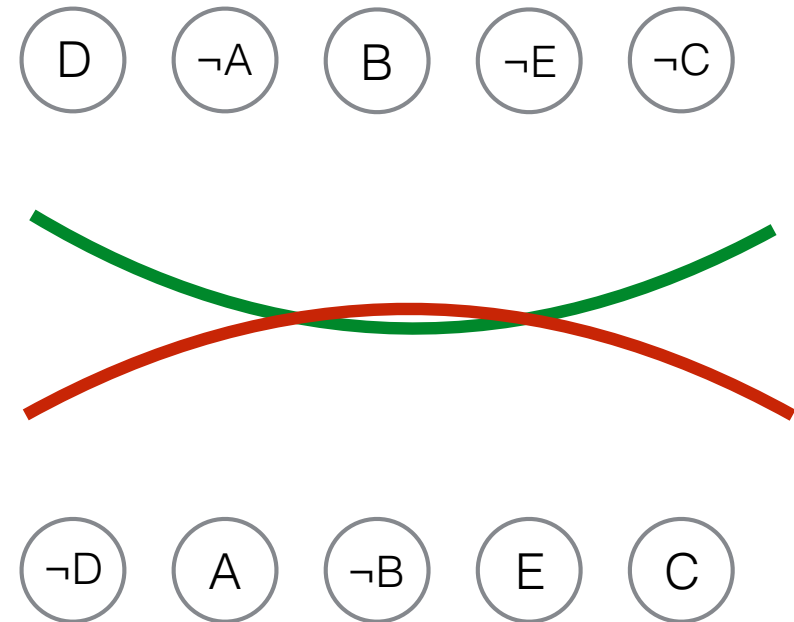


Informatics 1

Lecture 6 Satisfiability – 2-SAT

Michael Fourman

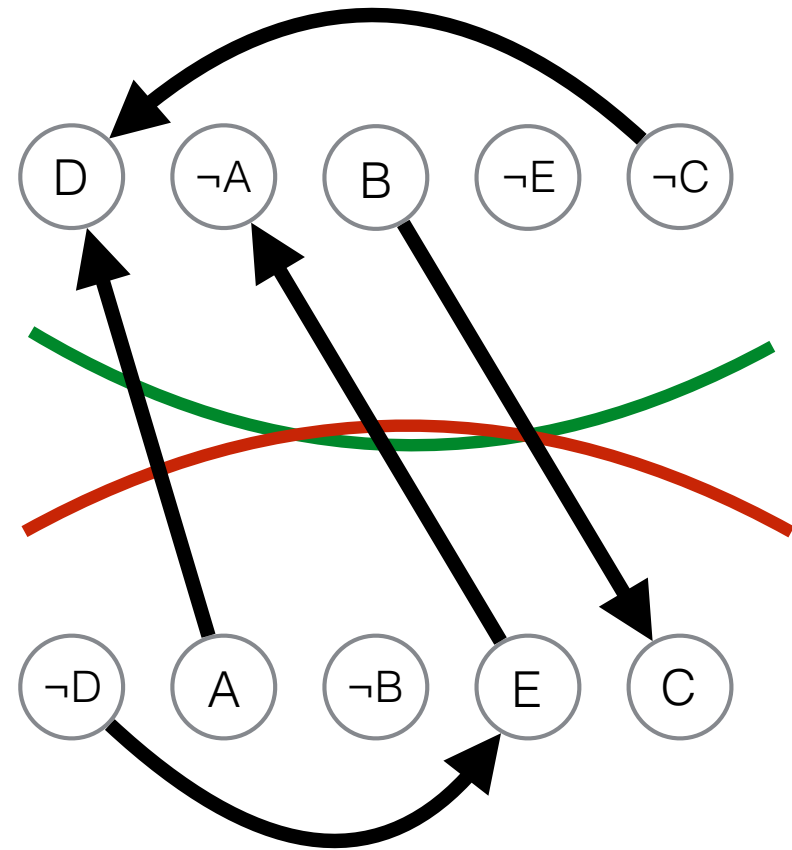
A **valuation**, or **state**, makes some atoms true and the rest false. Once we have a valuation, for each atom, we can compute the truth value of every expression. If an atom is true its negation is false, and vice versa.



We draw a line to visualise a valuation, placing the true literals above the line, and the false literals below it.

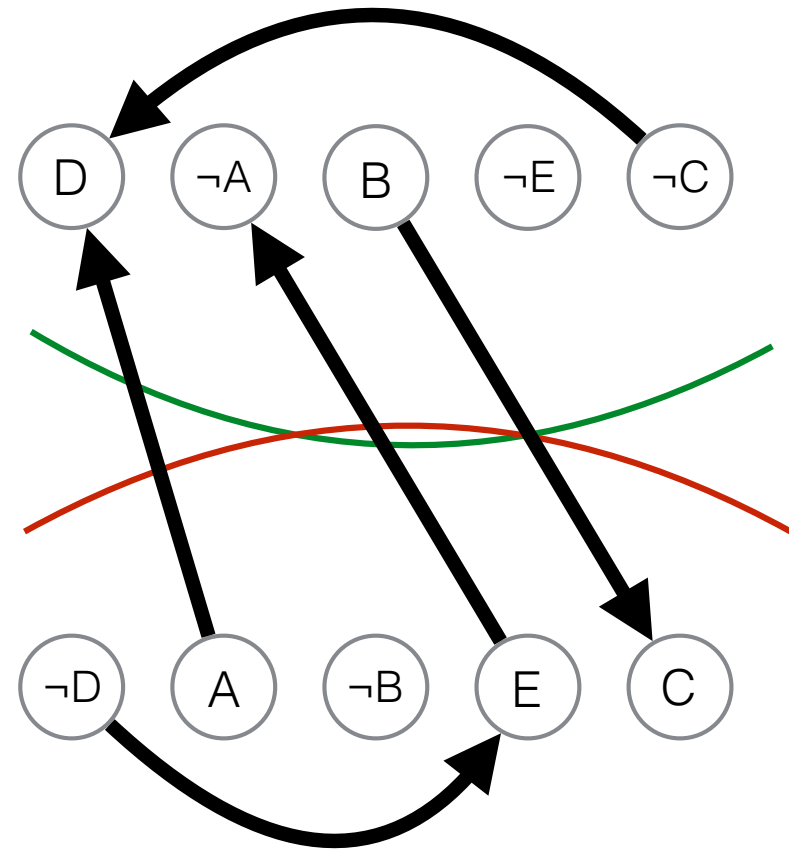
We draw a line to visualise a valuation, placing the true literals above the line, and the false literals below it.

An implication between literals is represented by an arrow.



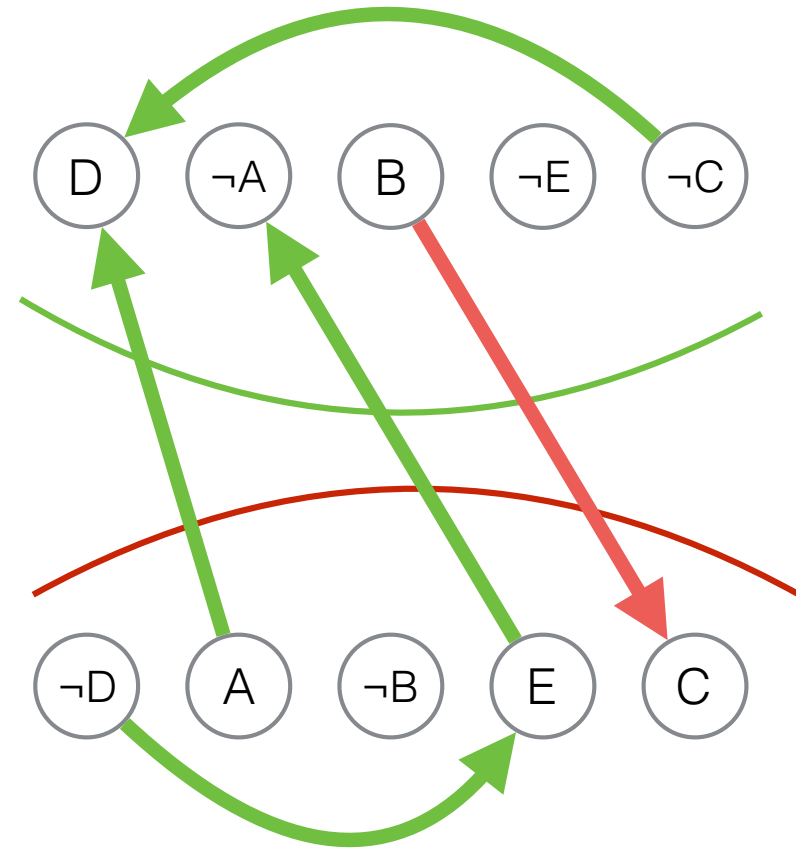
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An implication between literals is represented by an arrow.



We draw a line to visualise a valuation, placing the true literals above the line, and the false literals below it.

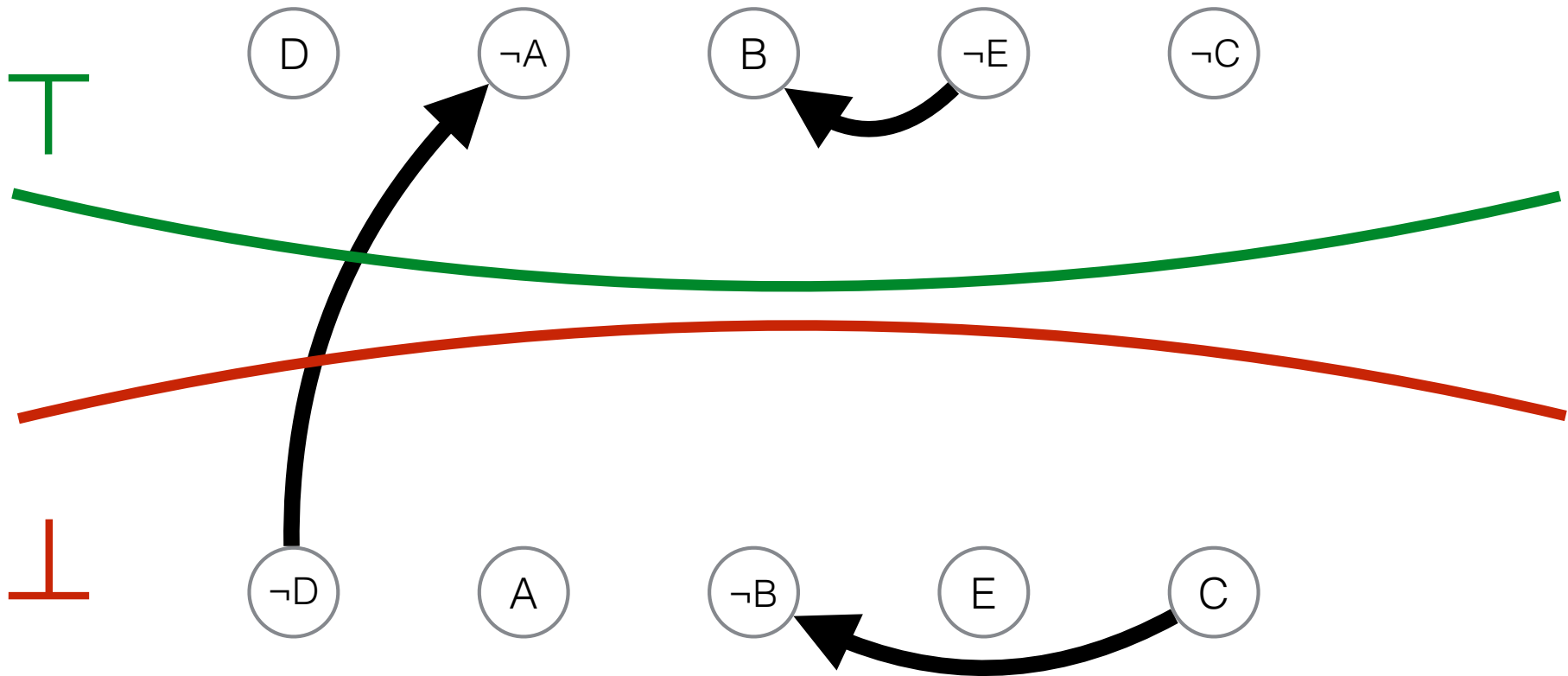
An implication between literals is represented by an arrow.



The valuation makes the implication true, unless the arrow goes from true to false.

$X \rightarrow Y$

if X is \top then Y is \top

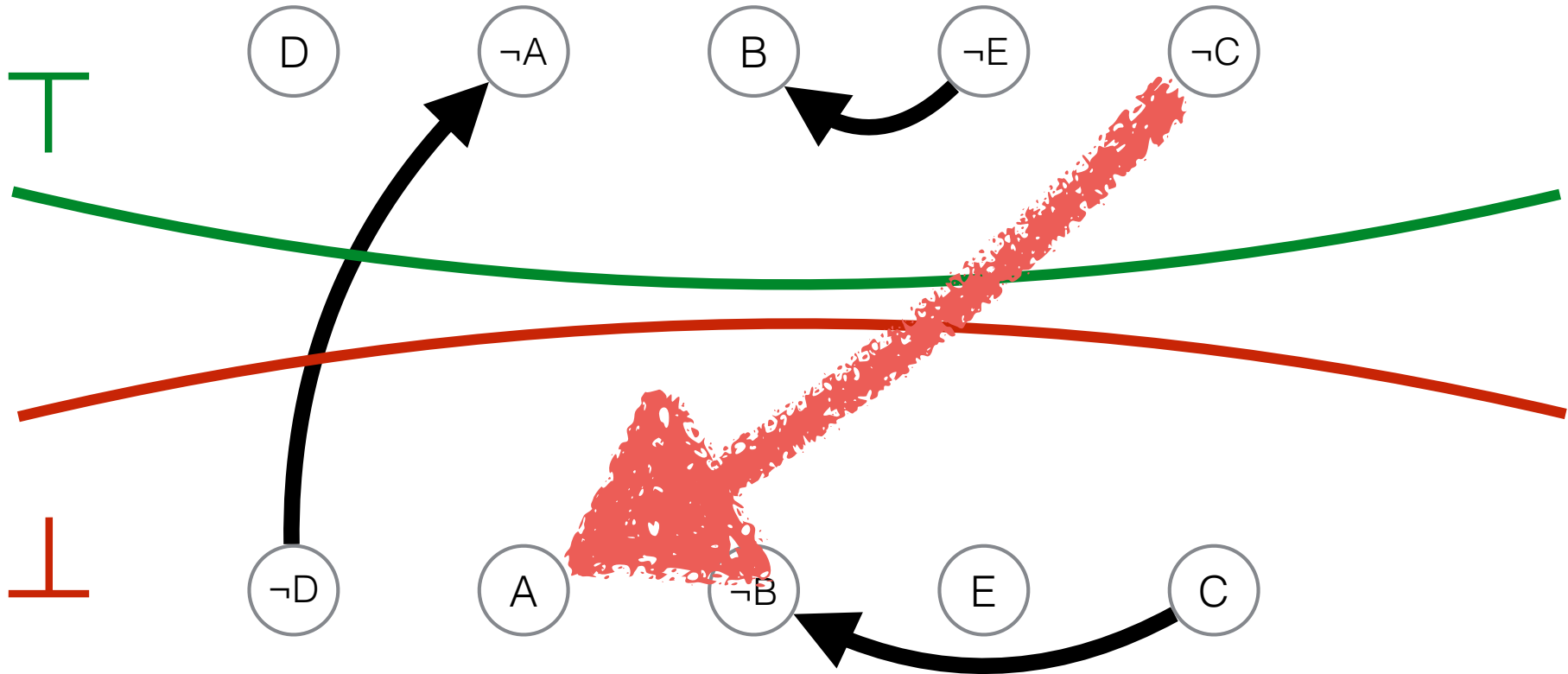


$X \rightarrow Y$

if Y is \perp then X is \perp

$X \rightarrow Y$

if X is \top then Y is \top



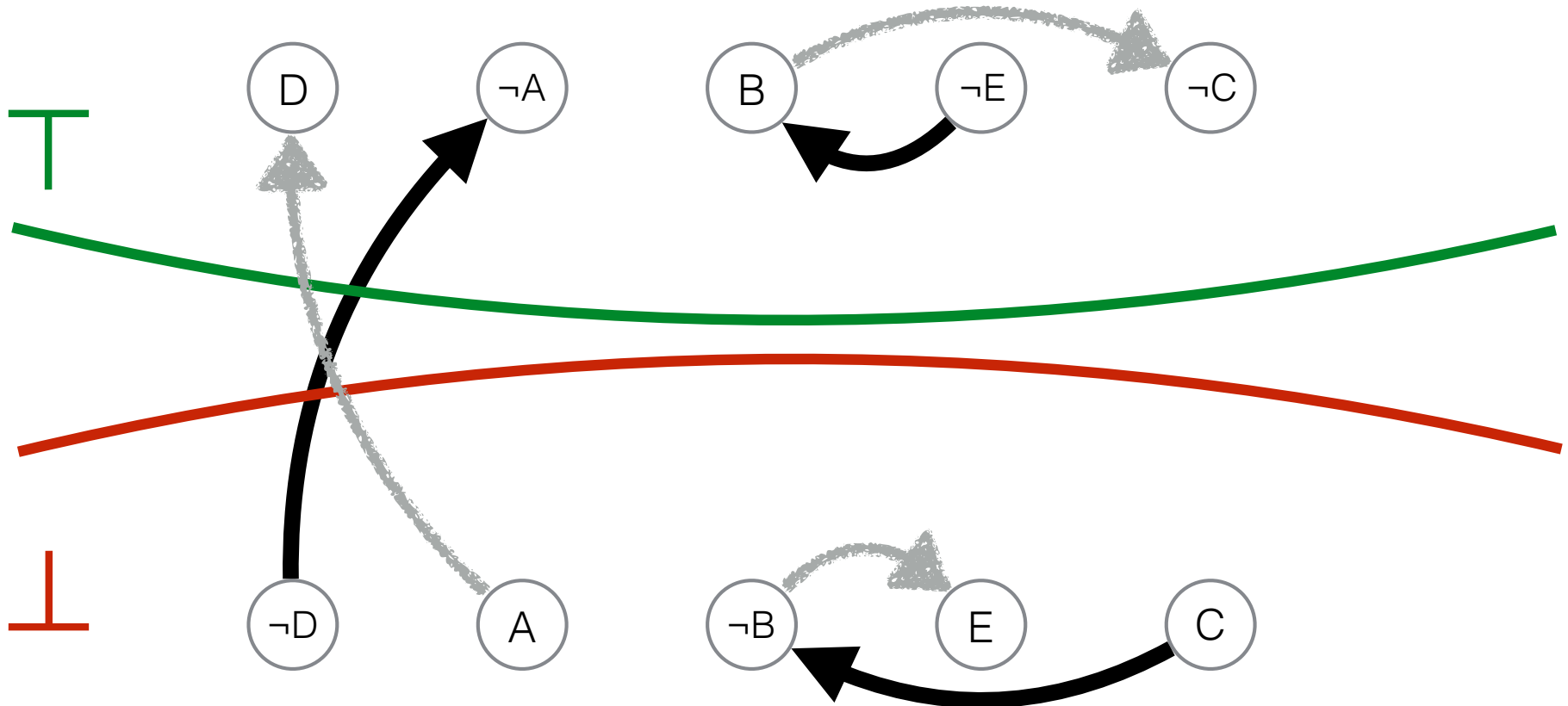
$X \rightarrow Y$

if Y is \perp then X is \perp

$$\neg X \vee Y \equiv X \rightarrow Y \equiv \neg Y \rightarrow \neg X$$

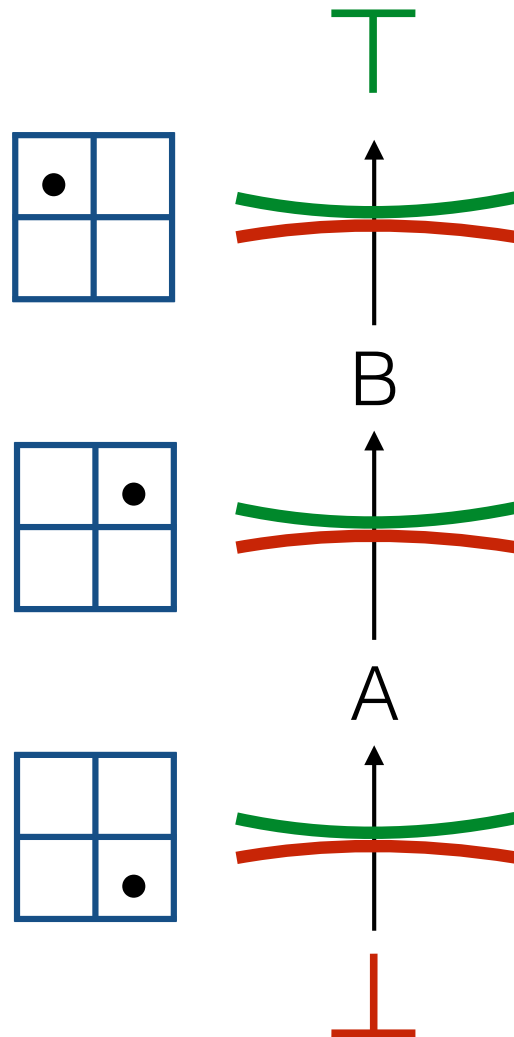
$$X \vee Y \equiv \neg X \rightarrow Y \equiv \neg Y \rightarrow X$$

$$\neg X \vee \neg Y \equiv X \rightarrow \neg Y \equiv Y \rightarrow \neg X$$



$A \rightarrow B$

| | |
|---|---|
| • | • |
| | • |



Suppose $A \rightarrow B$
 there are three possible
 truth valuations for A and B
 (we exclude only $(A = \top, B = \perp)$)

Propositions are ordered
 by $x \leq y$ iff $x \rightarrow y = \top$
 Any valid truth assignment
 must draw a line
 between \perp and \top

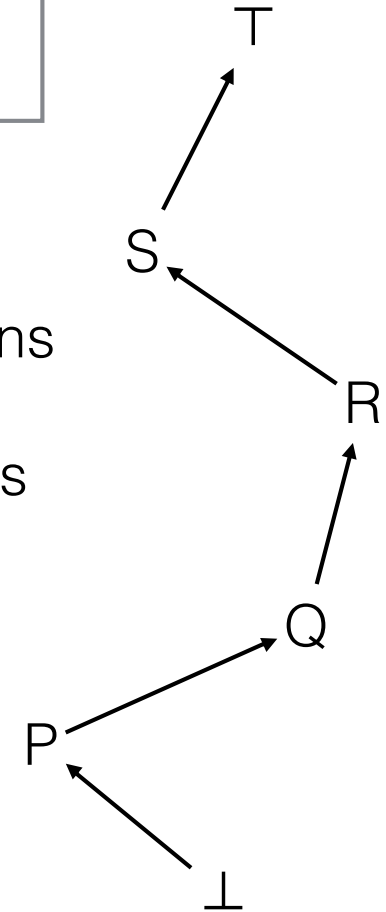
T
↑
Z
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Y
↑
X
↑
●
●
●
↑
C
↑
B
↑
A
↑
⊥

T
↑
Z
↑
Y
↑
X
↑
•
•
•
↑
C
↑
B
↑
A
↑
⊥

$$\begin{array}{c} P \rightarrow Q \\ \wedge \\ Q \rightarrow R \\ \wedge \\ R \rightarrow S \end{array}$$

$$\begin{array}{c} \neg P \vee Q \\ \wedge \\ \neg Q \vee R \\ \wedge \\ \neg R \vee S \end{array}$$

If we have a chain of $n-1$ implications between n variables we can draw the line in $n+1$ places making any number, from 0 to n , of these variables true.

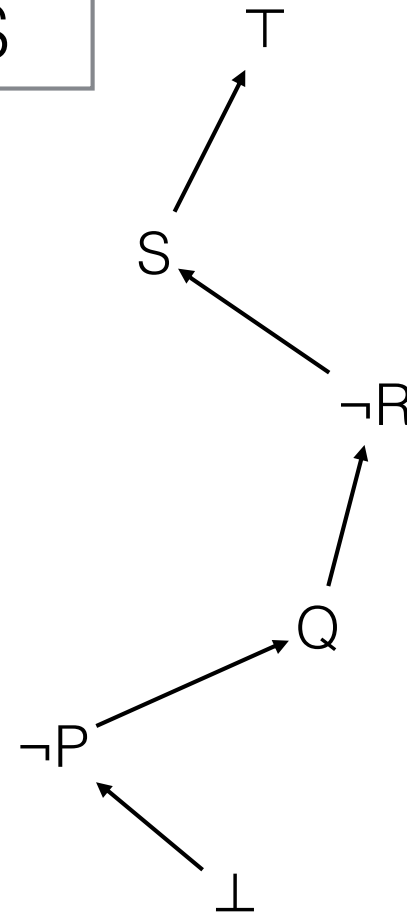


We can draw the line
before each letter, or after them all.

$$\begin{array}{c} \neg P \rightarrow Q \\ \wedge \\ Q \rightarrow \neg R \\ \wedge \\ \neg R \rightarrow S \end{array}$$

$$\begin{array}{c} P \vee Q \\ \wedge \\ \neg Q \vee \neg R \\ \wedge \\ R \vee S \end{array}$$

If some of the variables are negated we can do the same (but making the negated variables false when they fall above the line and true when they fall below)



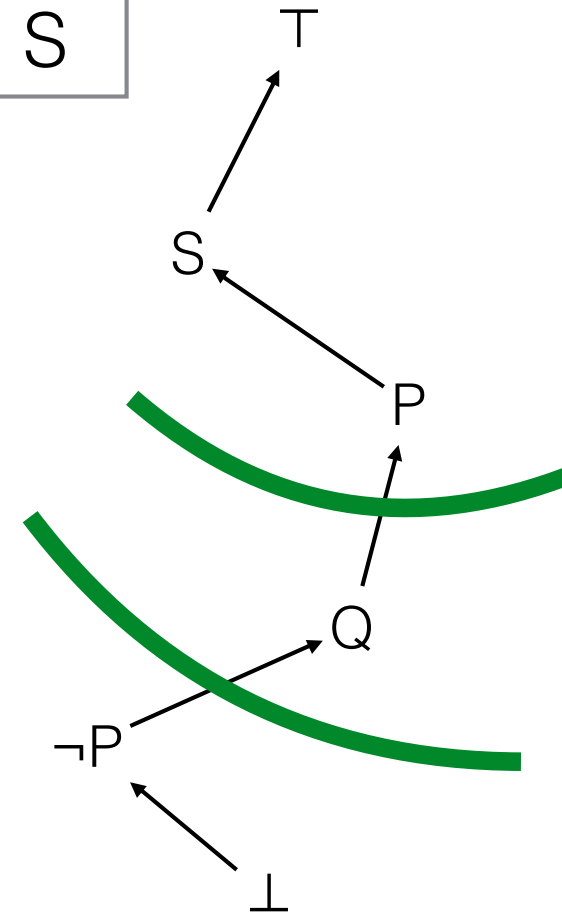
$$\begin{array}{c} \neg P \rightarrow Q \\ \wedge \\ Q \rightarrow P \\ \wedge \\ P \rightarrow S \end{array}$$

$$\begin{array}{c} P \vee Q \\ \wedge \\ \neg Q \vee P \\ \wedge \\ \neg P \vee S \end{array}$$

If a variable appears together with its negation, we have to draw the line between them.

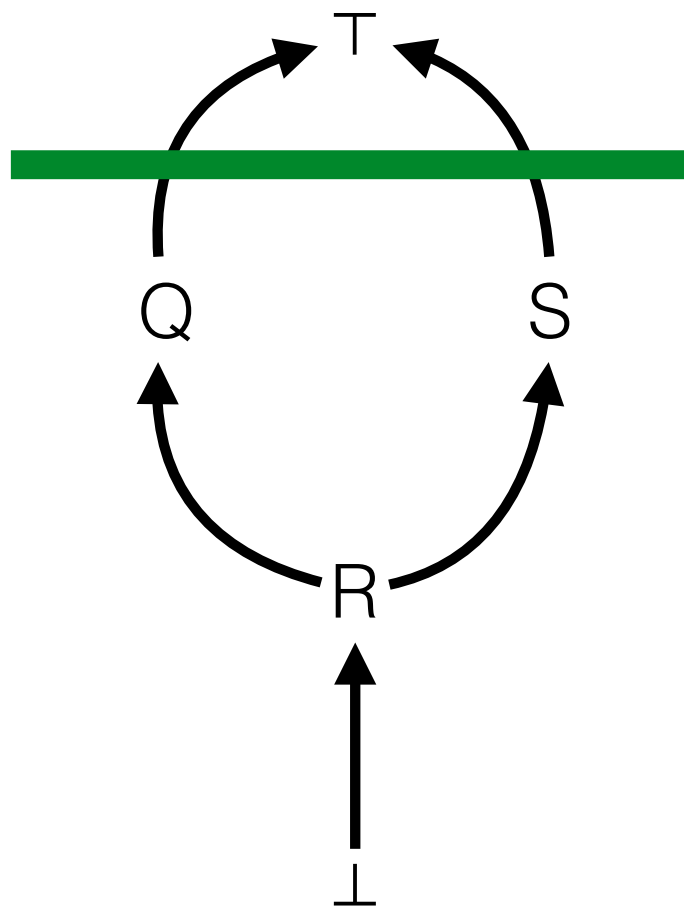
Here, P must be true.

$(\neg P \rightarrow P) \rightarrow P$
is a tautology



$$\begin{array}{c} R \rightarrow Q \\ \wedge \\ R \rightarrow S \end{array}$$

$$\begin{array}{c} \neg R \vee Q \\ \wedge \\ \neg R \vee S \end{array}$$

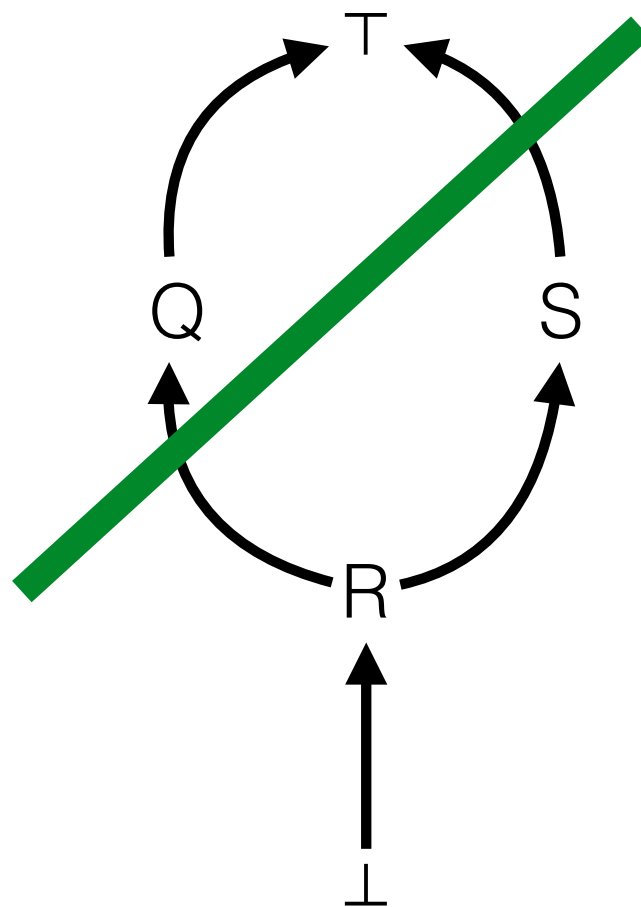
 $\neg R \quad \neg Q \quad \neg S$


$$\begin{array}{c} R \rightarrow Q \\ \wedge \\ R \rightarrow S \end{array}$$

$$\begin{array}{c} \neg R \vee Q \\ \wedge \\ \neg R \vee S \end{array}$$

 $\neg R$

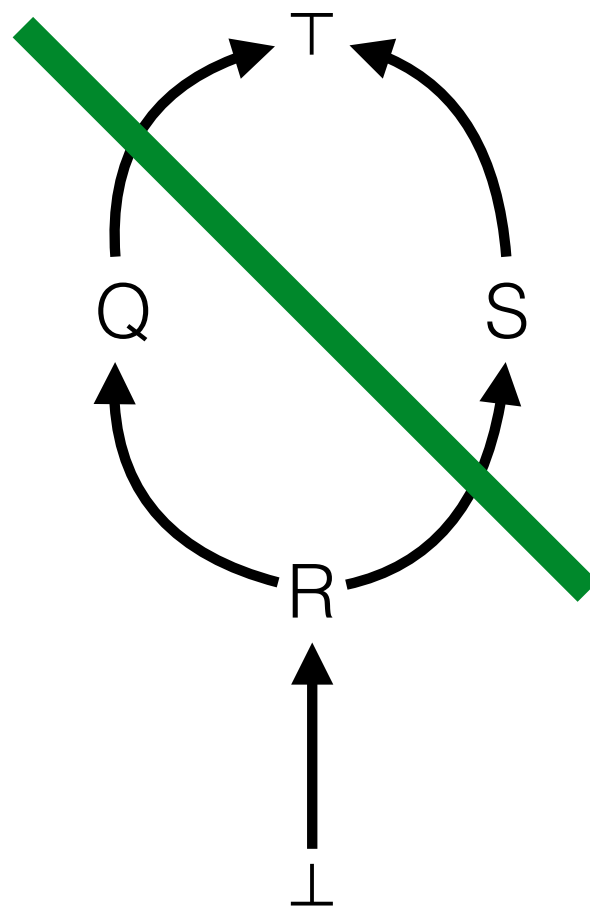
Q

 $\neg S$ 

$$\begin{array}{c} R \rightarrow Q \\ \wedge \\ R \rightarrow S \end{array}$$

$$\begin{array}{c} \neg R \vee Q \\ \wedge \\ \neg R \vee S \end{array}$$

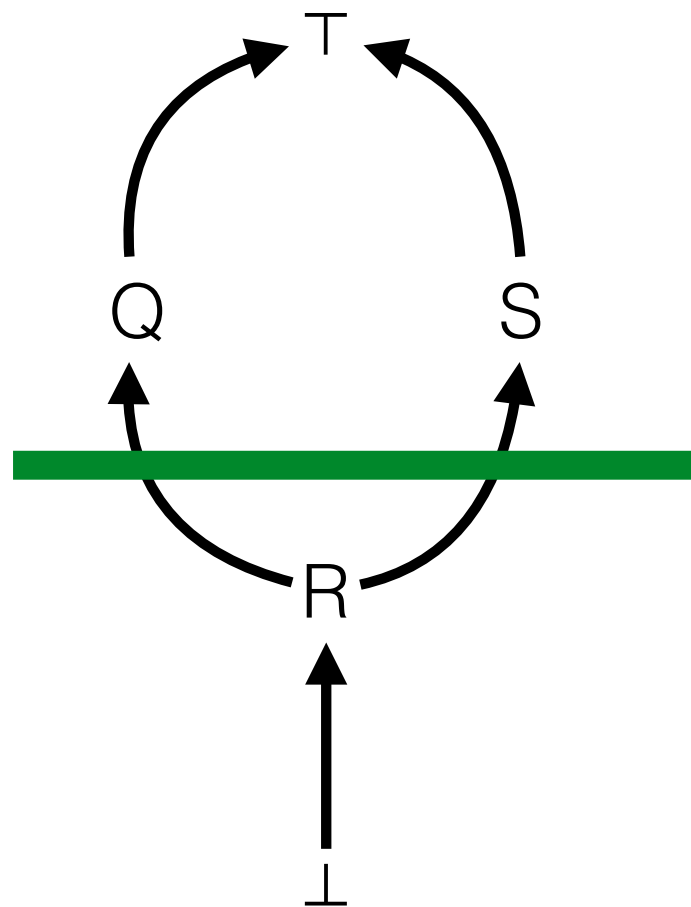
$\neg R$ $\neg Q$ S



$$\begin{array}{c} R \rightarrow Q \\ \wedge \\ R \rightarrow S \end{array}$$

$$\begin{array}{c} \neg R \vee Q \\ \wedge \\ \neg R \vee S \end{array}$$

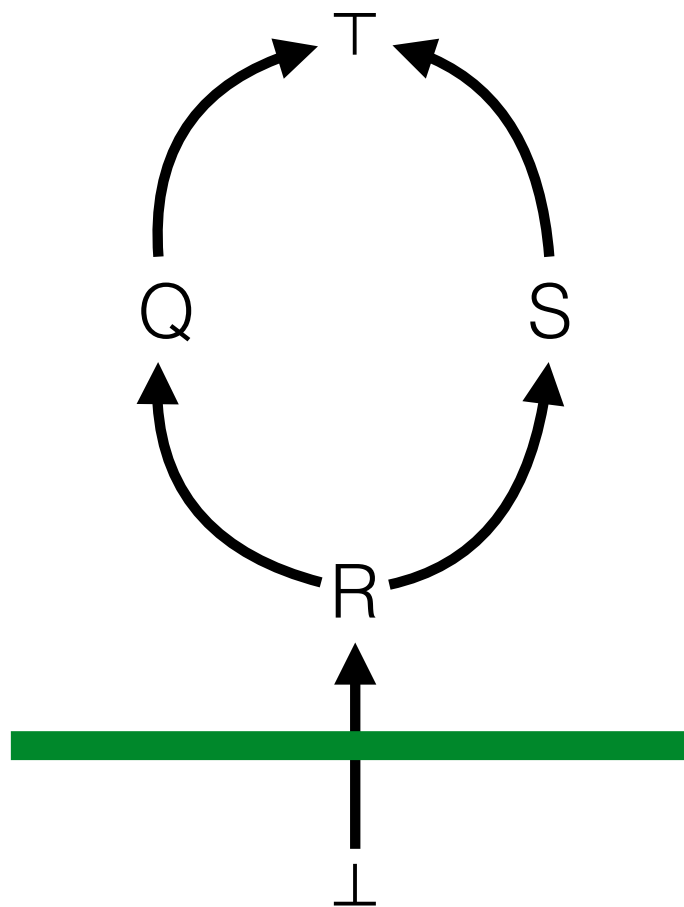
$\neg R$ Q S



$$\begin{array}{c} R \rightarrow Q \\ \wedge \\ R \rightarrow S \end{array}$$

$$\begin{array}{c} \neg R \vee Q \\ \wedge \\ \neg R \vee S \end{array}$$

R Q S



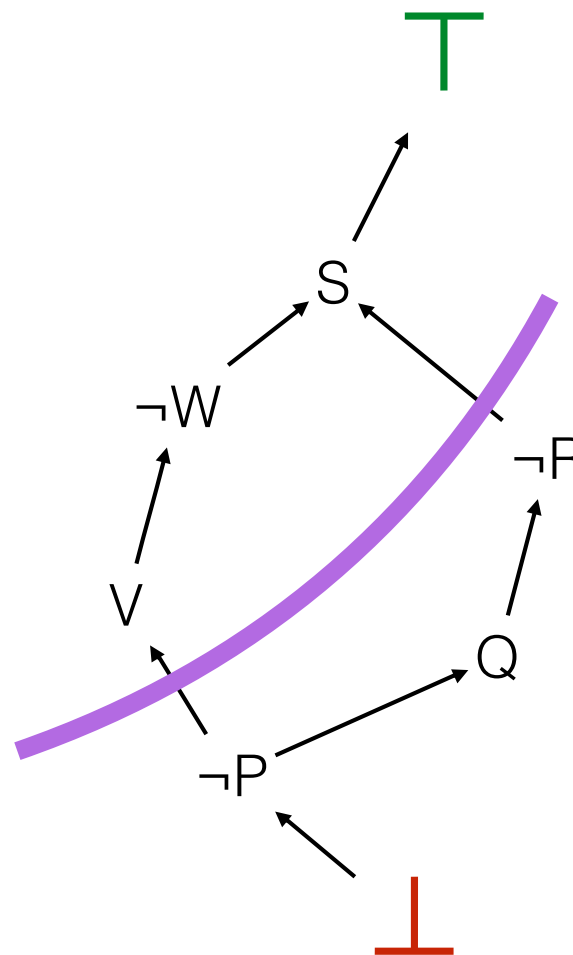
$$\begin{array}{cc}
 \neg P \rightarrow V & \neg P \rightarrow Q \\
 \wedge & \wedge \\
 V \rightarrow \neg W & Q \rightarrow \neg R \\
 \wedge & \wedge \\
 \neg W \rightarrow S & \neg R \rightarrow S
 \end{array}$$

$$\begin{array}{cc}
 P \vee V & P \vee Q \\
 \wedge & \wedge \\
 \neg V \vee \neg W & \neg Q \vee \neg R \\
 \wedge & \wedge \\
 W \vee S & R \vee S
 \end{array}$$

The same trick works if our implications form a partial order.

But we have more options since we can draw a wavy line.

The **arrow rule** says that, whenever our line cuts an arrow, then the head must be on the side of true and the tail on the side of false.

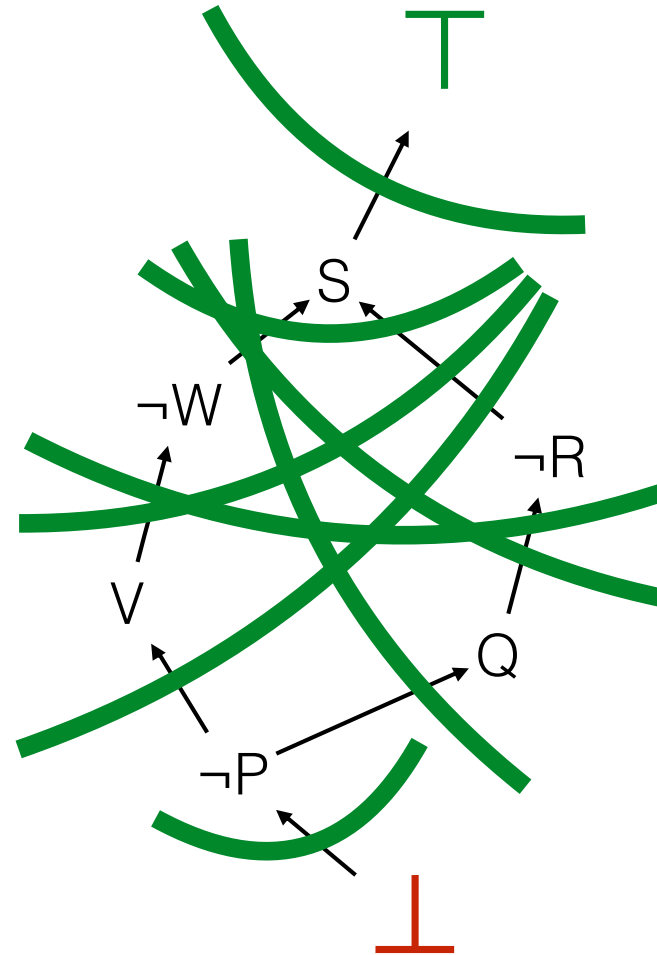


The same trick works if our implications form a partial order.

But we have more options since we can draw a wavy line.

Not all of the valid truth assignments are represented in this diagram.

How many are missing?



Binary constraints

You may not take both Archeology and Chemistry

If you take Biology you must take Chemistry

You must take Biology or Archeology

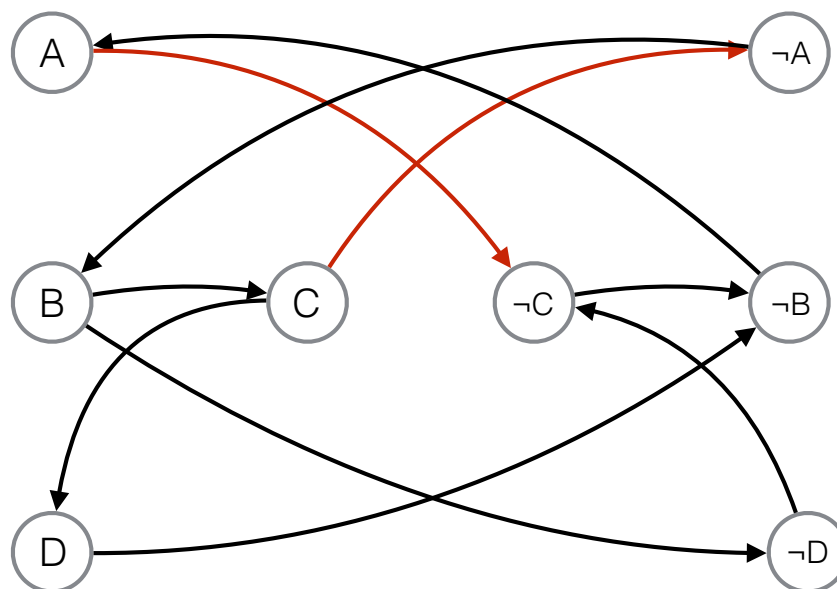
If you take Chemistry you must take Divinity

You may not take both Divinity and Biology

$$(\neg A \vee \neg C) \wedge (\neg B \vee C) \wedge (B \vee A) \wedge (\neg C \vee D) \wedge (\neg D \vee \neg B)$$

$$(A \rightarrow \neg C) \wedge (B \rightarrow C) \wedge (\neg B \rightarrow A) \wedge (C \rightarrow D) \wedge (D \rightarrow \neg B)$$

$$(\neg A \vee \neg C) \wedge (\neg B \vee C) \wedge (B \vee A) \wedge (\neg C \vee D) \wedge (\neg D \vee \neg B)$$

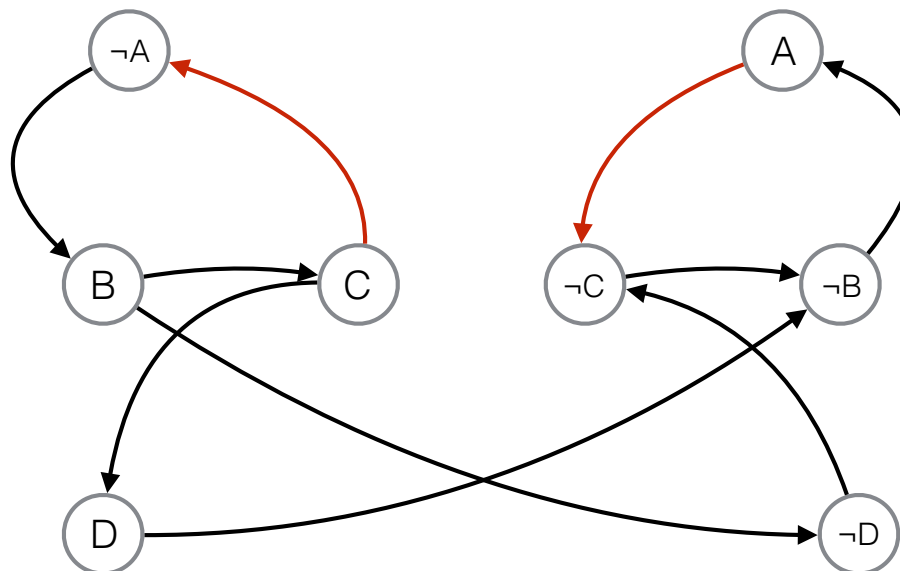


$$(\neg A \vee \neg C) \wedge (\neg B \vee C) \wedge (B \vee A) \wedge (\neg C \vee D) \wedge (\neg D \vee \neg B)$$

$$\equiv$$

$$(A \rightarrow \neg C) \wedge (B \rightarrow C) \wedge (\neg B \rightarrow A) \wedge (C \rightarrow D) \wedge (D \rightarrow \neg B)$$

$$(\neg A \vee \neg C) \wedge (\neg B \vee C) \wedge (B \vee A) \wedge (\neg C \vee D) \wedge (\neg D \vee \neg B)$$

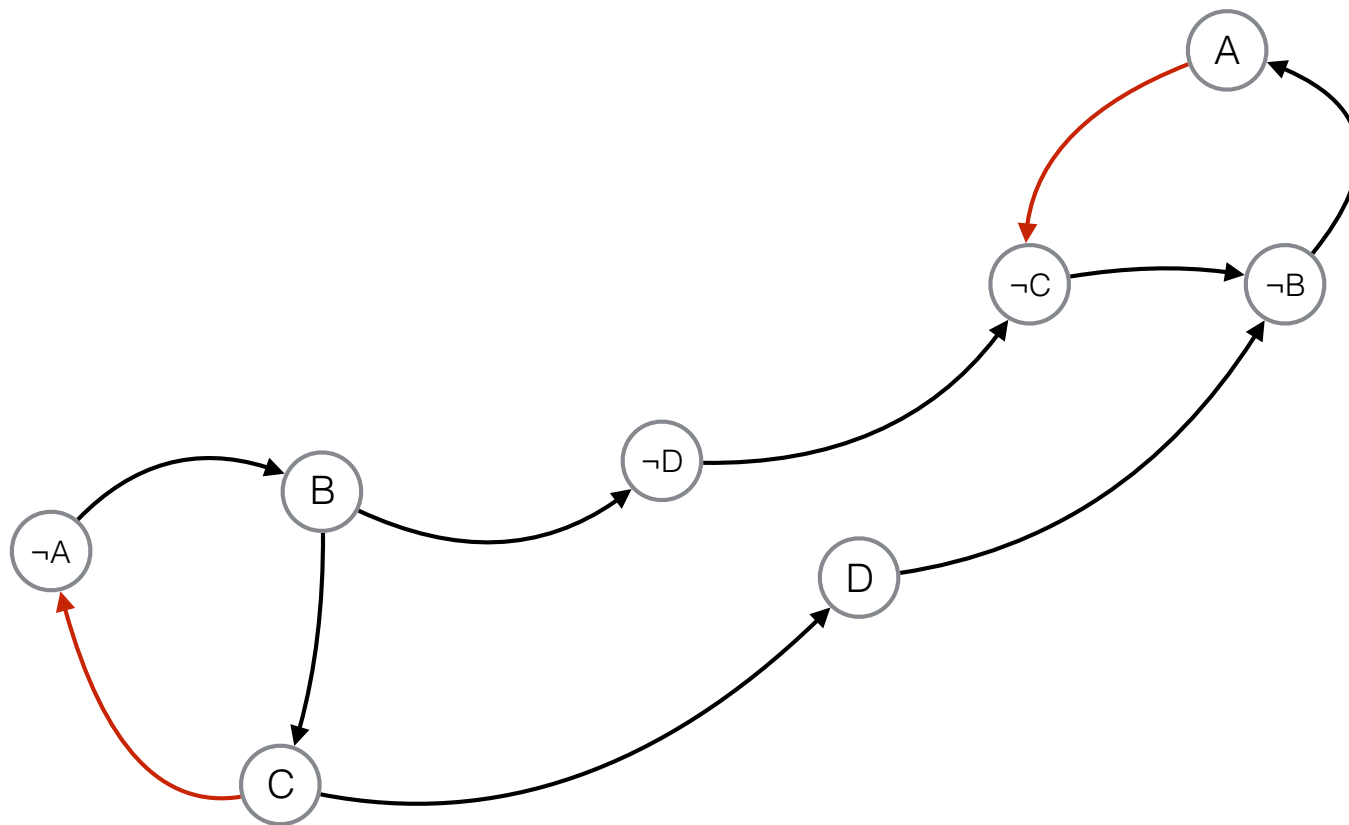


$$(\neg A \vee \neg C) \wedge (\neg B \vee C) \wedge (B \vee A) \wedge (\neg C \vee D) \wedge (\neg D \vee \neg B)$$

$$\equiv$$

$$(A \rightarrow \neg C) \wedge (B \rightarrow C) \wedge (\neg B \rightarrow A) \wedge (C \rightarrow D) \wedge (D \rightarrow \neg B)$$

$$(\neg A \vee \neg C) \wedge (\neg B \vee C) \wedge (B \vee A) \wedge (\neg C \vee D) \wedge (\neg D \vee \neg B)$$



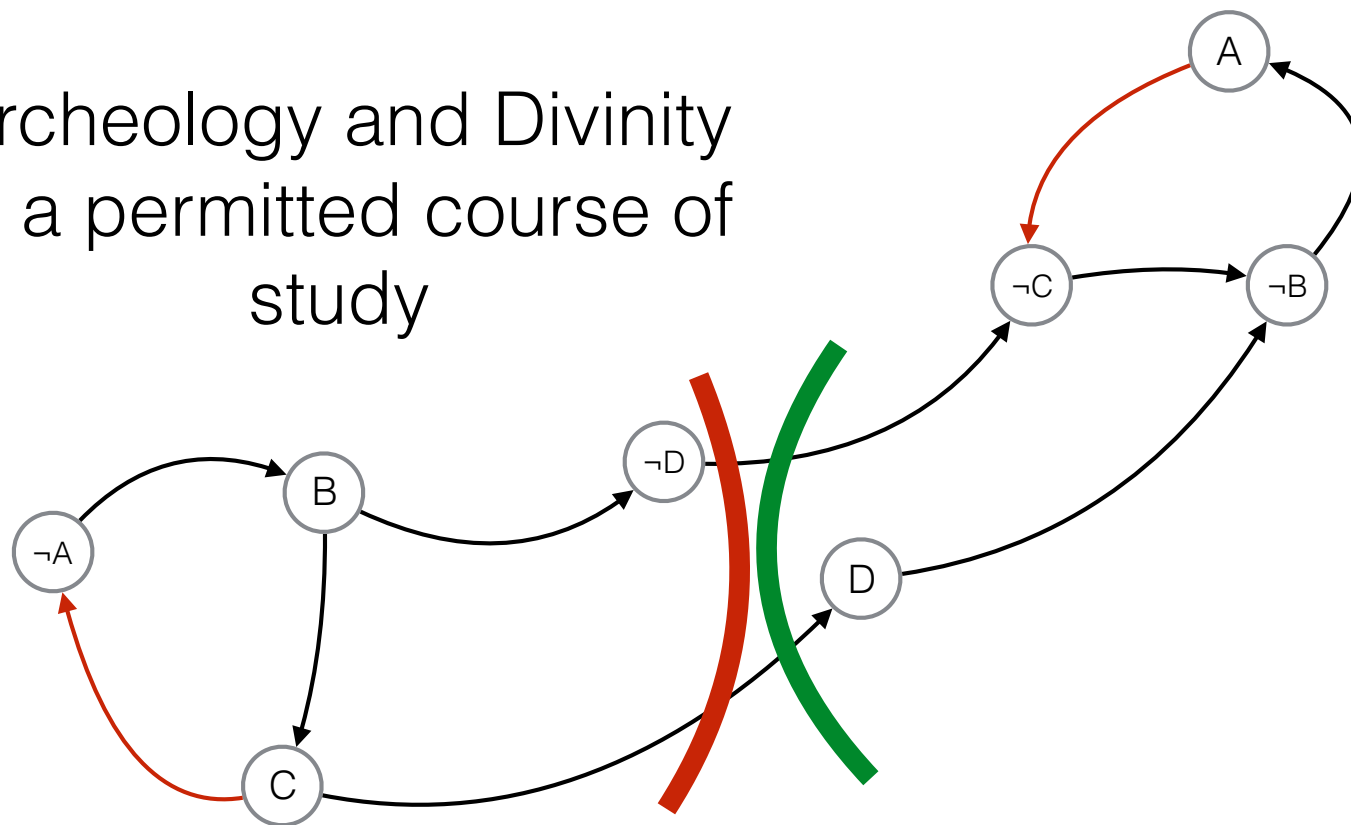
$$(\neg A \vee \neg C) \wedge (\neg B \vee C) \wedge (B \vee A) \wedge (\neg C \vee D) \wedge (\neg D \vee \neg B)$$

$$\equiv$$

$$(A \rightarrow \neg C) \wedge (B \rightarrow C) \wedge (\neg B \rightarrow A) \wedge (C \rightarrow D) \wedge (D \rightarrow \neg B)$$

$$(\neg A \vee \neg C) \wedge (\neg B \vee C) \wedge (B \vee A) \wedge (\neg C \vee D) \wedge (\neg D \vee \neg B)$$

Archeology and Divinity
is a permitted course of
study



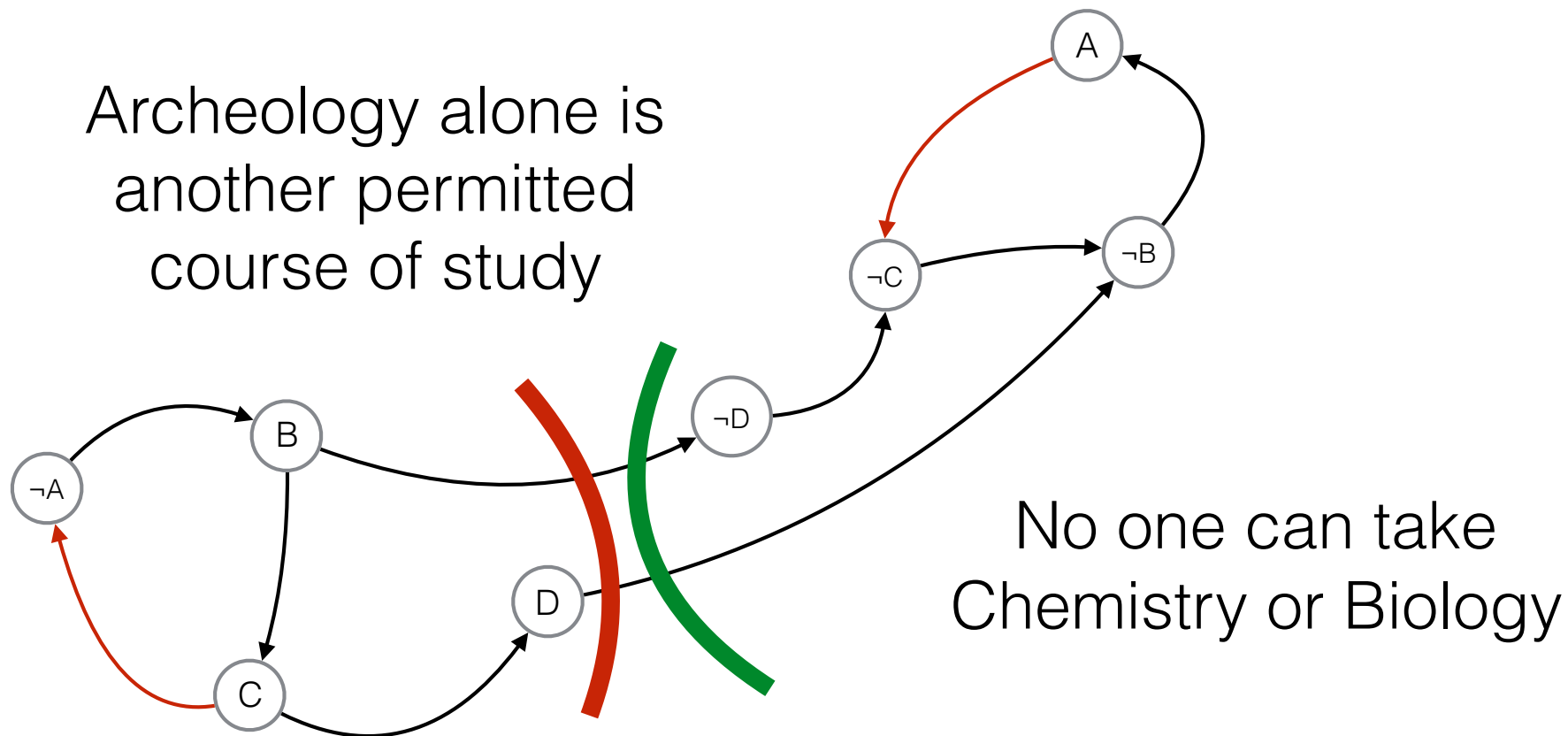
$$(\neg A \vee \neg C) \wedge (\neg B \vee C) \wedge (B \vee A) \wedge (\neg C \vee D) \wedge (\neg D \vee \neg B)$$

≡

$$(A \rightarrow \neg C) \wedge (B \rightarrow C) \wedge (\neg B \rightarrow A) \wedge (C \rightarrow D) \wedge (D \rightarrow \neg B)$$

$$(\neg A \vee \neg C) \wedge (\neg B \vee C) \wedge (B \vee A) \wedge (\neg C \vee D) \wedge (\neg D \vee \neg B)$$

Archeology alone is
another permitted
course of study



No one can take
Chemistry or Biology

$$(\neg A \vee \neg C) \wedge (\neg B \vee C) \wedge (B \vee A) \wedge (\neg C \vee D) \wedge (\neg D \vee \neg B)$$

≡

$$(A \rightarrow \neg C) \wedge (B \rightarrow C) \wedge (\neg B \rightarrow A) \wedge (C \rightarrow D) \wedge (D \rightarrow \neg B)$$