1.	$A \to B$	Premise
2.	$\sim B$	Premise
3.	$\sim A$	Modus tollens (1,2)
4.	$\sim A \to (C \wedge D)$	Premise
5.	$C \wedge D$	Modus ponens (3,4)
6.	C	Decomposing a conjunction (5)

1.	$P \wedge Q$	Premise
2.	P	Decomposing a conjunction (1)
3.	Q	Decomposing a conjunction (1)
4.	$P \to \sim (Q \wedge R)$	Premise
5.	$\sim (Q \wedge R)$	Modus ponens (3,4)
6.	$\sim Q \vee \sim R$	DeMorgan (5)
7.	$\sim R$	Disjunctive syllogism (3,6)
8.	$S \to R$	Premise
9.	$\sim S$	Modus tollens (7,8) $\square$

Lecture 17:Inference

Michael Fourman

The 9 Elementary Valid Arg't Forms

1. Modus Ponens (MP)
P → Q
P
Q

2. Modus Tollens (MT)
P → Q
~ Q
~ P

3. Hypothetical Syllogism (HS)
P → Q
Q → R
P → R

4. Disjunctive Syllogism (DS)
P v Q
~ P
Q

5. Constructive Dilemma (CD)
(P → Q) & (R → S)
P v R
Q v S

6. Absorption (Abs)
P → Q
P → (P & Q)

7. Simplification (Simp)
P & Q
P

8. Conjunction (Conj)
P
Q
P & Q

9. Addition (Add)
P
P v Q

10 Logically Equivalent Expressions

10. De Morgan's Theorums (DeM)
~ (P & Q) ≡ (~ P v ~ Q)
~ (P v Q) ≡ (~ P & ~ Q)

11. Commutation (Com)
(P v Q) ≡ (Q v P)
(P & Q) ≡ (Q & P)

12. Association (Assoc)
[P v (Q v R)] ≡ [(P v Q) v R]
[P & (Q & R)] ≡ [(P & Q) & R]

13. Distribution (Dist)
[P & (Q v R)] ≡ [(P & Q) v (P & R)]
[P v (Q & R)] ≡ [(P v Q) & (P v R)]

14. Double Negation (DN)
~ ~ P ≡ P

15. Transposition (Trans)
(P → Q) ≡ (~ Q → ~ P)

16. Material Implication (Impl)
(P → Q) ≡ (~ P v Q)

17. Material Equivalence (Equiv)
(P ≡ Q) ≡ [(P → Q) & (Q → P)]
(P ≡ Q) ≡ [(P & Q) v (~ P & ~ Q)]

18. Exportation (Exp)
[(P & Q) → R] ≡ [P → (Q → R)]

19. Tautology (Taut)
P ≡ (P v P)
P ≡ (P & P)

https://www.youtube.com/watch?v=Lvcnx6-0GhA

An argument is

a connected series of statements to establish a proposition.

Is this a valid argument?

- Assumptions:

  If the races are fixed or the gambling houses are crooked, then the tourist trade will decline.
  If the tourist trade declines then the police force will be happy.
  The police force is never happy.

- Conclusion:

  The races are not fixed

The argument is ***valid*** *iff*
if the assumptions are all true then the conclusion is true

RF	the Races are Fixed
GC	the Gambling houses are Crooked
TT	the Tourist Trade will decline
PH	the Police force will be Happy

Assumptions:

- If the races are fixed or the gambling houses are crooked then the tourist trade will decline. $(\mathrm{RF} \vee \mathrm{GC}) \rightarrow \mathrm{TT}$
- If the tourist trade declines then the police force will be happy. $\mathrm{TT} \rightarrow \mathrm{PH}$
- The police force is never happy. $\neg\mathrm{PH}$

Conclusion:

- The races are not fixed. $\neg\mathrm{RF}$

The argument is valid iff the following entailment is valid:

$$(\mathrm{RF} \vee \mathrm{GC}) \rightarrow \mathrm{TT},\ \mathrm{TT} \rightarrow \mathrm{PH},\ \neg\mathrm{PH} \models \neg\mathrm{RF}$$

We could check the validity of the entailment by checking all sixteen assignments of truth values to the four basic propositions.

Can we do do less work?

Consider our example

$$(\mathrm{RF} \vee \mathrm{GC}) \rightarrow \mathrm{TT},\ \mathrm{TT} \rightarrow \mathrm{PH},\ \neg\mathrm{PH} \models \neg\mathrm{RF}$$

Remember that an entailment is valid unless there is a counterexample.

A counterexample is an assignment of truth values that makes everything on the left true, and everything on the right false.

A counterexample is an assignment of truth values that makes everything on the left true, and everything on the right false.

The basic idea:

for each entailment $\Gamma \models \Delta$ show that if there is a counterexample to this entailment then there is a counterexample to some simpler entailment.

Consider:

$$(\mathrm{RF} \vee \mathrm{GC}) \rightarrow \mathrm{TT},\ \mathrm{TT} \rightarrow \mathrm{PH},\ \neg\mathrm{PH} \models \neg\mathrm{RF} \tag{1}$$

$$(\mathrm{RF} \vee \mathrm{GC}) \rightarrow \mathrm{TT},\ \mathrm{TT} \rightarrow \mathrm{PH},\ \mathrm{RF} \models \mathrm{PH} \tag{2}$$

Any counterexample to (1) is a counterexample to (2) (and *vice versa*).

Consider:

$$(\mathrm{RF} \vee \mathrm{GC}) \to \mathrm{TT},\ \mathrm{TT} \to \mathrm{PH},\ \neg\mathrm{PH} \models \neg\mathrm{RF} \tag{1}$$

$$(\mathrm{RF} \vee \mathrm{GC}) \to \mathrm{TT},\ \mathrm{TT} \to \mathrm{PH},\ \mathrm{RF} \models \mathrm{PH} \tag{2}$$

Any counterexample to (1) is a counterexample to (2) (and *vice versa*).

(2) is simpler - there are fewer logical operators

If (2) is valid, there is no counterexample, so (1) is also valid

We write this as a rule

$$\frac{(\mathrm{RF} \vee \mathrm{GC}) \to \mathrm{TT},\ \mathrm{TT} \to \mathrm{PH},\ \mathrm{RF} \models \mathrm{PH}}{(\mathrm{RF} \vee \mathrm{GC}) \to \mathrm{TT},\ \mathrm{TT} \to \mathrm{PH},\ \neg\mathrm{PH} \models \neg\mathrm{RF}}$$

There is a counterexample to the conclusion,
iff there is a counterexample to the assumption.

Therefore ∴

If the assumption of the rule (above the line) is valid,
then the conclusion (below the line) is valid.

$$\frac{(\mathrm{RF} \vee \mathrm{GC}) \to \mathrm{TT},\ \mathrm{TT} \to \mathrm{PH},\ \mathrm{RF} \models \mathrm{PH}}{(\mathrm{RF} \vee \mathrm{GC}) \to \mathrm{TT},\ \mathrm{TT} \to \mathrm{PH},\ \neg\mathrm{PH} \models \neg\mathrm{RF}}$$

Now consider

$$(\mathrm{RF} \vee \mathrm{GC}) \to \mathrm{TT},\ \mathrm{TT} \to \mathrm{PH},\ \mathrm{RF} \models \mathrm{PH} \tag{2}$$

Any counterexample will make TT $\to$ PH true so it will

either make TT false, in which case it is a counterexample to

$$(\mathrm{RF} \vee \mathrm{GC}) \to \mathrm{TT},\ \mathrm{RF} \models \mathrm{PH}, \mathrm{TT} \tag{3}$$

or make PH true, in which case it is a counterexample to

$$(\mathrm{RF} \vee \mathrm{GC}) \to \mathrm{TT},\ \mathrm{PH},\ \mathrm{RF} \models \mathrm{PH} \tag{4}$$

(or both).

There is a counter-example to (2) iff there is a counter-example to (at least) one of (3), (4).

Now consider

$$(\mathrm{RF} \vee \mathrm{GC}) \to \mathrm{TT},\ \mathrm{TT} \to \mathrm{PH},\ \mathrm{RF} \models \mathrm{PH} \tag{2}$$

Any counterexample will make TT $\to$ PH true so it will

either make TT false, in which case it is a counterexample to

$$(\mathrm{RF} \vee \mathrm{GC}) \to \mathrm{TT},\ \mathrm{RF} \models \mathrm{PH}, \mathrm{TT} \tag{3}$$

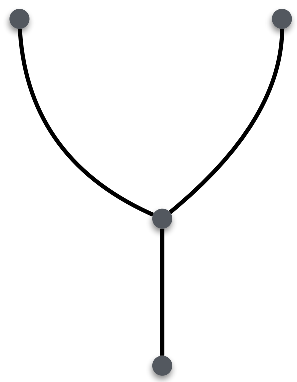
or make PH true, in which case it is a counterexample to

$$(\mathrm{RF} \vee \mathrm{GC}) \to \mathrm{TT},\ \mathrm{PH},\ \mathrm{RF} \models \mathrm{PH} \tag{4}$$

This gives a rule:

$$\frac{(\mathrm{RF} \vee \mathrm{GC}) \to \mathrm{TT},\ \mathrm{RF} \models \mathrm{PH}, \mathrm{TT} \quad (\mathrm{RF} \vee \mathrm{GC}) \to \mathrm{TT},\ \mathrm{PH},\ \mathrm{RF} \models \mathrm{PH}}{(\mathrm{RF} \vee \mathrm{GC}) \to \mathrm{TT},\ \mathrm{TT} \to \mathrm{PH},\ \mathrm{RF} \models \mathrm{PH}}$$

There is a counter-example to the conclusion iff there is a counter-example to (at least) one of the assumptions.



Putting these two rules together we start to build a *proof tree*

$$\dfrac{\dfrac{(\mathrm{RF} \vee \mathrm{GC}) \to \mathrm{TT},\ \mathrm{RF} \models \mathrm{PH}, \mathrm{TT} \quad (\mathrm{RF} \vee \mathrm{GC}) \to \mathrm{TT},\ \mathrm{PH},\ \mathrm{RF} \models \mathrm{PH}}{(\mathrm{RF} \vee \mathrm{GC}) \to \mathrm{TT},\ \mathrm{TT} \to \mathrm{PH},\ \mathrm{RF} \models \mathrm{PH}}}{(\mathrm{RF} \vee \mathrm{GC}) \to \mathrm{TT},\ \mathrm{TT} \to \mathrm{PH},\ \neg\mathrm{PH} \models \neg\mathrm{RF}}$$

If we have a counterexample to the conclusion then we have a counterexample to at least one of the assumptions

Now consider

$$(\mathrm{RF} \vee \mathrm{GC}) \rightarrow \mathrm{TT},\ \mathrm{PH},\ \mathrm{RF} \models \mathrm{PH} \tag{4}$$

Any counterexample would make PH true and make PH false, but this is impossible, so there are no counterexamples.
We draw a line over (4) to make a rule with no assumptions.

$$\frac{}{(\mathrm{RF} \vee \mathrm{GC}) \rightarrow \mathrm{TT},\ \mathrm{PH},\ \mathrm{RF} \models \mathrm{PH}}$$

We still have the key property:

- there is a counterexample to the conclusion iff there is a counterexample to (at least) one of the assumptions

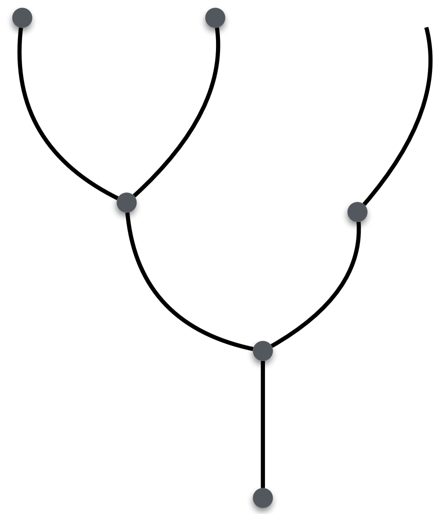
Only one assumption remains

$$\dfrac{\dfrac{(\mathrm{RF} \vee \mathrm{GC}) \to \mathrm{TT},\ \mathrm{RF} \models \mathrm{PH}, \mathrm{TT} \quad \overline{(\mathrm{RF} \vee \mathrm{GC}) \to \mathrm{TT},\ \mathrm{PH},\ \mathrm{RF} \models \mathrm{PH}}}{(\mathrm{RF} \vee \mathrm{GC}) \to \mathrm{TT},\ \mathrm{TT} \to \mathrm{PH},\ \mathrm{RF} \models \mathrm{PH}}}{(\mathrm{RF} \vee \mathrm{GC}) \to \mathrm{TT},\ \mathrm{TT} \to \mathrm{PH},\ \neg\mathrm{PH} \models \neg\mathrm{RF}}$$

If we have a counterexample to the conclusion then we have a counterexample to at least one of the assumptions.

Our next step should be familiar.

We follow a pattern used earlier

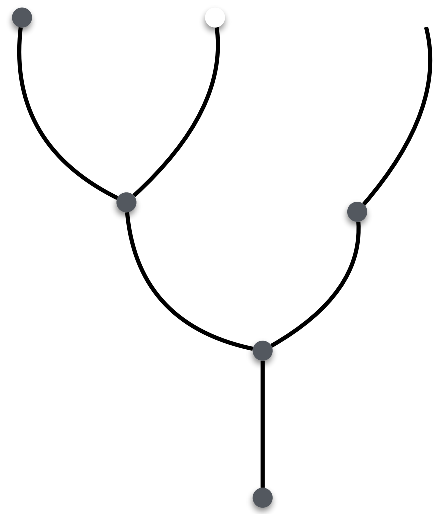


$$\frac{\Gamma \vdash A, \Delta \quad \Gamma, B \vdash \Delta}{\Gamma, A \to B \vdash \Delta} \; (\to L)$$

$$\left(\begin{array}{r} \text{with } \Gamma = \mathrm{RF},\ \ A = \mathrm{RF} \vee \mathrm{GC}, \\ B = \mathrm{TT},\ \ \Delta = \mathrm{PH},\ \mathrm{TT} \end{array} \right)$$

$$\dfrac{\dfrac{\mathrm{RF} \models \mathrm{PH}, \mathrm{TT},\ \mathrm{RF} \vee \mathrm{GC} \quad \mathrm{TT},\ \mathrm{RF} \models \mathrm{PH}, \mathrm{TT}}{(\mathrm{RF} \vee \mathrm{GC}) \to \mathrm{TT},\ \mathrm{RF} \models \mathrm{PH}, \mathrm{TT}} \qquad \dfrac{}{(\mathrm{RF} \vee \mathrm{GC}) \to \mathrm{TT},\ \mathrm{PH},\ \mathrm{RF} \models \mathrm{PH}}}{\dfrac{(\mathrm{RF} \vee \mathrm{GC}) \to \mathrm{TT},\ \mathrm{TT} \to \mathrm{PH},\ \mathrm{RF} \models \mathrm{PH}}{(\mathrm{RF} \vee \mathrm{GC}) \to \mathrm{TT},\ \mathrm{TT} \to \mathrm{PH},\ \neg\mathrm{PH} \models \neg\mathrm{RF}}}$$

Another pattern
we used earlier



$$\overline{\Gamma, A \vdash \Delta, A} \ (I)$$

$$(\text{with } A = \mathrm{TT},\ \Gamma = \mathrm{RF},\ \Delta = \mathrm{PH})$$

$$\dfrac{\dfrac{\dfrac{\mathrm{RF} \models \mathrm{PH}, \mathrm{TT},\ \mathrm{RF} \vee \mathrm{GC} \qquad \overline{\mathrm{TT},\ \mathrm{RF} \models \mathrm{PH}, \mathrm{TT}}}{(\mathrm{RF} \vee \mathrm{GC}) \to \mathrm{TT},\ \mathrm{RF} \models \mathrm{PH}, \mathrm{TT}} \qquad \overline{(\mathrm{RF} \vee \mathrm{GC}) \to \mathrm{TT},\ \mathrm{PH},\ \mathrm{RF} \models \mathrm{PH}}}{(\mathrm{RF} \vee \mathrm{GC}) \to \mathrm{TT},\ \mathrm{TT} \to \mathrm{PH},\ \mathrm{RF} \models \mathrm{PH}}}{(\mathrm{RF} \vee \mathrm{GC}) \to \mathrm{TT},\ \mathrm{TT} \to \mathrm{PH},\ \neg\mathrm{PH} \models \neg\mathrm{RF}}$$

Now consider

$$\mathrm{RF} \models \mathrm{PH}, \mathrm{TT},\ \mathrm{RF} \vee \mathrm{GC}$$

Any counterexample will make both RF and GC false, so it is a counterexample to

$$\mathrm{RF} \models \mathrm{PH}, \mathrm{TT},\ \mathrm{RF}, \mathrm{GC}$$

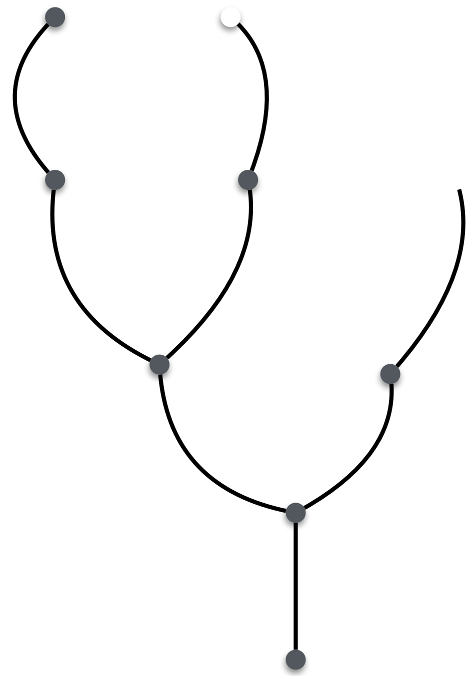
This gives a rule

$$\frac{\mathrm{RF} \models \mathrm{PH}, \mathrm{TT},\ \mathrm{RF}, \mathrm{GC}}{\mathrm{RF} \models \mathrm{PH}, \mathrm{TT},\ \mathrm{RF} \vee \mathrm{GC}}$$

A valuation is a counter-example to the conclusion iff it is a counter-example to the assumption.

The pattern for this rule is

$$\frac{\Gamma \vdash A, B, \Delta}{\Gamma \vdash A \vee B, \Delta}\ (\vee R)$$

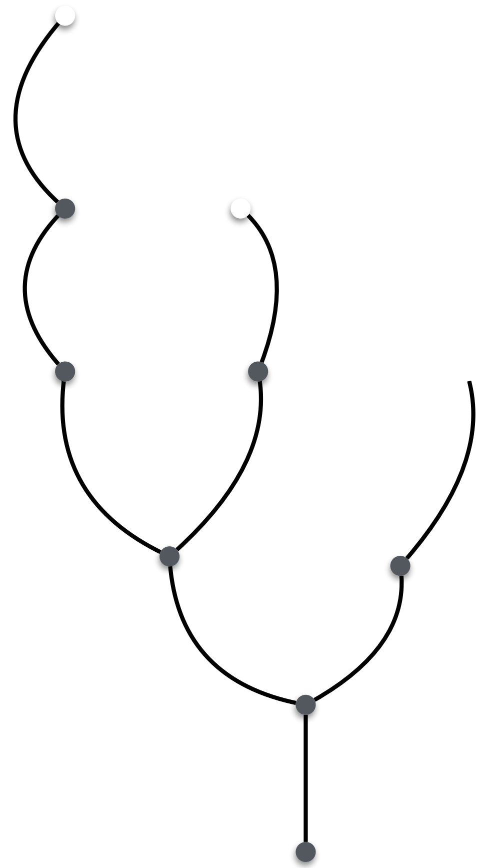


Our proof is almost done

$$\cfrac{\cfrac{\cfrac{\cfrac{\mathrm{RF} \models \mathrm{PH}, \mathrm{TT},\ \mathrm{RF},\ \mathrm{GC}}{\mathrm{RF} \models \mathrm{PH}, \mathrm{TT},\ \mathrm{RF} \vee \mathrm{GC}} \quad \cfrac{}{\mathrm{TT},\ \mathrm{RF} \models \mathrm{PH}, \mathrm{TT}}}{(\mathrm{RF} \vee \mathrm{GC}) \to \mathrm{TT},\ \mathrm{RF} \models \mathrm{PH}, \mathrm{TT}} \quad \cfrac{}{(\mathrm{RF} \vee \mathrm{GC}) \to \mathrm{TT},\ \mathrm{PH},\ \mathrm{RF} \models \mathrm{PH}}}{(\mathrm{RF} \vee \mathrm{GC}) \to \mathrm{TT},\ \mathrm{TT} \to \mathrm{PH},\ \mathrm{RF} \models \mathrm{PH}}}{(\mathrm{RF} \vee \mathrm{GC}) \to \mathrm{TT},\ \mathrm{TT} \to \mathrm{PH},\ \neg\mathrm{PH} \models \neg\mathrm{RF}}$$

Immediate!

$$\frac{}{\Gamma, A \vdash \Delta, A}\ (I)$$



$$\cfrac{\cfrac{\cfrac{\cfrac{}{\mathrm{RF} \models \mathrm{PH}, \mathrm{TT},\ \mathrm{RF},\ \mathrm{GC}}}{\mathrm{RF} \models \mathrm{PH}, \mathrm{TT},\ \mathrm{RF} \vee \mathrm{GC}} \quad \cfrac{}{\mathrm{TT},\ \mathrm{RF} \models \mathrm{PH}, \mathrm{TT}}}{(\mathrm{RF} \vee \mathrm{GC}) \to \mathrm{TT},\ \mathrm{RF} \models \mathrm{PH}, \mathrm{TT}} \qquad \cfrac{}{(\mathrm{RF} \vee \mathrm{GC}) \to \mathrm{TT},\ \mathrm{PH},\ \mathrm{RF} \models \mathrm{PH}}}{\cfrac{(\mathrm{RF} \vee \mathrm{GC}) \to \mathrm{TT},\ \mathrm{TT} \to \mathrm{PH},\ \mathrm{RF} \models \mathrm{PH}}{(\mathrm{RF} \vee \mathrm{GC}) \to \mathrm{TT},\ \mathrm{TT} \to \mathrm{PH},\ \neg\mathrm{PH} \models \neg\mathrm{RF}}}$$

1924

Gentzen's Rules (I)

1945

$$\frac{}{\Gamma, A \vdash \Delta, A}\ (I)$$

$$\frac{\Gamma, A, B \vdash \Delta}{\Gamma, A \wedge B \vdash \Delta}\ (\wedge L) \qquad \frac{\Gamma \vdash A, B, \Delta}{\Gamma \vdash A \vee B, \Delta}\ (\vee R)$$

$$\frac{\Gamma, A \vdash \Delta \quad \Gamma, B \vdash \Delta}{\Gamma, A \vee B \vdash \Delta}\ (\vee L) \qquad \frac{\Gamma \vdash A, \Delta \quad \Gamma \vdash B, \Delta}{\Gamma \vdash A \wedge B, \Delta}\ (\wedge R)$$

a **sequent,** Γ ⊢ Δ

where Γ and Δ are finite sets of expressions

is **valid** iff

whenever every expression in Γ is true

some expression in Δ is true

Gerhard Karl Erich Gentzen (November 24, 1909 – August 4, 1945)

Gentzen's Rules (I)

$$\frac{}{\Gamma, A \vdash \Delta, A}\ (I)$$

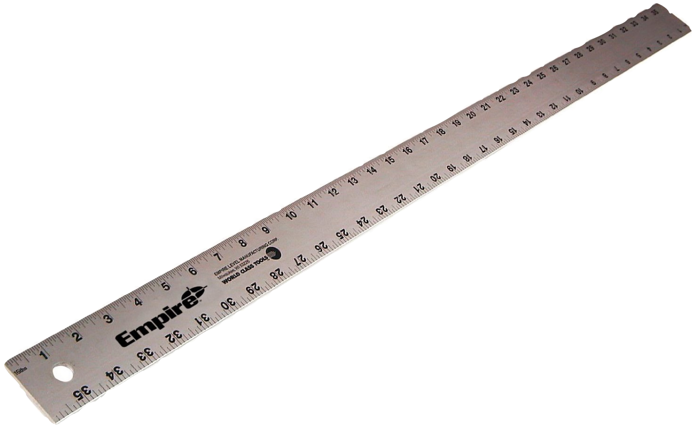
$$\frac{\Gamma, A, B \vdash \Delta}{\Gamma, A \wedge B \vdash \Delta}\ (\wedge L) \qquad \frac{\Gamma \vdash A, B, \Delta}{\Gamma \vdash A \vee B, \Delta}\ (\vee R)$$

$$\frac{\Gamma, A \vdash \Delta \quad \Gamma, B \vdash \Delta}{\Gamma, A \vee B \vdash \Delta}\ (\vee L) \qquad \frac{\Gamma \vdash A, \Delta \quad \Gamma \vdash B, \Delta}{\Gamma \vdash A \wedge B, \Delta}\ (\wedge R)$$

a **counterexample** to the **sequent** $\Gamma \vdash \Delta$,
is a valuation that makes
every expression in Γ true
and
every expression in Δ false

(a sequent is valid iff it has no counterexample)

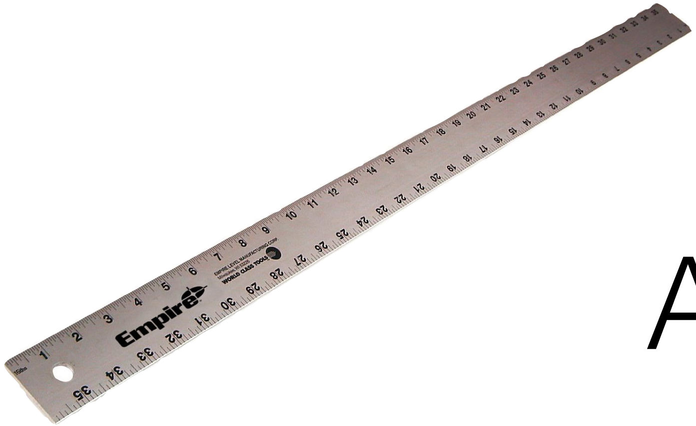
$$\dfrac{\dfrac{\dfrac{}{A, B \vdash A, B}\,(I)}{A \wedge B \vdash A, B}\,(\wedge L)}{A \wedge B \vdash A \vee B}\,(\vee R)$$



A rule

$$\frac{\Gamma, A \vdash B, \Delta}{\Gamma \vdash A \to B, \Delta}\ (\to R)$$

A valuation is a counterexample to the top line
iff it is a counterexample to the bottom line



Another rule

$$\frac{\Gamma \vdash A, \Delta \quad \Gamma, B \vdash \Delta}{\Gamma, A \to B \vdash \Delta} \; (\to L)$$

A valuation is a counterexample to the bottom line
iff it is a counterexample to
at least one of the entailments on the top line

a valuation is a counterexample to the conclusion ***iff*** *it is a counterexample to at least one assumption*

$$\frac{}{\Gamma, A \vdash \Delta, A}\ (I)$$

$$\frac{\Gamma, A, B \vdash \Delta}{\Gamma, A \wedge B \vdash \Delta}\ (\wedge L) \qquad \frac{\Gamma \vdash A, B, \Delta}{\Gamma \vdash A \vee B, \Delta}\ (\vee R)$$

$$\frac{\Gamma, A \vdash \Delta \quad \Gamma, B \vdash \Delta}{\Gamma, A \vee B \vdash \Delta}\ (\vee L) \qquad \frac{\Gamma \vdash A, \Delta \quad \Gamma \vdash B, \Delta}{\Gamma \vdash A \wedge B, \Delta}\ (\wedge R)$$

$$\frac{\Gamma \vdash A, \Delta \quad \Gamma, B \vdash \Delta}{\Gamma, A \to B \vdash \Delta}\ (\to L) \qquad \frac{\Gamma, A \vdash B, \Delta}{\Gamma \vdash A \to B, \Delta}\ (\to R)$$

$$\frac{\Gamma \vdash A, \Delta}{\Gamma, \neg A \vdash \Delta}\ (\neg L) \qquad \frac{\Gamma, A \vdash \Delta}{\Gamma \vdash \neg A, \Delta}\ (\neg R)$$

a valuation is a counterexample to the conclusion ***iff*** *it is a counterexample to at least one assumption*

$$\frac{}{\Gamma, A \vdash \Delta, A}\ (I)$$

$$\frac{\Gamma, A, B \vdash \Delta}{\Gamma, A \wedge B \vdash \Delta}\ (\wedge L) \qquad \frac{\Gamma \vdash A, B, \Delta}{\Gamma \vdash A \vee B, \Delta}\ (\vee R)$$

$$\frac{\Gamma, A \vdash \Delta \quad \Gamma, B \vdash \Delta}{\Gamma, A \vee B \vdash \Delta}\ (\vee L) \qquad \frac{\Gamma \vdash A, \Delta \quad \Gamma \vdash B, \Delta}{\Gamma \vdash A \wedge B, \Delta}\ (\wedge R)$$

$$\frac{\Gamma \vdash A, \Delta \quad \Gamma, B \vdash \Delta}{\Gamma, A \to B \vdash \Delta}\ (\to L) \qquad \frac{\Gamma, A \vdash B, \Delta}{\Gamma \vdash A \to B, \Delta}\ (\to R)$$

$$\frac{\Gamma \vdash A, \Delta}{\Gamma, \neg A \vdash \Delta}\ (\neg L) \qquad \frac{\Gamma, A \vdash \Delta}{\Gamma \vdash \neg A, \Delta}\ (\neg R)$$

$$\frac{}{\Gamma, A \vdash \Delta, A}\ (I)$$

$$\frac{\Gamma \vdash A, \Delta \quad \Gamma, B \vdash \Delta}{\Gamma, A \to B \vdash \Delta}\ (\to L) \qquad \frac{\Gamma, A \vdash B, \Delta}{\Gamma \vdash A \to B, \Delta}\ (\to R)$$

$$\frac{??}{A \to (B \to C) \vdash B \to (A \to C)}$$

$$\frac{}{\Gamma, A \vdash \Delta, A}\ (I)$$

$$\frac{\Gamma \vdash A, \Delta \quad \Gamma, B \vdash \Delta}{\Gamma, A \to B \vdash \Delta}\ (\to L) \qquad \frac{\Gamma, A \vdash B, \Delta}{\Gamma \vdash A \to B, \Delta}\ (\to R)$$

this goal
$$\frac{??}{A \to (B \to C) \vdash B \to (A \to C)}$$

$$\Gamma, \quad A \to \quad B \quad \vdash \quad \Delta$$

matches the conclusion of $(\to L)$

where

- Γ is empty
- Δ is $B \to (A \to C)$
- A is A
- B is $B \to C$

$$\frac{}{\Gamma, A \vdash \Delta, A}\ (I)$$

$$\frac{\Gamma \vdash A, \Delta \quad \Gamma, B \vdash \Delta}{\Gamma, A \to B \vdash \Delta}\ (\to L) \qquad \frac{\Gamma, A \vdash B, \Delta}{\Gamma \vdash A \to B, \Delta}\ (\to R)$$

this goal : $$\frac{??}{A \to (B \to C) \vdash B \to (A \to C)}$$

matches $\qquad \Gamma \qquad \vdash A \to \qquad B \qquad , \Delta$

which is the conclusion of $(\to R)$
where

Γ is $A \to (B \to C)$
Δ is empty
A is B
B is $A \to C$

$$\frac{}{\Gamma, A \vdash \Delta, A}\ (I)$$

$$\frac{\Gamma \vdash A, \Delta \quad \Gamma, B \vdash \Delta}{\Gamma, A \to B \vdash \Delta}\ (\to L) \qquad \frac{\Gamma, A \vdash B, \Delta}{\Gamma \vdash A \to B, \Delta}\ (\to R)$$

$$\frac{\overbrace{A \to (B \to C)}^{\Gamma}, \overbrace{B}^{A} \vdash \overbrace{A \to C}^{B}\ \overbrace{}^{, \Delta}}{A \to (B \to C) \vdash B \to (A \to C)}\ (\to R)$$

this goal
matches the conclusion of $(\to R)$
where

- Γ is $A \to (B \to C)$
- Δ is empty
- A is B
- B is $A \to C$

KEEP CALM & FOLLOW THE RULES

$$\frac{}{\Gamma, A \vdash \Delta, A}\ (I)$$

$$\frac{\Gamma \vdash A, \Delta \quad \Gamma, B \vdash \Delta}{\Gamma, A \to B \vdash \Delta}\ (\to L) \qquad \frac{\Gamma, A \vdash B, \Delta}{\Gamma \vdash A \to B, \Delta}\ (\to R)$$

$$\dfrac{\dfrac{\dfrac{??}{A \to (B \to C), B, A \vdash C}}{A \to (B \to C), B \vdash A \to C}\ (\to R)}{A \to (B \to C) \vdash B \to (A \to C)}\ (\to R)$$

$$\frac{}{\Gamma, A \vdash \Delta, A}\;(I)$$

$$\frac{\Gamma \vdash A, \Delta \quad \Gamma, B \vdash \Delta}{\Gamma, A \to B \vdash \Delta}\;(\to L) \qquad \frac{\Gamma, A \vdash B, \Delta}{\Gamma \vdash A \to B, \Delta}\;(\to R)$$

$$\dfrac{\dfrac{\dfrac{\dfrac{}{B, A \vdash A, C}\;(I) \quad \dfrac{??}{B \to C, B, A \vdash C}}{A \to (B \to C), B, A \vdash C}\;(\to L)}{A \to (B \to C), B \vdash A \to C}\;(\to R)}{A \to (B \to C) \vdash B \to (A \to C)}\;(\to R)$$

$$\frac{}{\Gamma, A \vdash \Delta, A}\ (I)$$

$$\frac{\Gamma \vdash A, \Delta \quad \Gamma, B \vdash \Delta}{\Gamma, A \to B \vdash \Delta}\ (\to L) \qquad \frac{\Gamma, A \vdash B, \Delta}{\Gamma \vdash A \to B, \Delta}\ (\to R)$$

$$\dfrac{\dfrac{\dfrac{\dfrac{}{B, A \vdash A, C}\ (I) \qquad \dfrac{\dfrac{}{B, A \vdash B, C}\ (I) \quad \dfrac{}{C, B, A \vdash C}\ (I)}{B \to C, B, A \vdash C}\ (\to L)}{A \to (B \to C), B, A \vdash C}\ (\to L)}{A \to (B \to C), B \vdash A \to C}\ (\to R)}{A \to (B \to C) \vdash B \to (A \to C)}\ (\to R)$$