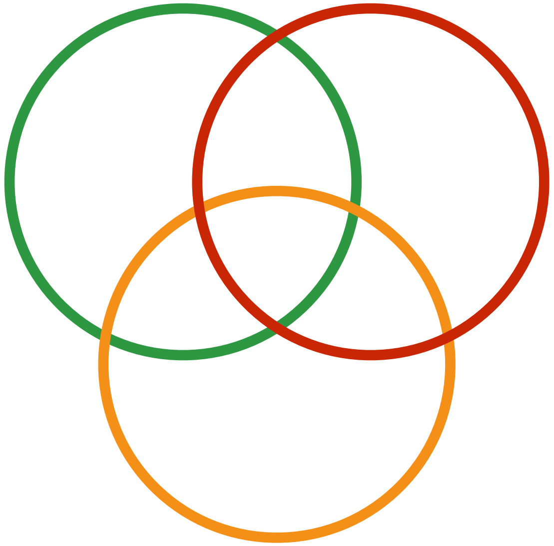


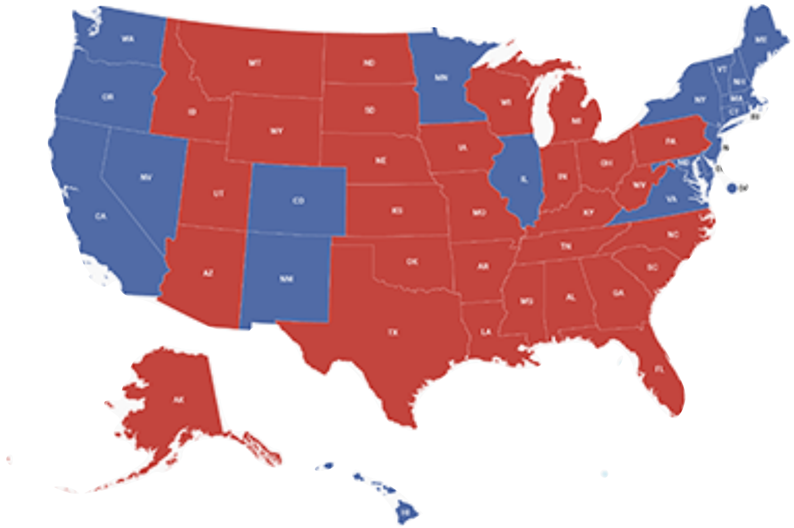
Informatics 1

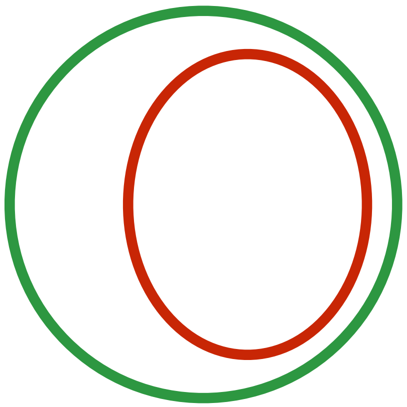
Computation and Logic

Sets of States: Venn Diagrams and Truth Tables

Michael Fourman
@mp4man

p	q	$p \vee q$	$p \wedge q$	$p \to q$	$p \leftrightarrow q$
T	T	T	T	T	T
T	F	T	F	F	F
F	T	T	F	T	F
F	F	F	F	T	T





sets and subsets

$$A \subseteq U \;\; \textbf{iff} \;\; \textbf{for all } x \in A.\; x \in U$$

universe $\boldsymbol{U}$ a set

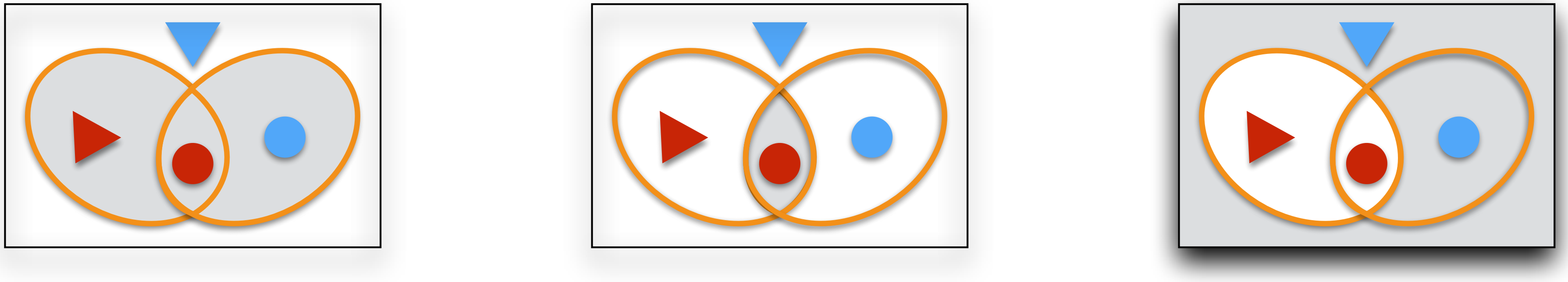
subsets $A, B \subseteq U$

comprehension

For any set X
and any property P
$\{x \in X \mid P(x)\}$ is a set,
whose members are those $x \in X$ such that $P(x)$.

$$y \in \{x \in X \mid P(x)\} \textbf{ iff } y \in X \textbf{ and } P(y)$$

Operations on Sets



$$x \in A \cup B \textbf{ iff } x \in A \textbf{ or } x \in B \qquad \text{(union)}$$

$$x \in A \cap B \textbf{ iff } x \in A \textbf{ and } x \in B \qquad \text{(intersection)}$$

for $x \in U$

$$x \in U \setminus A \textbf{ iff } x \notin A \qquad \text{(complement)}$$

Singletons

For any X

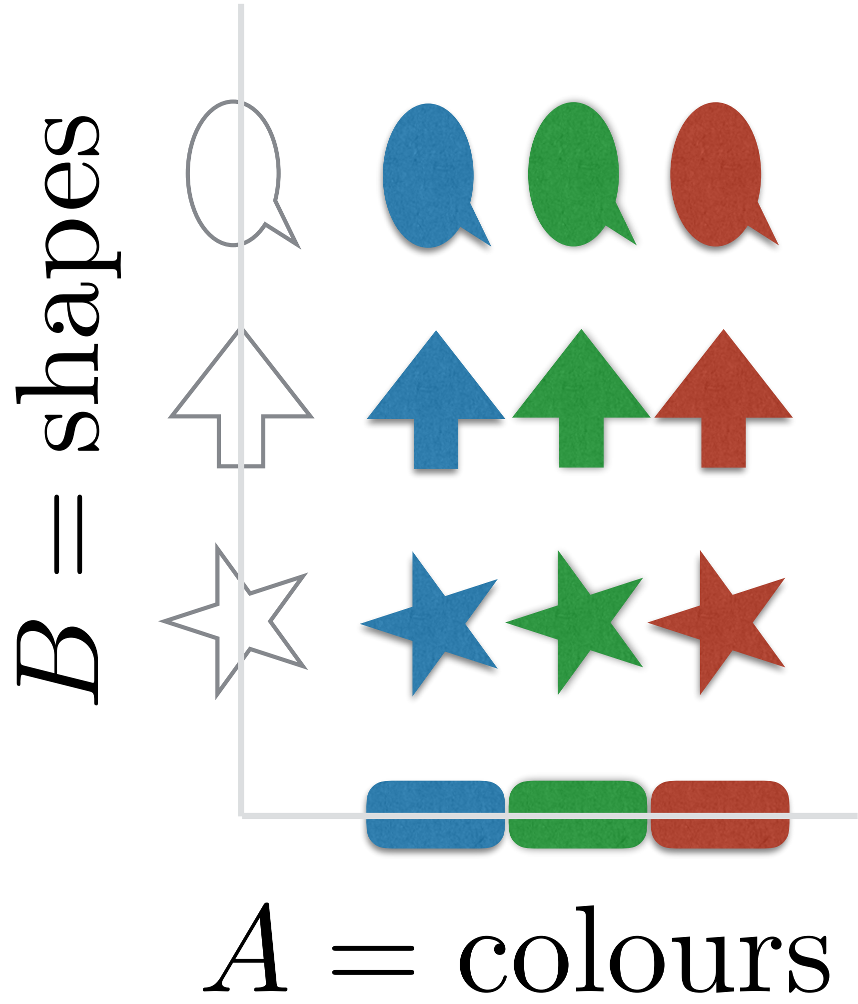
$$\{X\} \text{ is a set, whose only member is } X.$$

We can take unions of singletons to construct any non-empty finite set.

products : sets of pairs

$$A \times B \ = \{\langle x, y\rangle \mid x \in A \textbf{ and } y \in B\}$$

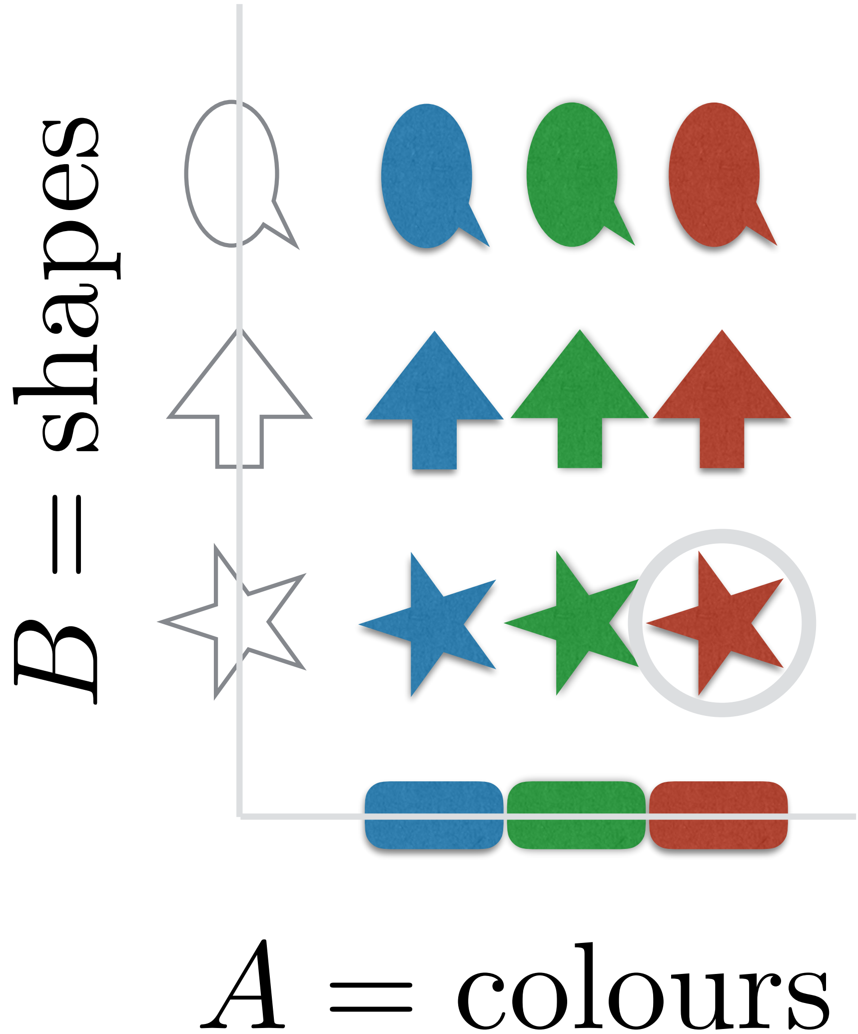
$$|A \times B| = |A| \times |B|$$



$$A \times B = \{\langle x, y\rangle \mid x \in A \textbf{ and } y \in B\}$$

$$|A \times B| = |A| \times |B|$$

$$\langle \text{red}, \text{star}\rangle$$



sets and properties

universe U a set

true-false properties $P, Q : U \Rightarrow \{\top, \bot\}$

every property corresponds to a subset

$$[\![P]\!] = \{x \in U \mid P(x)\}$$

subsets and properties

every property corresponds to a subset

$$[\![P]\!] = \{x \in U \mid P(x)\}$$

every subset corresponds to a property

$$P(x) \ \textbf{iff} \ x \in [\![P]\!]$$

Powerset

the subsets of a set form a set

$$A \in \wp X \ \textbf{ iff } \ A \subseteq X$$

if X has n elements
how many subsets does it have?
how big is $\wp X$?

Powerset

if X has n elements

how many subsets does it have?

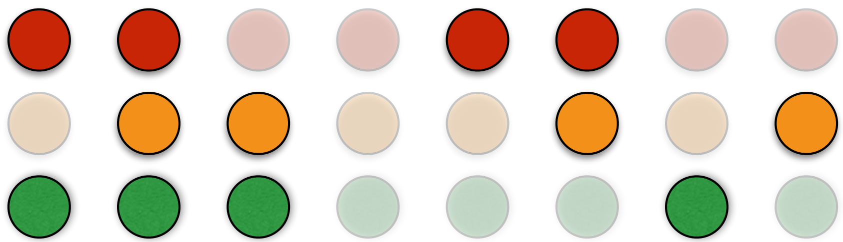
how big is $\wp X$?

$$\textbf{if } |X| = n$$

$$\textbf{then } |\wp X| = 2^n$$

example: traffic lights

if we have a set of three lights, red, amber, green
then a **state** of the lights
is a subset of the set {red, amber, green}
whose members are those lights that are on

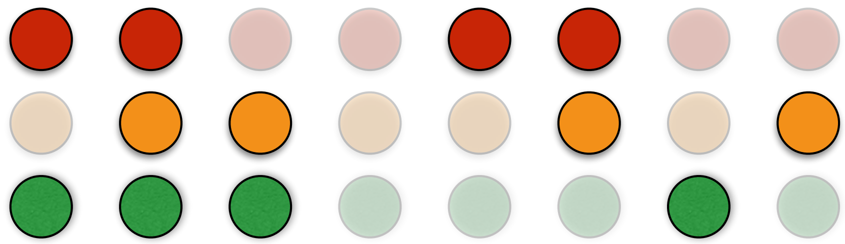


there are 8 possible states

example: traffic lights

a **state** of the signal

is a subset of the set $L = \{\text{red}, \text{amber}, \text{green}\}$



there are 3 natural properties of this set of 8 states

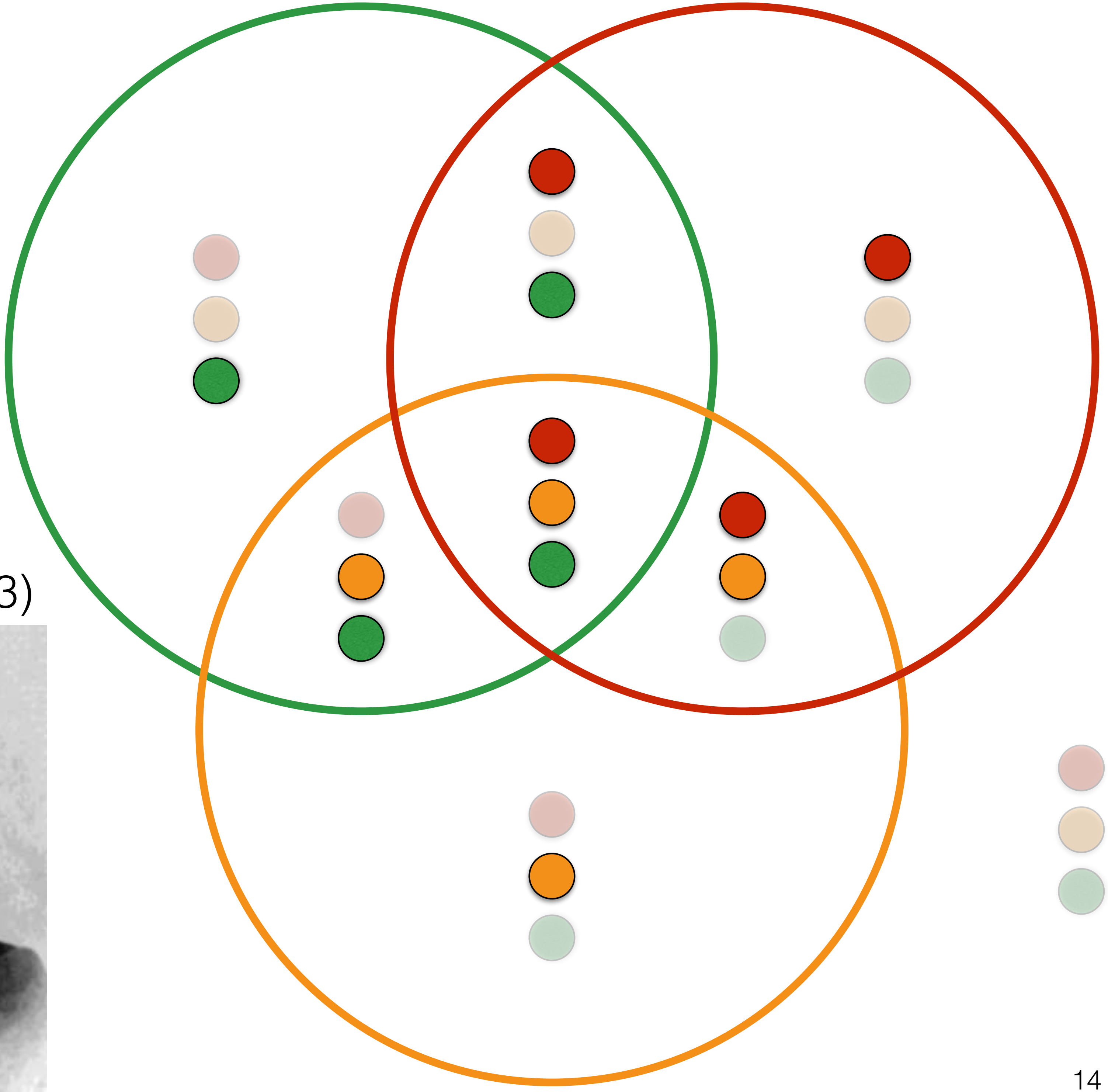
$$\textbf{for } S \subseteq L$$

$$R(S) \textbf{ iff } \text{red} \in S$$

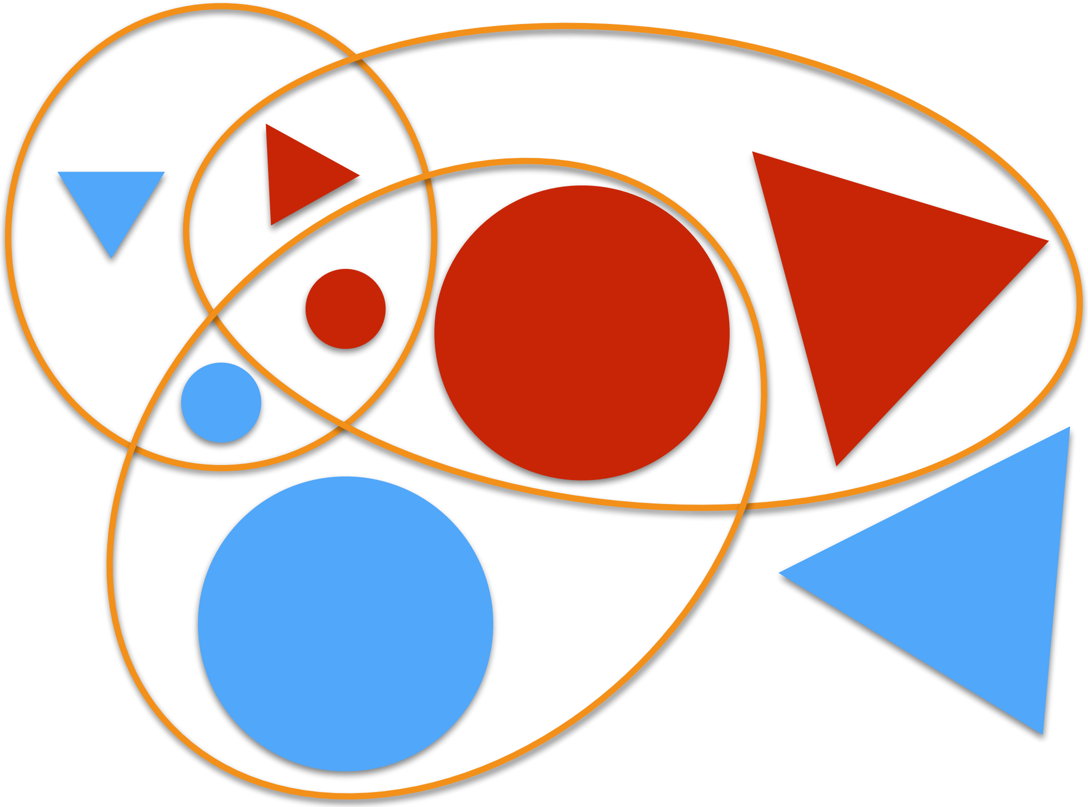
$$A(S) \textbf{ iff } \text{amber} \in S$$

$$G(S) \textbf{ iff } \text{green} \in S$$

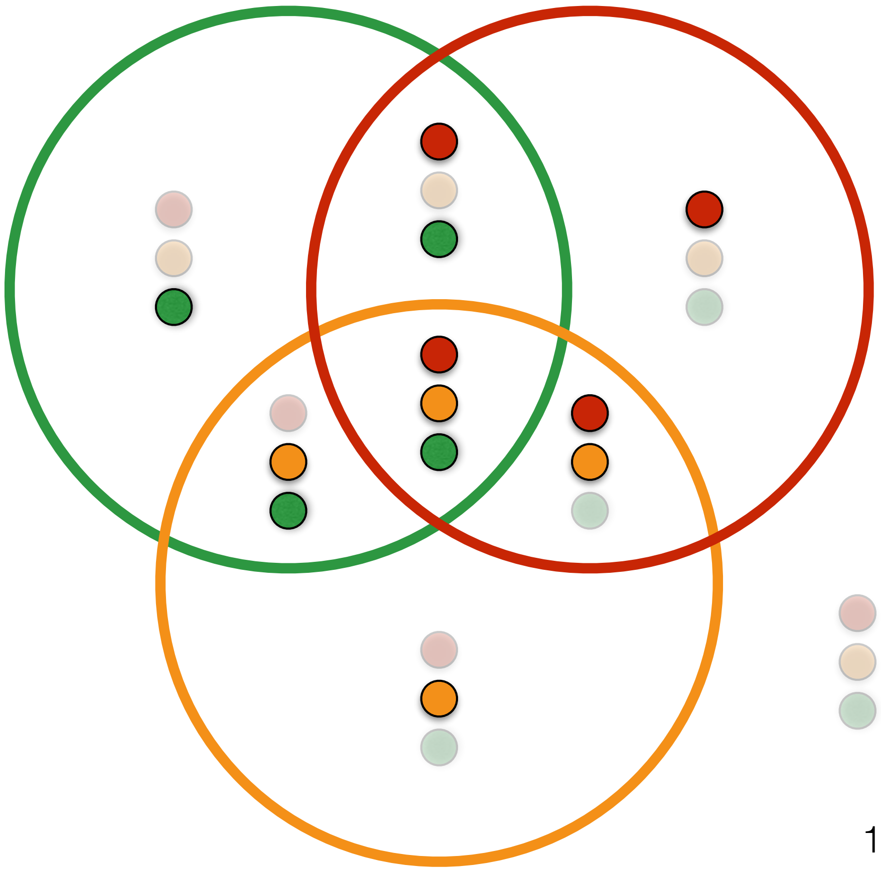
Venn (1834–1923)



properties are extensional



red	red
small	green
disc	amber



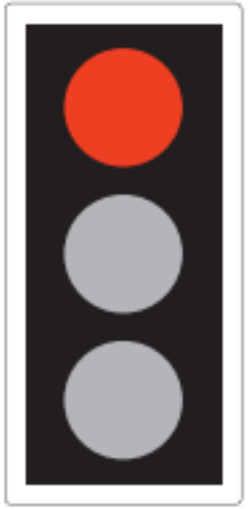
As with all sciences, informatics is concerned with the mathematical modelling of the real world.

Most branches of engineering use a continuum model. They neglect the fact that real substances are composed of discrete molecules and model matter from the start as a smoothed-out continuum.

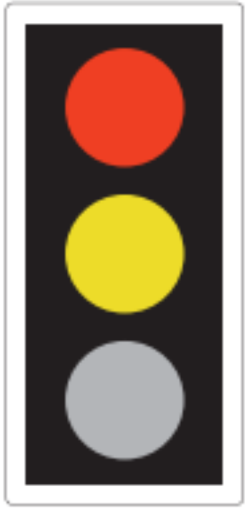
Digital systems take an opposite approach. They also neglect the complexity of reality – but they are engineered so that we can use discrete models to design and reason about their behaviour.

We build these models using logic and set theory.

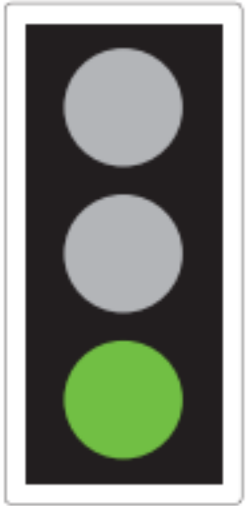
Traffic Light Signals



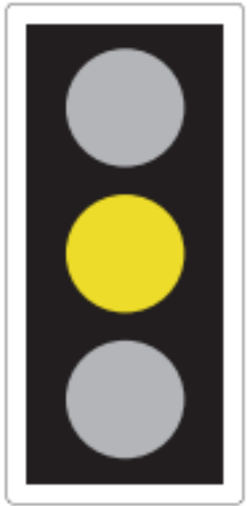
RED means 'Stop'. Wait behind the stop line on the carriageway



RED AND AMBER also means 'Stop'. Do not pass through or start until GREEN shows



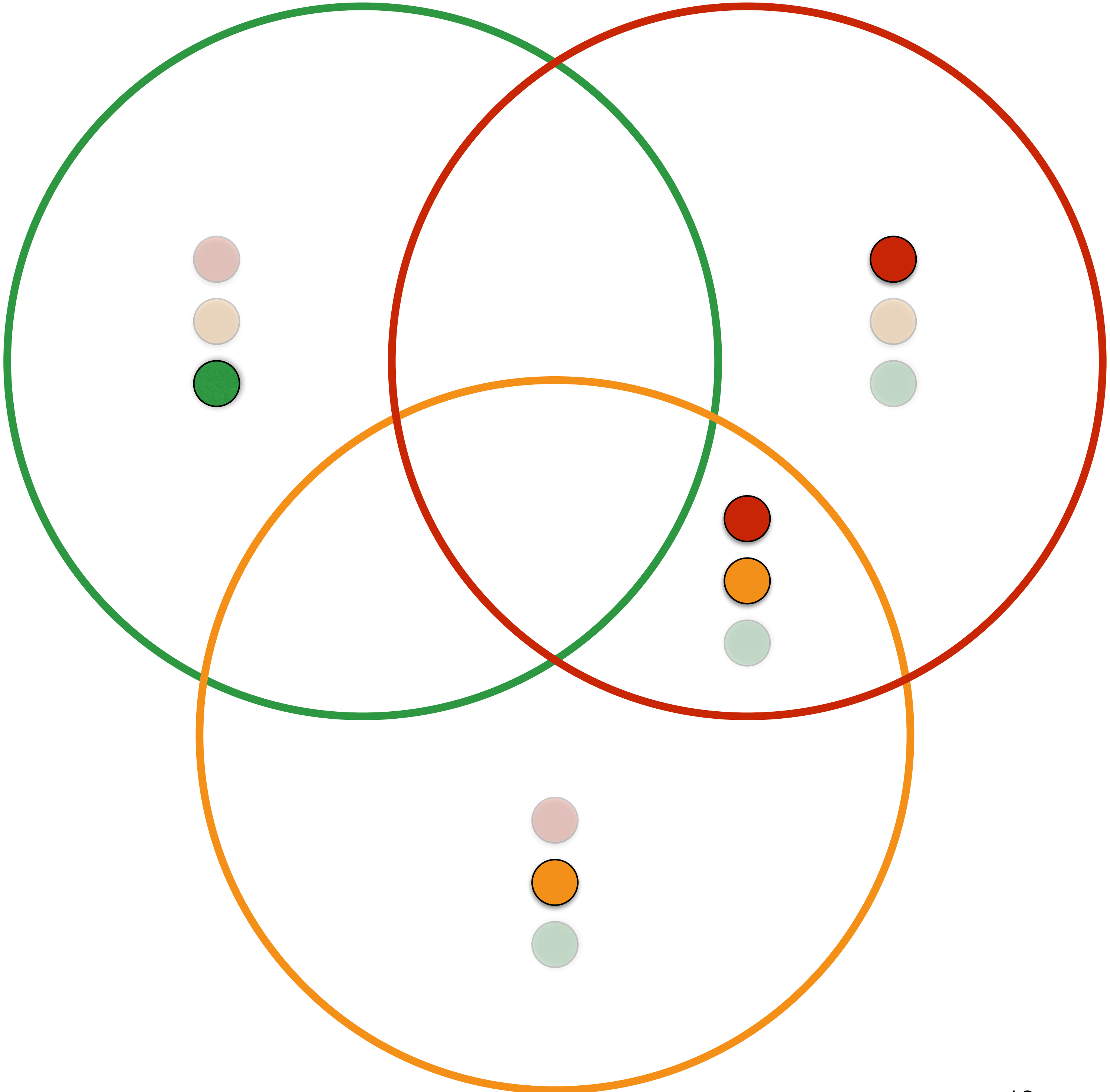
GREEN means you may go on if the way is clear. Take special care if you intend to turn left or right and give way to pedestrians who are crossing



AMBER means 'Stop' at the stop line. You may go on only if the AMBER appears after you have crossed the stop line or are so close to it that to pull up might cause an accident

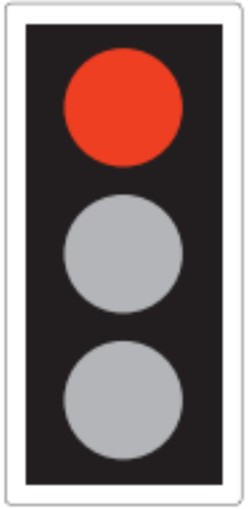
How can we characterise these three states?

Traffic Light



Venn (1834–1923)

Traffic Light Signals



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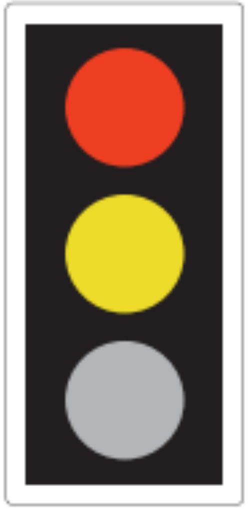

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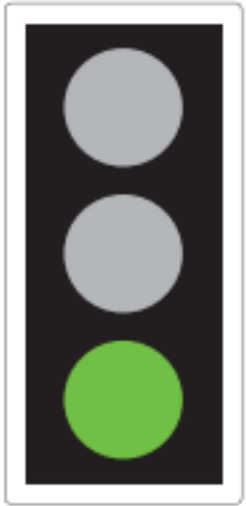

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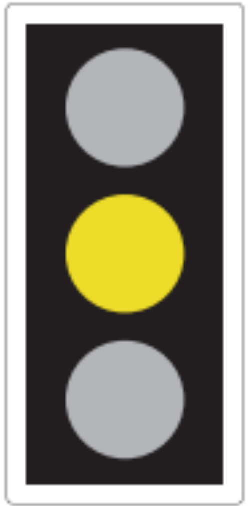

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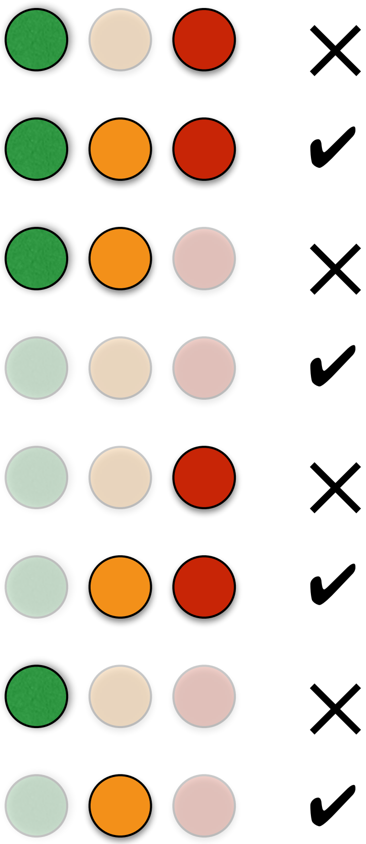
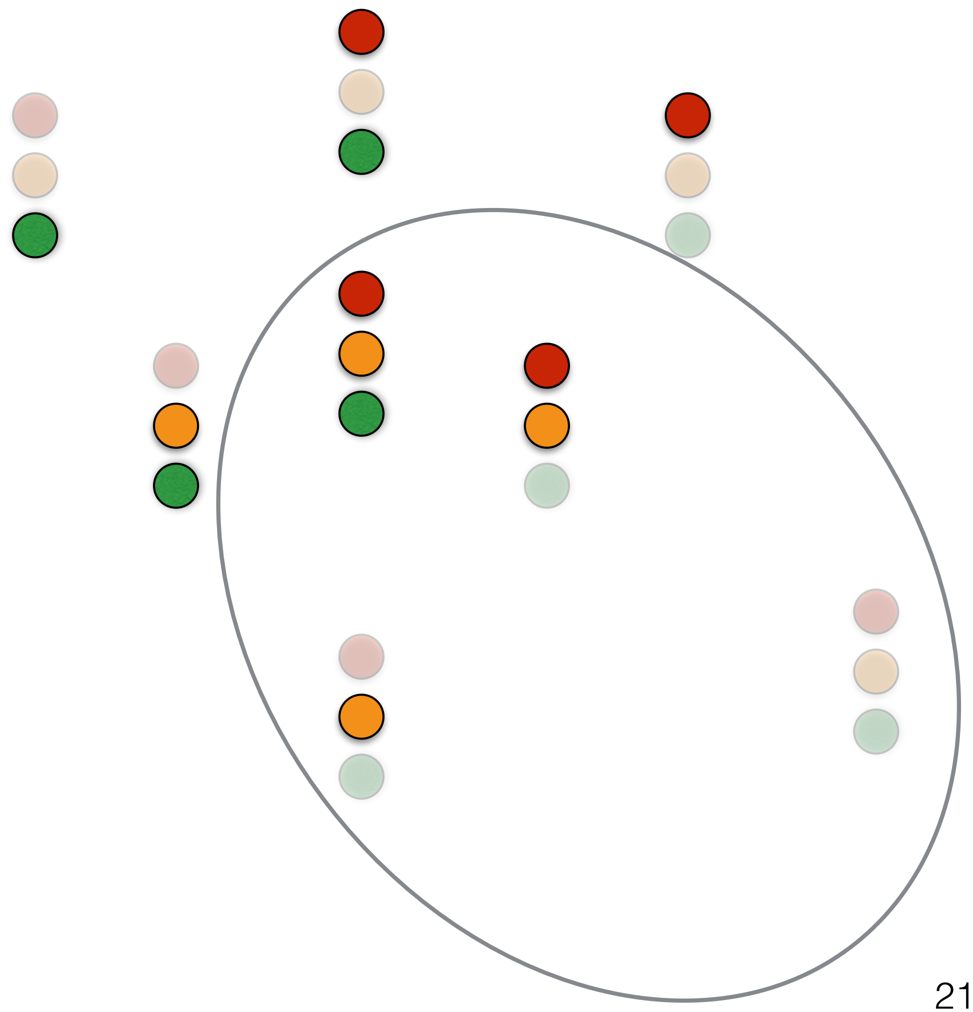
$$G \oplus (R \vee A)$$


ABS

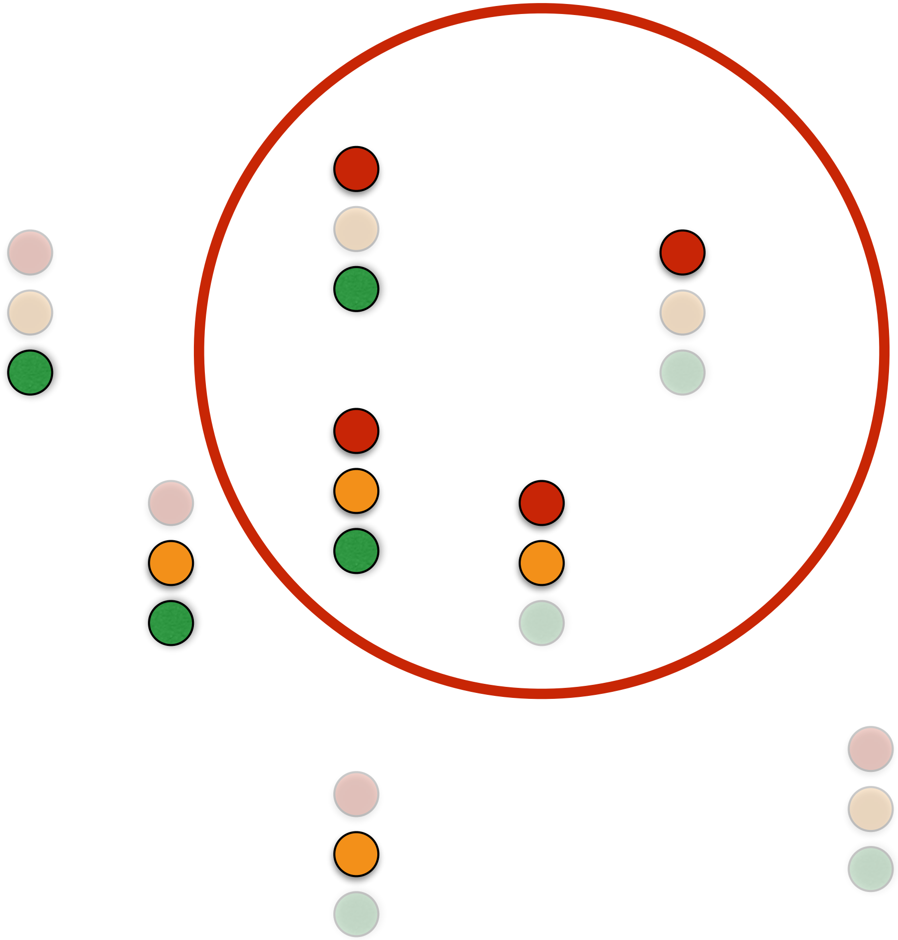
A **truth table** represents sets of states as a functions from states to truth values





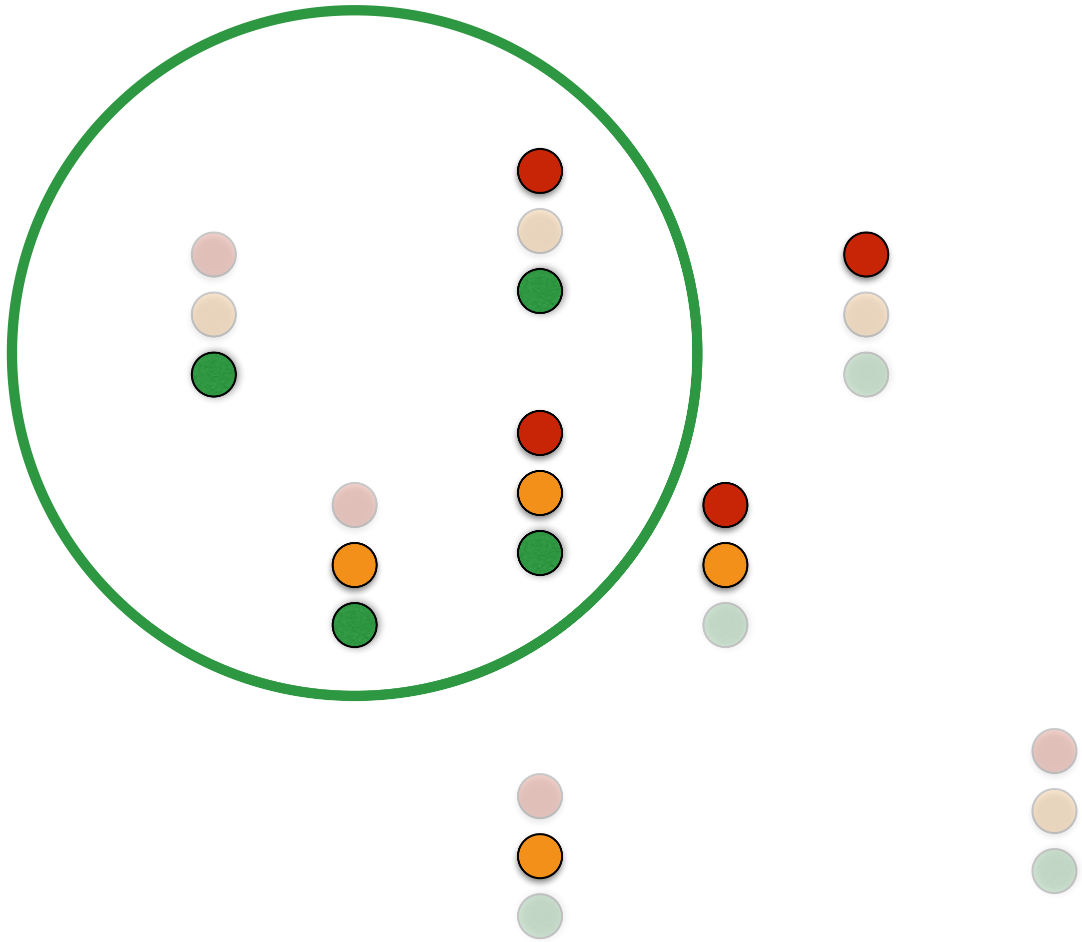
$$\{x \mid R(x)\}$$

R	A	G
0	0	0
0	0	1
0	1	0
0	1	1
1	0	0
1	0	1
1	1	0
1	1	1



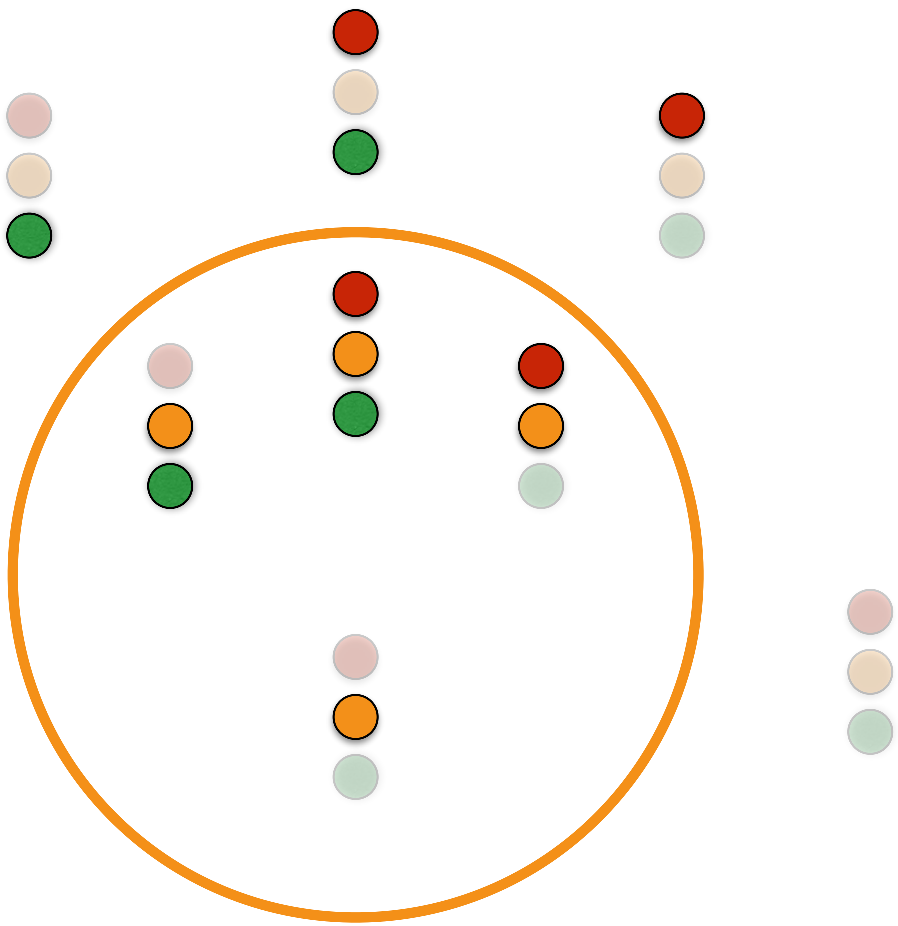
$$\{x \mid G(x)\}$$

R	A	**G**
0	0	**0**
0	0	**1**
0	1	**0**
0	1	**1**
1	0	**0**
1	0	**1**
1	1	**0**
1	1	**1**



$$\{x \mid A(x)\}$$

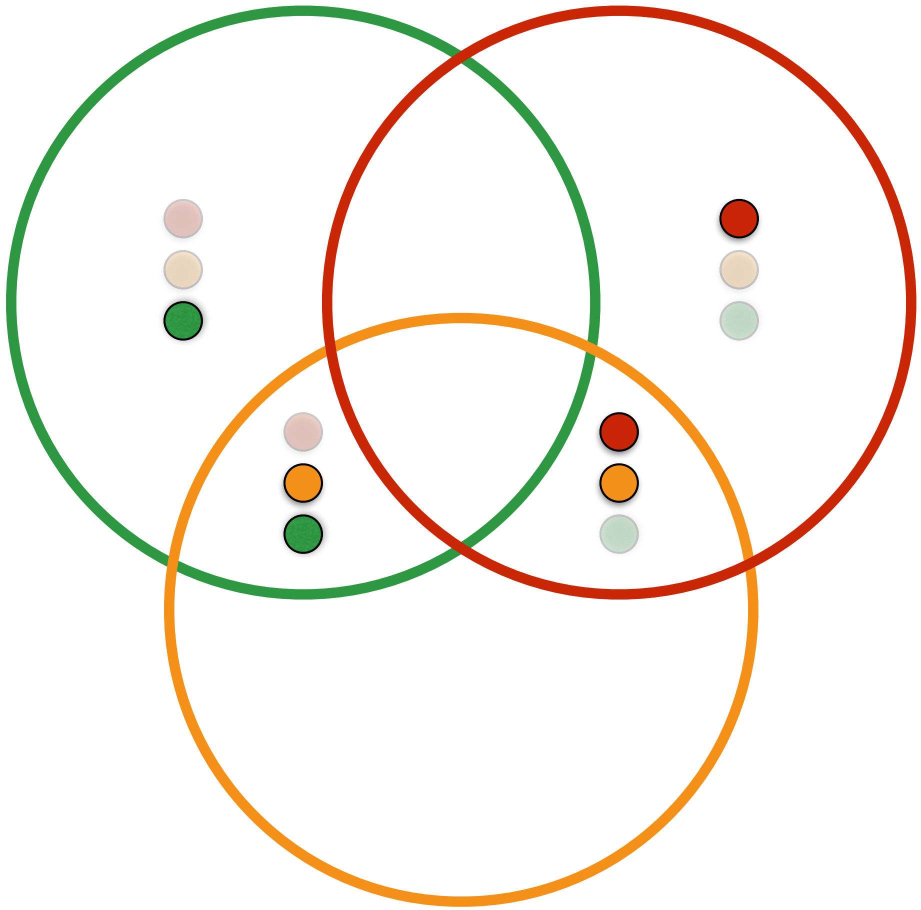
R	**A**	G
0	**0**	0
0	**0**	1
0	**1**	0
0	**1**	1
1	**0**	0
1	**0**	1
1	**1**	0
1	**1**	1



xor

$$\{x \mid G(x) \oplus R(x)\}$$

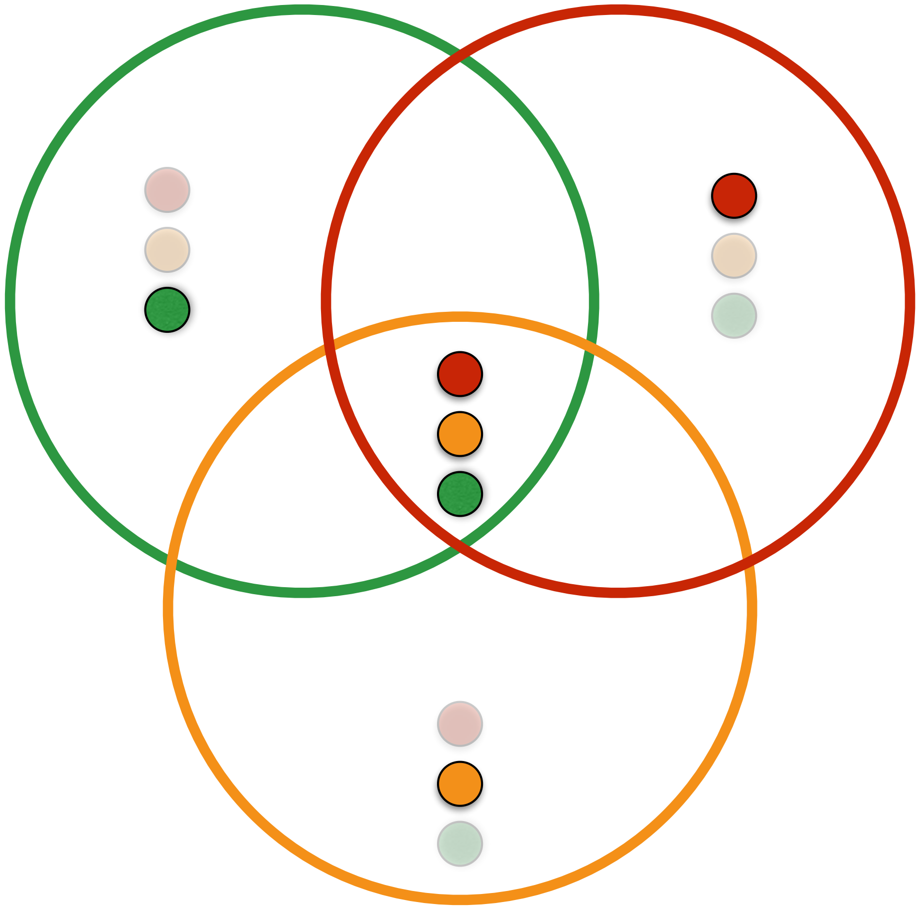
R	A	G	**G ⊕ R**
0	0	0	**0**
0	0	1	**1**
0	1	0	**0**
0	1	1	**1**
1	0	0	**1**
1	0	1	**0**
1	1	0	**1**
1	1	1	**0**



xor

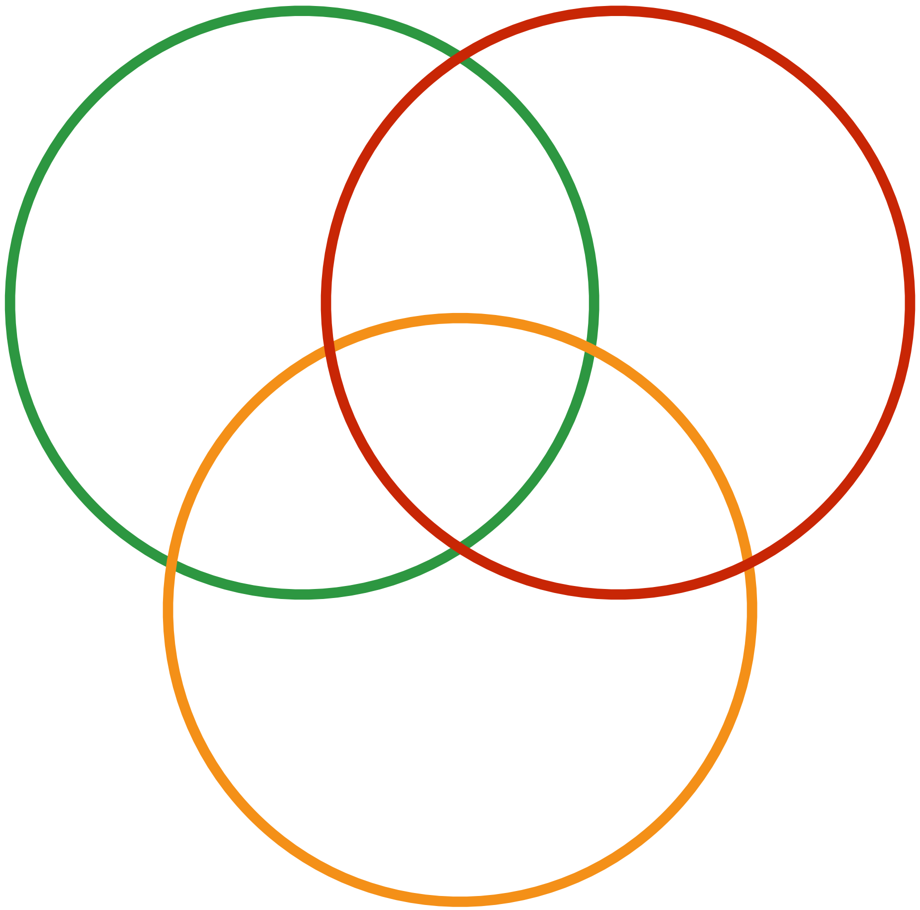
$$\{x \mid (G(x) \oplus R(x)) \oplus A(x)\}$$

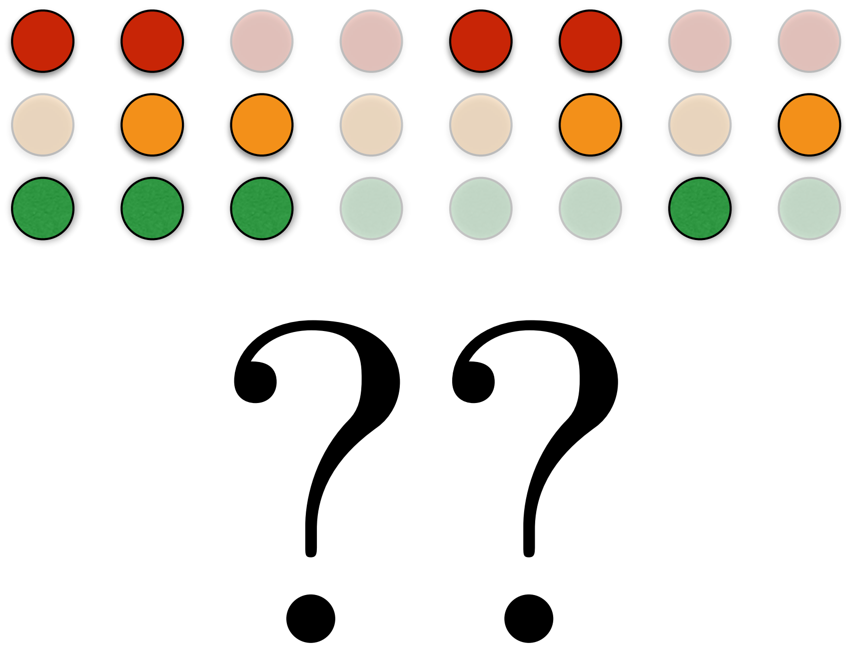
R	A	G	**G ⊕ R**	**(G ⊕ R) ⊕ A**
0	0	0	**0**	**0**
0	0	1	**1**	**1**
0	1	0	**0**	**1**
0	1	1	**1**	**0**
1	0	0	**1**	**1**
1	0	1	**0**	**0**
1	1	0	**1**	**0**
1	1	1	**0**	**1**



xor

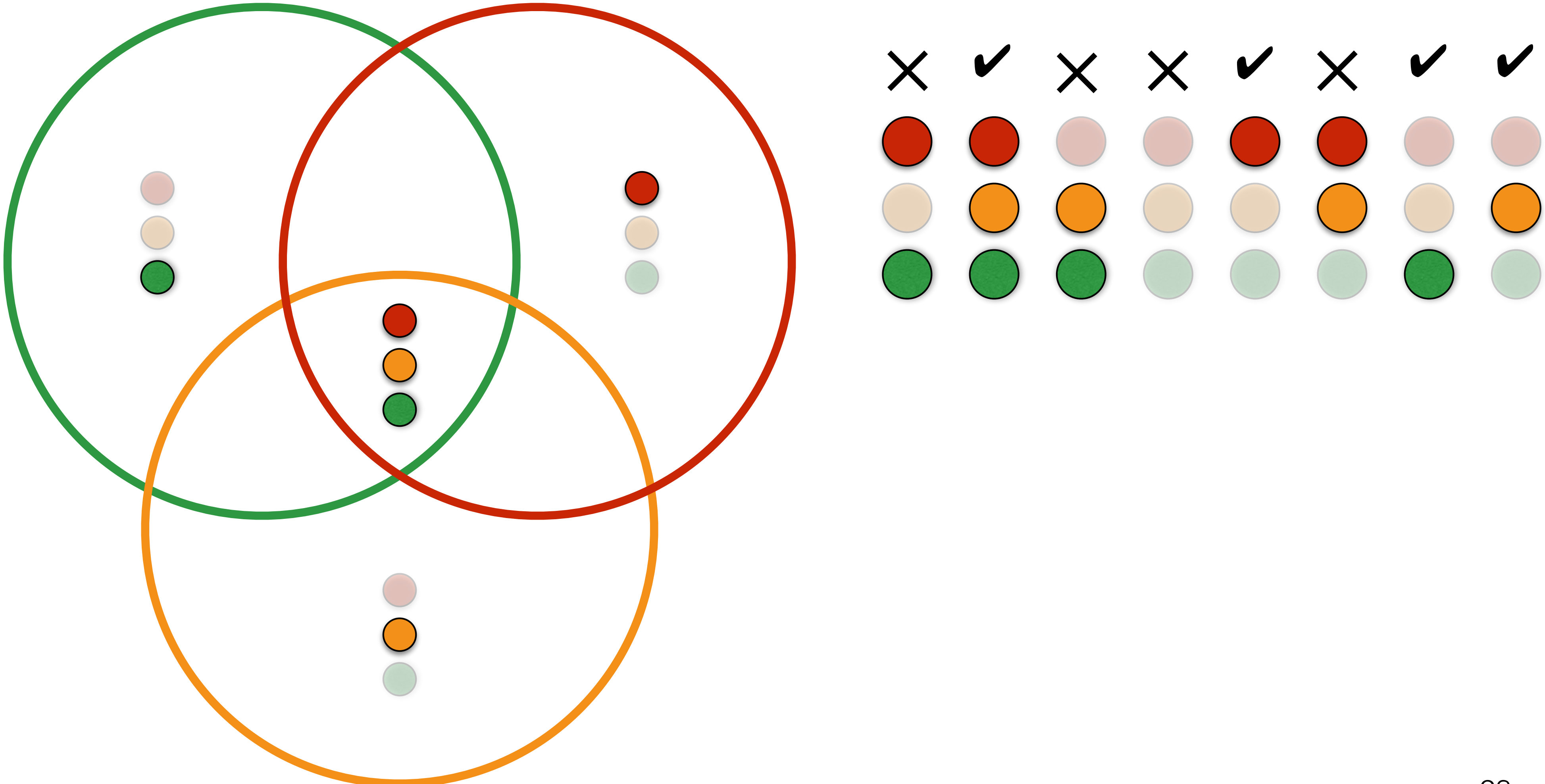
$$\{x \mid G(x) \oplus (R(x) \oplus A(x))\}$$





xor

$$\{x \mid G(x) \oplus (R(x) \oplus A(x))\}$$



xor

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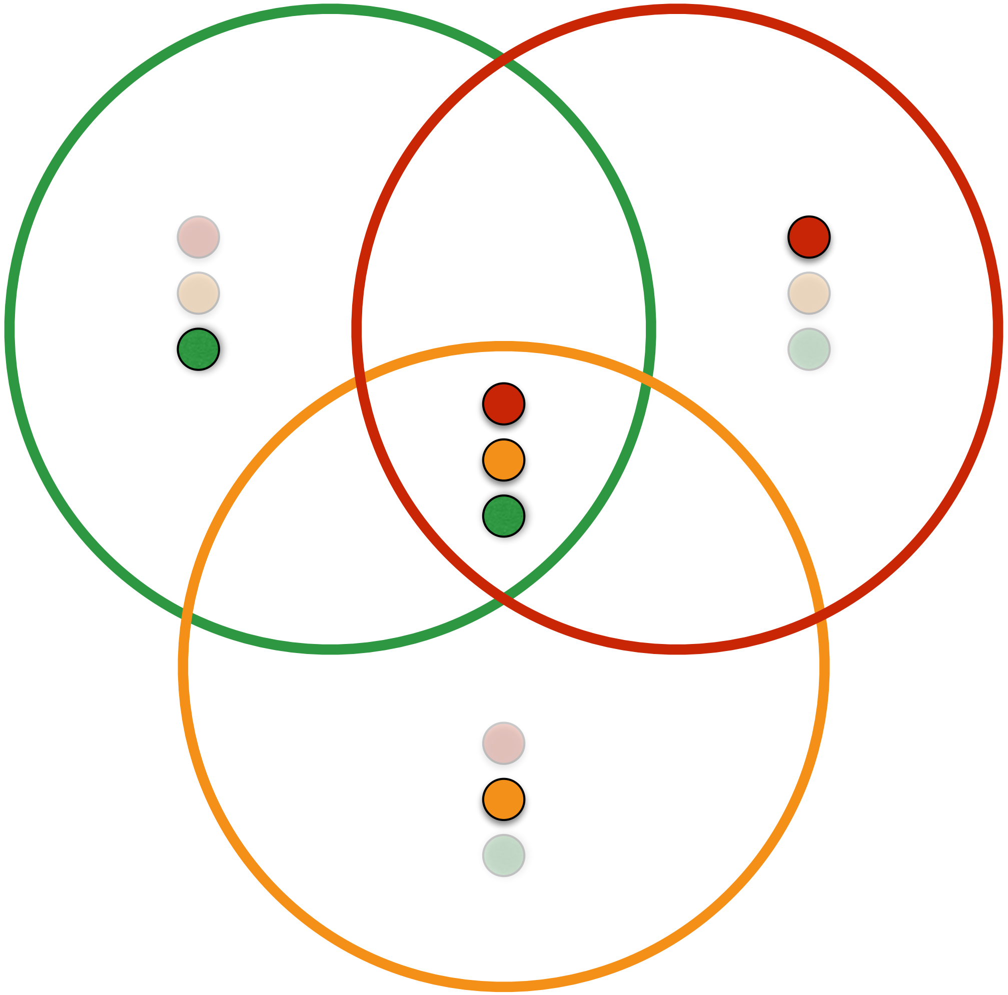
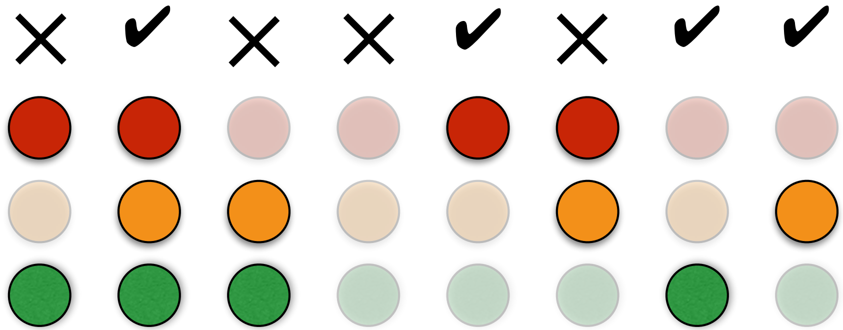
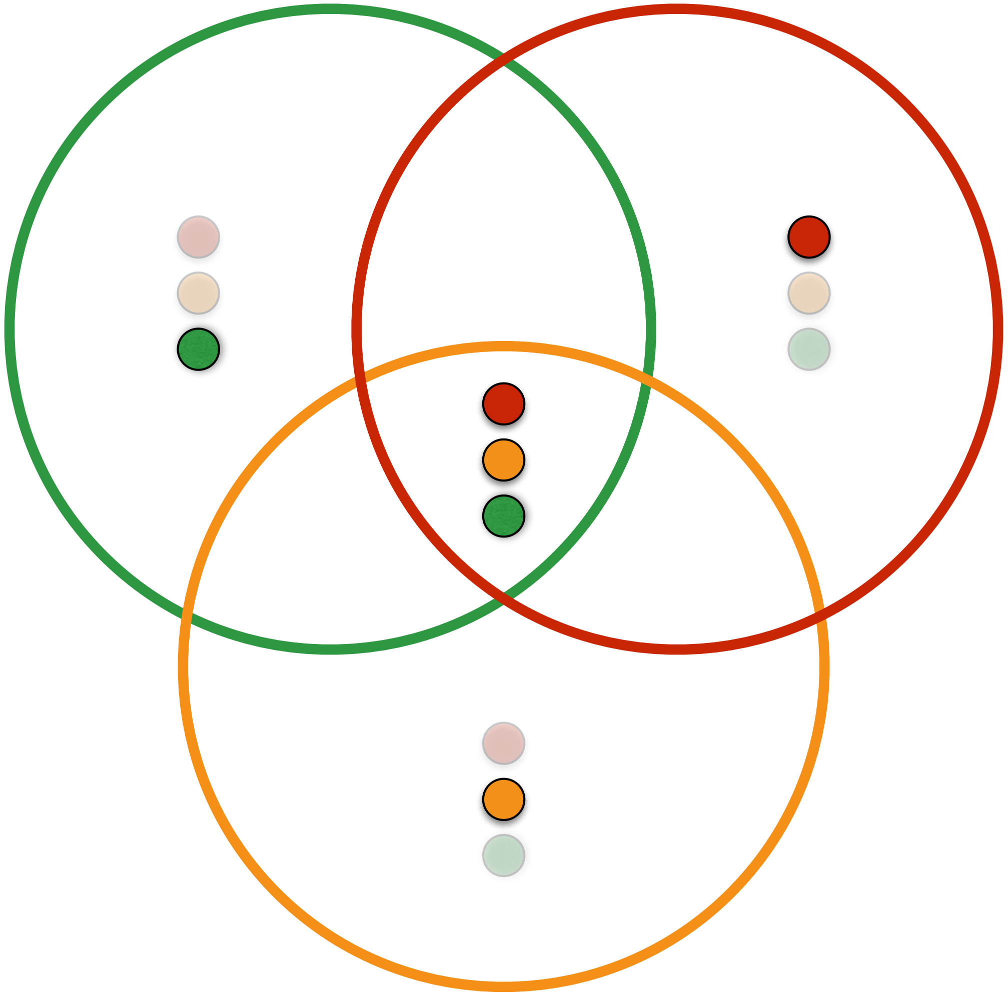




xor

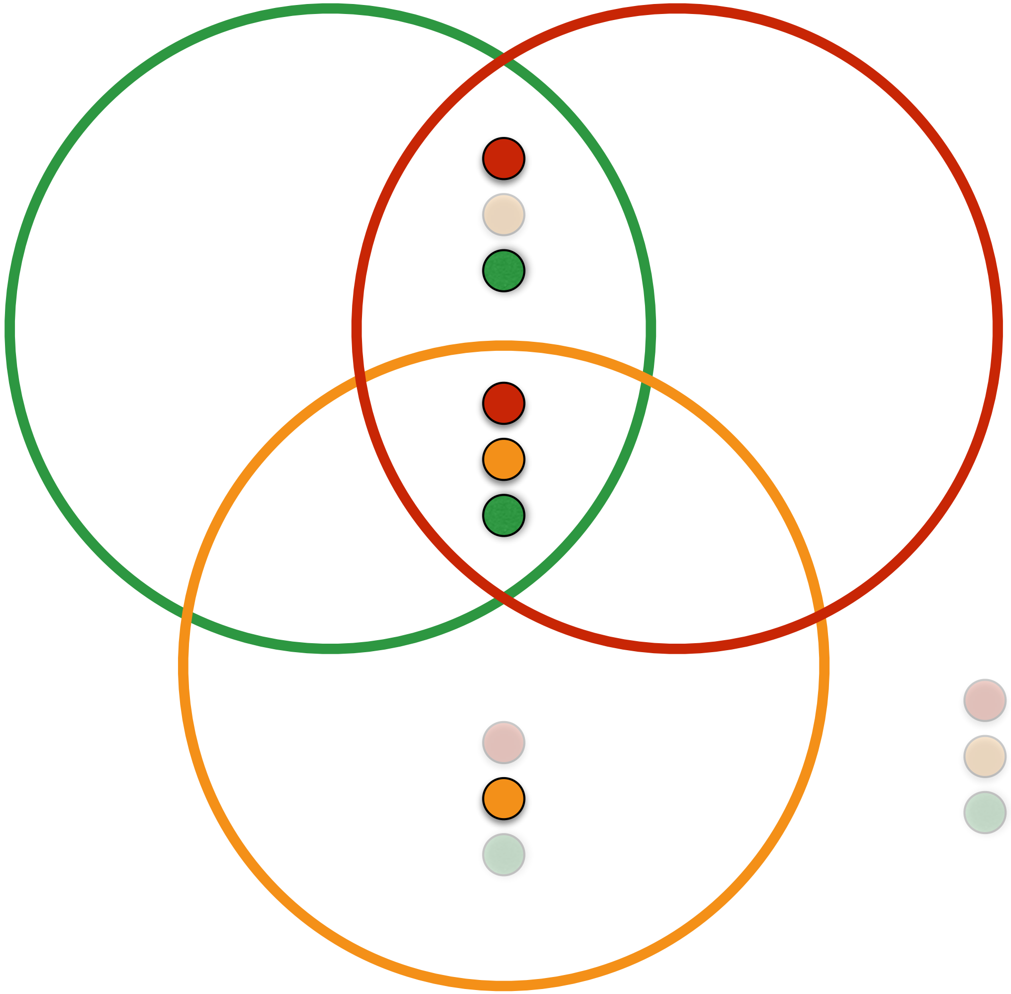
$$\{x \mid G(x) \oplus R(x) \oplus A(x)\}$$

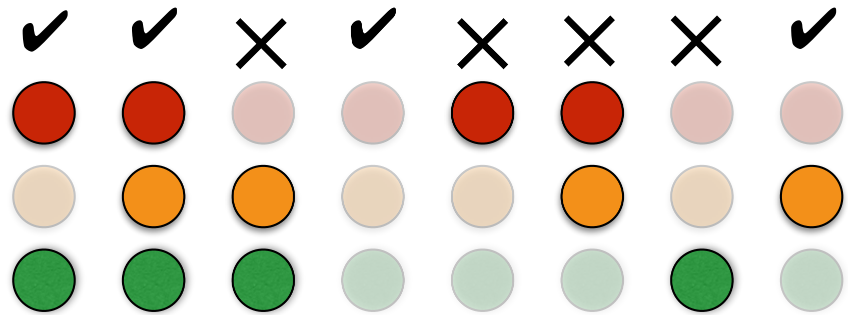


R	A	G	**R ⊕ A**	**R ⊕ A ⊕ G**
0	0	0	**0**	**0**
0	0	1	**0**	**1**
0	1	0	**1**	**1**
0	1	1	**1**	**0**
1	0	0	**1**	**1**
1	0	1	**1**	**0**
1	1	0	**0**	**0**
1	1	1	**0**	**1**

iff

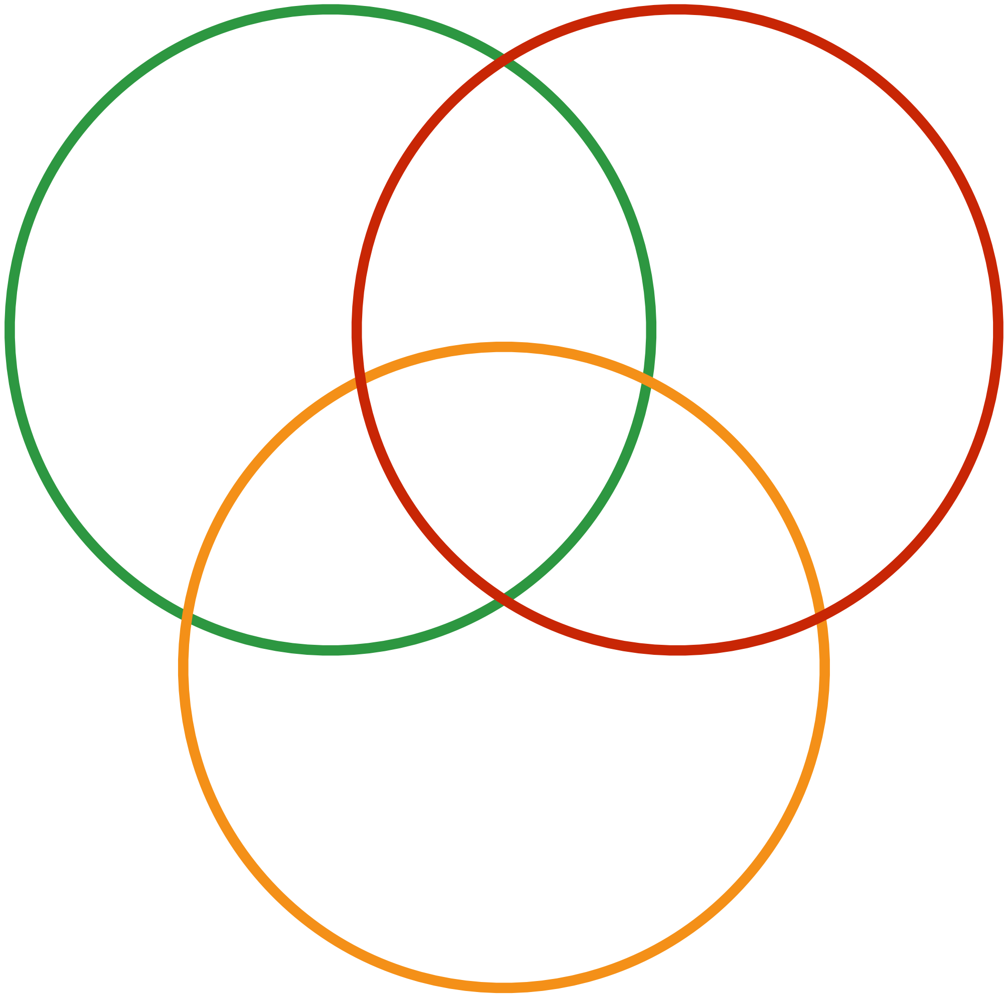
$$\{x \mid G(x) \leftrightarrow R(x)\}$$

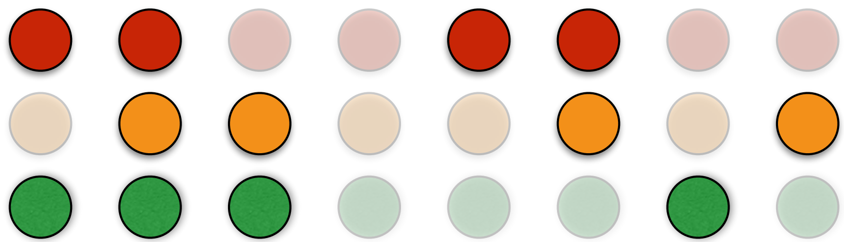




iff

$$\{x \mid G(x) \leftrightarrow (R(x) \leftrightarrow A(x))\}$$

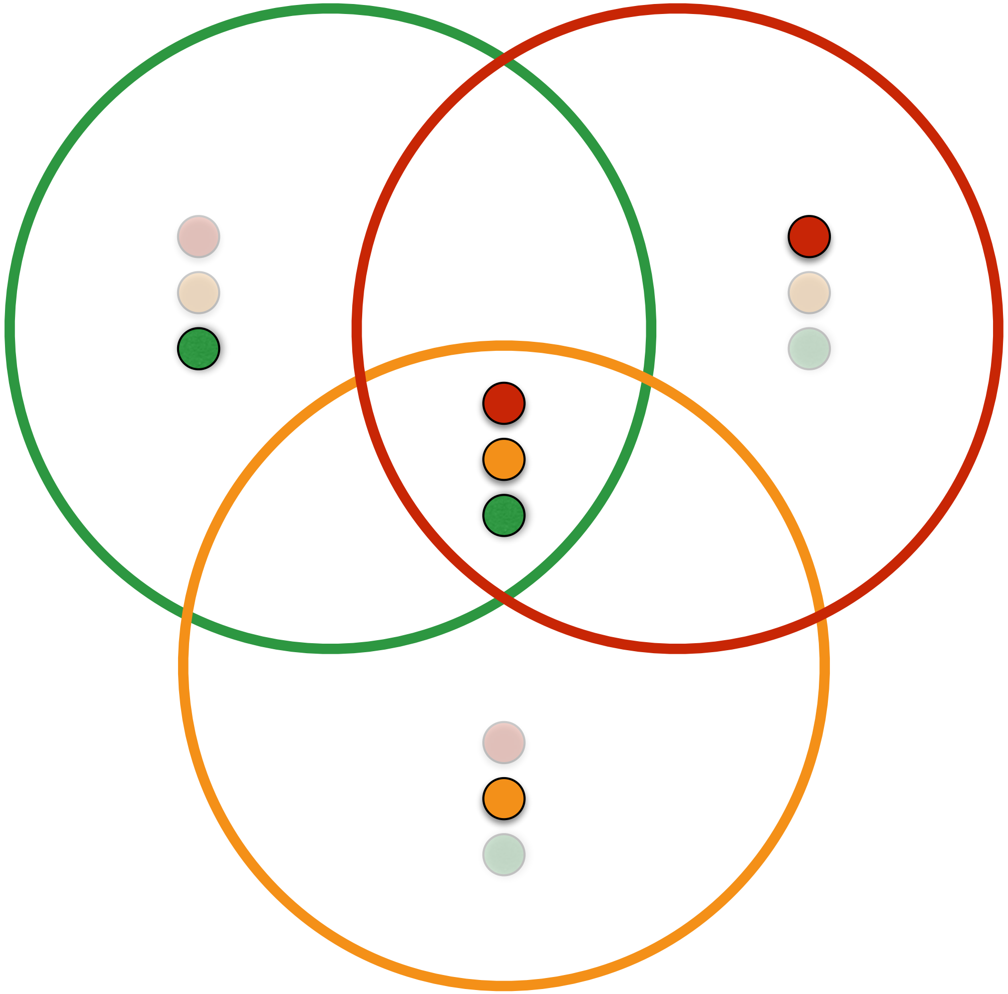


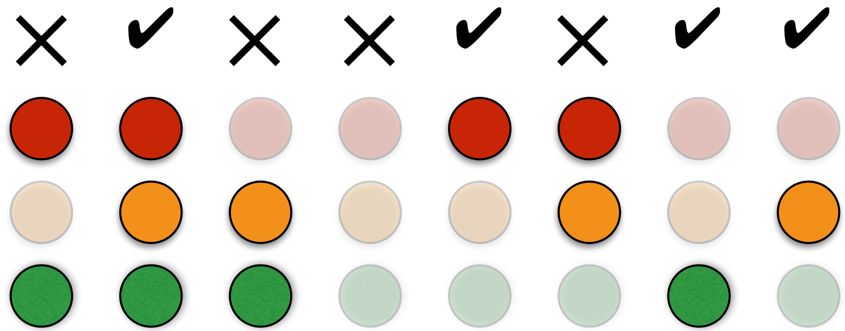


??

iff

$$\{x \mid G(x) \leftrightarrow R(x) \leftrightarrow A(x)\}$$





$$G(x) \leftrightarrow R(x) \leftrightarrow A(x)$$

$$\equiv$$

$$G(x) \oplus R(x) \oplus A(x)$$