Lecture18: Gentzen

a valuation is a counterexample to the conclusion ***iff*** *it is a counterexample to at least one assumption*

$$\frac{}{\Gamma, A \vdash \Delta, A}\ (I)$$

$$\frac{\Gamma, A, B \vdash \Delta}{\Gamma, A \wedge B \vdash \Delta}\ (\wedge L) \qquad \frac{\Gamma \vdash A, B, \Delta}{\Gamma \vdash A \vee B, \Delta}\ (\vee R)$$

$$\frac{\Gamma, A \vdash \Delta \quad \Gamma, B \vdash \Delta}{\Gamma, A \vee B \vdash \Delta}\ (\vee L) \qquad \frac{\Gamma \vdash A, \Delta \quad \Gamma \vdash B, \Delta}{\Gamma \vdash A \wedge B, \Delta}\ (\wedge R)$$

$$\frac{\Gamma \vdash A, \Delta \quad \Gamma, B \vdash \Delta}{\Gamma, A \to B \vdash \Delta}\ (\to L) \qquad \frac{\Gamma, A \vdash B, \Delta}{\Gamma \vdash A \to B, \Delta}\ (\to R)$$

$$\frac{\Gamma \vdash A, \Delta}{\Gamma, \neg A \vdash \Delta}\ (\neg L) \qquad \frac{\Gamma, A \vdash \Delta}{\Gamma \vdash \neg A, \Delta}\ (\neg R)$$

$$\frac{}{\Gamma, A \vdash \Delta, A}\ (I)$$

$$\frac{\Gamma \vdash A, \Delta \quad \Gamma, B \vdash \Delta}{\Gamma, A \to B \vdash \Delta}\ (\to L) \qquad \frac{\Gamma, A \vdash B, \Delta}{\Gamma \vdash A \to B, \Delta}\ (\to R)$$

$$\frac{??}{A \to (B \to C) \vdash B \to (A \to C)}$$

$$\frac{}{\Gamma, A \vdash \Delta, A}\ (I)$$

$$\frac{\Gamma \vdash A, \Delta \quad \Gamma, B \vdash \Delta}{\Gamma, A \to B \vdash \Delta}\ (\to L) \qquad \frac{\Gamma, A \vdash B, \Delta}{\Gamma \vdash A \to B, \Delta}\ (\to R)$$

this goal

$$\frac{??}{A \to (B \to C) \vdash B \to (A \to C)}$$

$$\Gamma, \quad A \to \quad B \quad \vdash \quad \Delta$$

matches the conclusion of $(\to L)$

where

Γ is empty

Δ is $B \to (A \to C)$

A is A

B is $B \to C$

$$\frac{}{\Gamma, A \vdash \Delta, A}\ (I)$$

$$\frac{\Gamma \vdash A, \Delta \quad \Gamma, B \vdash \Delta}{\Gamma, A \to B \vdash \Delta}\ (\to L) \qquad \frac{\Gamma, A \vdash B, \Delta}{\Gamma \vdash A \to B, \Delta}\ (\to R)$$

this goal : $$\frac{??}{A \to (B \to C) \vdash B \to (A \to C)}$$

matches $\Gamma \quad \vdash A \to \quad B \quad , \Delta$

which is the conclusion of $(\to R)$
where

Γ is $A \to (B \to C)$

Δ is empty

A is B

B is $A \to C$

$$\frac{}{\Gamma, A \vdash \Delta, A}\ (I)$$

$$\frac{\Gamma \vdash A, \Delta \quad \Gamma, B \vdash \Delta}{\Gamma, A \to B \vdash \Delta}\ (\to L) \qquad \frac{\Gamma, A \vdash B, \Delta}{\Gamma \vdash A \to B, \Delta}\ (\to R)$$

$$\Gamma \qquad , A \vdash \qquad B \qquad , \Delta$$

$$\frac{A \to (B \to C), B \vdash A \to C}{A \to (B \to C) \vdash B \to (A \to C)}\ (\to R)$$

this goal
matches the conclusion of $(\to R)$
where

Γ is $A \to (B \to C)$

Δ is empty

A is B

B is $A \to C$

$$\frac{}{\Gamma, A \vdash \Delta, A}\ (I)$$

$$\frac{\Gamma \vdash A, \Delta \quad \Gamma, B \vdash \Delta}{\Gamma, A \to B \vdash \Delta}\ (\to L) \qquad \frac{\Gamma, A \vdash B, \Delta}{\Gamma \vdash A \to B, \Delta}\ (\to R)$$

$$\cfrac{\cfrac{\cfrac{??}{A \to (B \to C), B, A \vdash C}}{A \to (B \to C), B \vdash A \to C}\ (\to R)}{A \to (B \to C) \vdash B \to (A \to C)}\ (\to R)$$

KEEP
CALM
&
FOLLOW
THE RULES

$$\frac{}{\Gamma, A \vdash \Delta, A}\ (I)$$

$$\frac{\Gamma \vdash A, \Delta \quad \Gamma, B \vdash \Delta}{\Gamma, A \to B \vdash \Delta}\ (\to L) \qquad \frac{\Gamma, A \vdash B, \Delta}{\Gamma \vdash A \to B, \Delta}\ (\to R)$$

$$\dfrac{\dfrac{\dfrac{\dfrac{}{B, A \vdash A, C}\ (I) \quad \dfrac{??}{B \to C, B, A \vdash C}}{A \to (B \to C), B, A \vdash C}\ (\to L)}{A \to (B \to C), B \vdash A \to C}\ (\to R)}{A \to (B \to C) \vdash B \to (A \to C)}\ (\to R)$$

$$\frac{}{\Gamma, A \vdash \Delta, A}\ (I)$$

$$\frac{\Gamma \vdash A, \Delta \quad \Gamma, B \vdash \Delta}{\Gamma, A \to B \vdash \Delta}\ (\to L) \qquad \frac{\Gamma, A \vdash B, \Delta}{\Gamma \vdash A \to B, \Delta}\ (\to R)$$

$$\cfrac{\cfrac{\cfrac{}{B, A \vdash A, C}\ (I) \quad \cfrac{\cfrac{}{B, A \vdash B, C}\ (I) \quad \cfrac{}{C, B, A \vdash C}\ (I)}{B \to C, B, A \vdash C}\ (\to L)}{\cfrac{A \to (B \to C), B, A \vdash C}{A \to (B \to C), B \vdash A \to C}\ (\to R)}\ (\to L)}{A \to (B \to C) \vdash B \to (A \to C)}\ (\to R)$$

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$$\frac{??}{A \to (B \to C) \vdash B \to (C \to A)}$$

$$\dfrac{\dfrac{??}{A \to (B \to C), B \vdash C \to A}}{A \to (B \to C) \vdash B \to (C \to A)}\ (\to R)$$

$$\dfrac{\dfrac{\dfrac{??}{A \to (B \to C), B, C \vdash A}}{A \to (B \to C), B \vdash C \to A}\ (\to R)}{A \to (B \to C) \vdash B \to (C \to A)}\ (\to R)$$

$$\dfrac{\dfrac{\dfrac{B, C \vdash A \qquad \dfrac{??}{B \to C, B, C \vdash A}}{A \to (B \to C), B, C \vdash A}\,(\to L)}{A \to (B \to C), B \vdash C \to A}\,(\to R)}{A \to (B \to C) \vdash B \to (C \to A)}\,(\to R)$$

$$\dfrac{\dfrac{\dfrac{B, C \vdash A \qquad \dfrac{\dfrac{}{B, C \vdash B, A}\,(I) \qquad B, C \vdash A}{B \to C, B, C \vdash A}\,(\to L)}{A \to (B \to C), B, C \vdash A}\,(\to L)}{A \to (B \to C), B \vdash C \to A}\,(\to R)}{A \to (B \to C) \vdash B \to (C \to A)}\,(\to R)$$

$$\frac{\Gamma \vdash A, \Delta \quad \Gamma, B \vdash \Delta}{\Gamma, A \to B \vdash \Delta} \ (\to L)$$

a **counterexample** to the **sequent** $\Gamma \vdash A, \Delta$
is a **counterexample** to $\Gamma, A \to B \vdash \Delta$
(since if A is false then $A \to B$ is true)

a **counterexample** to the **sequent** $\Gamma, B \vdash \Delta$
is a **counterexample** to $\Gamma, A \to B \vdash \Delta$
(since if B is true then $A \to B$ is true)

$$\frac{\Gamma, A \vdash B, \Delta}{\Gamma \vdash A \to B, \Delta} \; (\to R)$$

a **counterexample** to Γ, A ⊢ B, Δ

is a **counterexample** to Γ ⊢ A→B, Δ

(if A is true and B false then A→B is false)

$$\frac{\Gamma \vdash A, \Delta \quad \Gamma, B \vdash \Delta}{\Gamma, A \to B \vdash \Delta}\ (\to L) \qquad \frac{\Gamma, A \vdash B, \Delta}{\Gamma \vdash A \to B, \Delta}\ (\to R)$$

for these rules,
a **counterexample** to any assumption
is a **counterexample** to the conclusion

counterexample

$$B, C \nvdash A \qquad B = \top, C = \top, A = \bot$$

$$\dfrac{B, C \vdash A \qquad \dfrac{\dfrac{}{B, C \vdash B, A}\,(I) \qquad B, C \vdash A}{B \to C, B, C \vdash A}\,(\to L)}{\dfrac{A \to (B \to C), B, C \vdash A}{\dfrac{A \to (B \to C), B \vdash C \to A}{A \to (B \to C) \vdash B \to (C \to A)}\,(\to R)}\,(\to R)}\,(\to L)$$

$$A \to (B \to C) = \top \qquad B \vdash C \to A = \bot$$

$$A \to (B \to C) \nvdash B \to (C \to A)$$

$$\frac{}{\Gamma, A \vdash \Delta, A}\ (I)$$

$$\frac{\Gamma, A, B \vdash \Delta}{\Gamma, A \wedge B \vdash \Delta}\ (\wedge L) \qquad \frac{\Gamma \vdash A, B, \Delta}{\Gamma \vdash A \vee B, \Delta}\ (\vee R)$$

$$\frac{\Gamma, A \vdash \Delta \quad \Gamma, B \vdash \Delta}{\Gamma, A \vee B \vdash \Delta}\ (\vee L) \qquad \frac{\Gamma \vdash A, \Delta \quad \Gamma \vdash B, \Delta}{\Gamma \vdash A \wedge B, \Delta}\ (\wedge R)$$

$$\frac{\Gamma \vdash A, \Delta \quad \Gamma, B \vdash \Delta}{\Gamma, A \rightarrow B \vdash \Delta}\ (\rightarrow L) \qquad \frac{\Gamma, A \vdash B, \Delta}{\Gamma \vdash A \rightarrow B, \Delta}\ (\rightarrow R)$$

$$\frac{\Gamma \vdash A, \Delta}{\Gamma, \neg A \vdash \Delta}\ (\neg L) \qquad \frac{\Gamma, A \vdash \Delta}{\Gamma \vdash \neg A, \Delta}\ (\neg R)$$

for all these (sound) rules,
a **counterexample** to any assumption
is a **counterexample** to the conclusion

$$\frac{}{\Gamma, A \vdash \Delta, A} \; (I)$$

$$\frac{\Gamma, A, B \vdash \Delta}{\Gamma, A \wedge B \vdash \Delta} \; (\wedge L) \qquad \frac{\Gamma \vdash A, B, \Delta}{\Gamma \vdash A \vee B, \Delta} \; (\vee R)$$

$$\frac{\Gamma, A \vdash \Delta \quad \Gamma, B \vdash \Delta}{\Gamma, A \vee B \vdash \Delta} \; (\vee L) \qquad \frac{\Gamma \vdash A, \Delta \quad \Gamma \vdash B, \Delta}{\Gamma \vdash A \wedge B, \Delta} \; (\wedge R)$$

$$\frac{\Gamma \vdash A, \Delta \quad \Gamma, B \vdash \Delta}{\Gamma, A \to B \vdash \Delta} \; (\to L) \qquad \frac{\Gamma, A \vdash B, \Delta}{\Gamma \vdash A \to B, \Delta} \; (\to R)$$

$$\frac{\Gamma \vdash A, \Delta}{\Gamma, \neg A \vdash \Delta} \; (\neg L) \qquad \frac{\Gamma, A \vdash \Delta}{\Gamma \vdash \neg A, \Delta} \; (\neg R)$$

Each of Gentzen’s rules is sound:

$\therefore$ if a sequent can be proved using these rules it is valid

¿ if a sequent is valid can it be proved ?

$$\frac{}{\Gamma, A \vdash \Delta, A}\ (I)$$

$$\frac{\Gamma, A, B \vdash \Delta}{\Gamma, A \wedge B \vdash \Delta}\ (\wedge L) \qquad \frac{\Gamma \vdash A, B, \Delta}{\Gamma \vdash A \vee B, \Delta}\ (\vee R)$$

$$\frac{\Gamma, A \vdash \Delta \quad \Gamma, B \vdash \Delta}{\Gamma, A \vee B \vdash \Delta}\ (\vee L) \qquad \frac{\Gamma \vdash A, \Delta \quad \Gamma \vdash B, \Delta}{\Gamma \vdash A \wedge B, \Delta}\ (\wedge R)$$

$$\frac{\Gamma \vdash A, \Delta \quad \Gamma, B \vdash \Delta}{\Gamma, A \to B \vdash \Delta}\ (\to L) \qquad \frac{\Gamma, A \vdash B, \Delta}{\Gamma \vdash A \to B, \Delta}\ (\to R)$$

$$\frac{\Gamma \vdash A, \Delta}{\Gamma, \neg A \vdash \Delta}\ (\neg L) \qquad \frac{\Gamma, A \vdash \Delta}{\Gamma \vdash \neg A, \Delta}\ (\neg R)$$

Each of Gentzen’s rules has the property that:

a counterexample to any of its assumptions
is also
a counterexample to its conclusion

if the search for a proof fails,
we can use this property to provide a counterexample to the conclusion

Gentzen's rules are sound and complete

we apply the rules, until we can do no more;
at each step there are fewer connectives
in each assumption than in the conclusion

eventually we run out of connectives,
at which point, only atoms remain
either $\Gamma \cap \Delta = \emptyset$
in which case we can construct a counterexample
or *some atom occurs in both* Γ *and* Δ
so, we can apply rule I to discharge the assumption

if *all assumptions are discharged we have a proof;*
otherwise,
any counterexample can be pushed down the tree to
show that the conclusion is not valid

KEEP CALM
&
FOLLOW THE RULES

$$\frac{}{\Gamma, A \vdash \Delta, A}\ (I)$$

$$\frac{\Gamma, A, B \vdash \Delta}{\Gamma, A \wedge B \vdash \Delta}\ (\wedge L) \qquad \frac{\Gamma \vdash A, B, \Delta}{\Gamma \vdash A \vee B, \Delta}\ (\vee R)$$

$$\frac{\Gamma, A \vdash \Delta \quad \Gamma, B \vdash \Delta}{\Gamma, A \vee B \vdash \Delta}\ (\vee L) \qquad \frac{\Gamma \vdash A, \Delta \quad \Gamma \vdash B, \Delta}{\Gamma \vdash A \wedge B, \Delta}\ (\wedge R)$$

$$\frac{\Gamma \vdash A, \Delta \quad \Gamma, B \vdash \Delta}{\Gamma, A \to B \vdash \Delta}\ (\to L) \qquad \frac{\Gamma, A \vdash B, \Delta}{\Gamma \vdash A \to B, \Delta}\ (\to R)$$

$$\frac{\Gamma \vdash A, \Delta}{\Gamma, \neg A \vdash \Delta}\ (\neg L) \qquad \frac{\Gamma, A \vdash \Delta}{\Gamma \vdash \neg A, \Delta}\ (\neg R)$$

*This shows that Gentzen's set of rules is **complete,** that is to say:*

if a sequent is valid then it has a proof

(without assumptions)