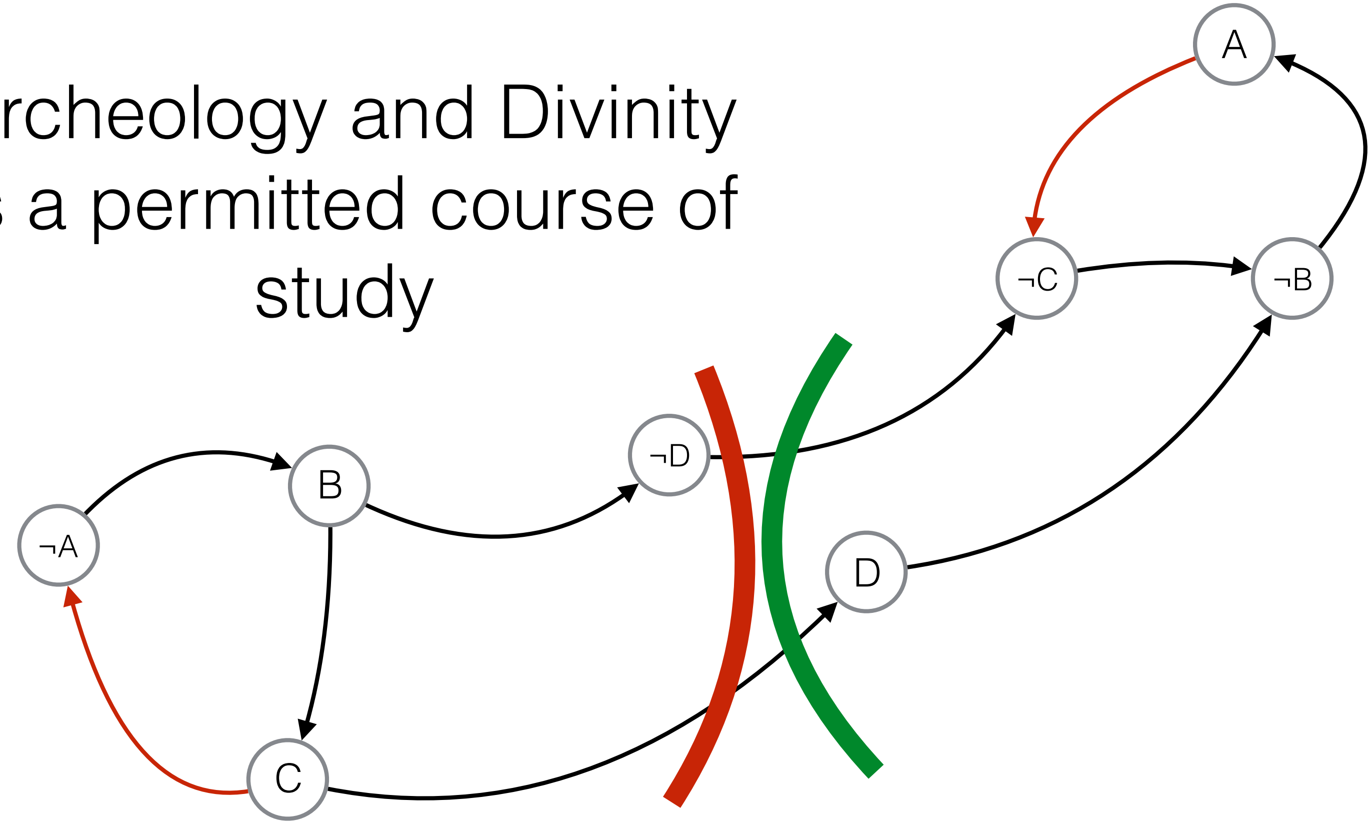
$(\neg A \lor \neg C) \land (\neg B \lor C) \land (B \lor A) \land (\neg C \lor D) \land (\neg D \lor \neg B)$

$$\equiv$$

$(A \rightarrow \neg C) \land (B \rightarrow C) \land (\neg B \rightarrow A) \land (C \rightarrow D) \land (D \rightarrow \neg B)$

$(\neg A \vee \neg C) \wedge (\neg B \vee C) \wedge (B \vee A) \wedge (\neg C \vee D) \wedge (\neg D \vee \neg B)$

Archeology and Divinity is a permitted course of study



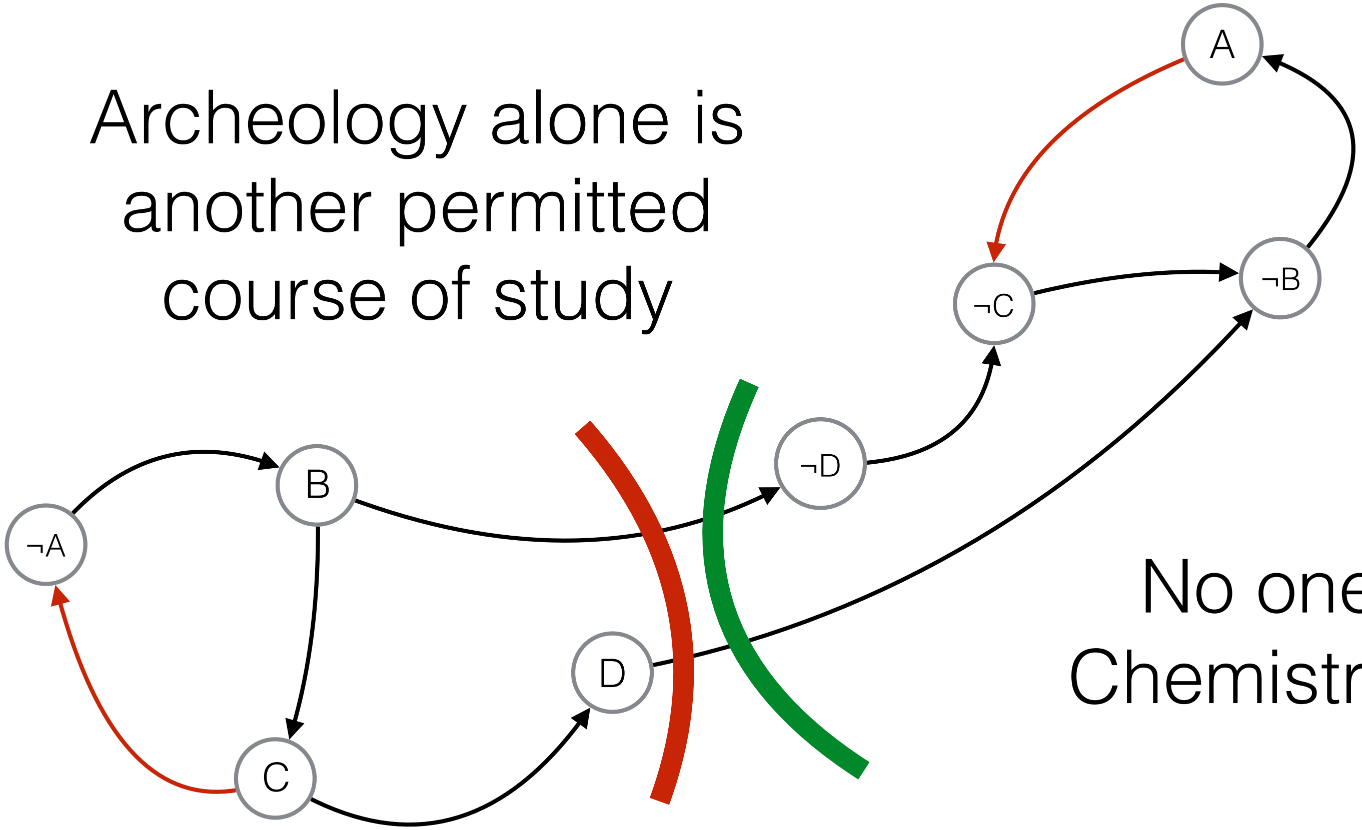
$$(\neg A \vee \neg C) \wedge (\neg B \vee C) \wedge (B \vee A) \wedge (\neg C \vee D) \wedge (\neg D \vee \neg B)$$

$$\equiv$$

$$(A \rightarrow \neg C) \wedge (B \rightarrow C) \wedge (\neg B \rightarrow A) \wedge (C \rightarrow D) \wedge (D \rightarrow \neg B)$$

$$(\neg A \vee \neg C) \wedge (\neg B \vee C) \wedge (B \vee A) \wedge (\neg C \vee D) \wedge (\neg D \vee \neg B)$$

Archeology alone is another permitted course of study



No one can take Chemistry or Biology

$$(\neg A \vee \neg C) \wedge (\neg B \vee C) \wedge (B \vee A) \wedge (\neg C \vee D) \wedge (\neg D \vee \neg B)$$

$$\equiv$$

$$(A \rightarrow \neg C) \wedge (B \rightarrow C) \wedge (\neg B \rightarrow A) \wedge (C \rightarrow D) \wedge (D \rightarrow \neg B)$$