1.	$A \to B$	Premise
2.	$\sim B$	Premise
3.	$\sim A$	Modus tollens (1,2)
4.	$\sim A \to (C \wedge D)$	Premise
5.	$C \wedge D$	Modus ponens (3,4)
6.	C	Decomposing a conjunction (5)

1.	$P \wedge Q$	Premise
2.	P	Decomposing a conjunction (1)
3.	Q	Decomposing a conjunction (1)
4.	$P \to \sim (Q \wedge R)$	Premise
5.	$\sim (Q \wedge R)$	Modus ponens (3,4)
6.	$\sim Q \vee \sim R$	DeMorgan (5)
7.	$\sim R$	Disjunctive syllogism (3,6)
8.	$S \to R$	Premise
9.	$\sim S$	Modus tollens (7,8) □

Lecture 17:Inference

Michael Fourman

The 9 Elementary Valid Arg't Forms

1. Modus Ponens (MP)
P → Q
<u>P</u>
Q

2. Modus Tollens (MT)
P → Q
<u>~ Q</u>
~ P

3. Hypothetical Syllogism (HS)
P → Q
<u>Q → R</u>
P → R

4. Disjunctive Syllogism (DS)
P v Q
<u>~ P</u>
Q

5. Constructive Dilemma (CD)
(P → Q) & (R → S)
<u>P v R</u>
Q v S

6. Absorption (Abs)
<u>P → Q</u>
P → (P & Q)

7. Simplification (Simp)
<u>P & Q</u>
P

8. Conjunction (Conj)
P
<u>Q</u>
P & Q

9. Addition (Add)
<u>P</u>
P v Q

10 Logically Equivalent Expressions

10. De Morgan's Theorums (DeM)
~ (P & Q) ≡ (~ P v ~ Q)
~ (P v Q) ≡ (~ P & ~ Q)

11. Commutation (Com)
(P v Q) ≡ (Q v P)
(P & Q) ≡ (Q & P)

12. Association (Assoc)
[P v (Q v R)] ≡ [(P v Q) v R]
[P & (Q & R)] ≡ [(P & Q) & R]

13. Distribution (Dist)
[P & (Q v R)] ≡ [(P & Q) v (P & R)]
[P v (Q & R)] ≡ [(P v Q) & (P v R)]

14. Double Negation (DN)
~ ~ P ≡ P

15. Transposition (Trans)
(P → Q) ≡ (~ Q → ~ P)

16. Material Implication (Impl)
(P → Q) ≡ (~ P v Q)

17. Material Equivalence (Equiv)
(P ≡ Q) ≡ [(P → Q) & (Q → P)]
(P ≡ Q) ≡ [(P & Q) v (~ P & ~ Q)]

18. Exportation (Exp)
[(P & Q) → R] ≡ [P → (Q → R)]

19. Tautology (Taut)
P ≡ (P v P)
P ≡ (P & P)

Is this a valid argument?

- Assumptions:

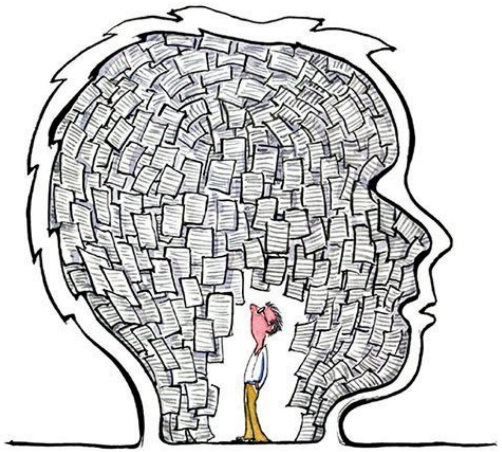
  If the races are fixed or the gambling houses are crooked, then the tourist trade will decline.
  If the tourist trade declines then the police force will be happy.
  The police force is never happy.

- Conclusion:

  The races are not fixed

Assumptions: If the races are fixed or the gambling houses are crooked, then the tourist trade will decline.
If the tourist trade declines then the police force will be happy.
The police force is never happy.

Conclusion: The races are not fixed.



$$\dfrac{\dfrac{\dfrac{(\mathrm{RF} \lor \mathrm{GC}) \to \mathrm{TT} \qquad \dfrac{\mathrm{TT} \to \mathrm{PH} \qquad \neg \mathrm{PH}}{\neg \mathrm{TT}}}{\neg(\mathrm{RF} \lor \mathrm{GC})}}{\neg \mathrm{RF} \land \neg \mathrm{GC}}}{\neg \mathrm{RF}}$$

we represent the argument by a deduction composed of sound deduction rules assumptions

$$\frac{X \rightarrow Y \qquad \neg Y}{\neg X}\ \mathit{modus\ tollendo\ tollens}$$

conclusion

A deduction rule is ***sound*** *if whenever its assumptions are true then its conclusion is true*

If we can deduce some conclusion from a set of assumptions, using only sound rules, and the assumptions are true then the conclusion is true; the argument is ***valid***

$$\frac{A \to B \quad \neg B}{\neg A}\ \textit{modus tollendo tollens} \qquad \frac{\neg A \quad A \vee B}{B}\ \textit{modus tollendo ponens}$$

$$\frac{A \quad \neg(A \wedge B)}{\neg B}\ \textit{modus ponendo tollens} \qquad \frac{A \quad A \to B}{B}\ \textit{modus ponendo ponens}$$

Can we find a finite set of sound rules sufficient to give a proof for any valid argument?

*A set of deduction rules that is sufficient to give a proof for any valid argument is said to be **complete***

Some deduction rules

Are these sound?

$$\frac{A \to B \quad \neg B}{\neg A}\ \mathit{modus\ tollendo\ tollens} \qquad \frac{\neg A \quad A \vee B}{B}\ \mathit{modus\ tollendo\ ponens}$$

$$\frac{A \quad \neg(A \wedge B)}{\neg B}\ \mathit{modus\ ponendo\ tollens} \qquad \frac{A \quad A \to B}{B}\ \mathit{modus\ ponendo\ ponens}$$

$$\frac{\neg A \vee B \quad \neg B}{\neg A}\ \mathit{modus\ tollendo\ tollens} \qquad \frac{\neg A \quad A \vee B}{B}\ \mathit{modus\ tollendo\ ponens}$$

$$\frac{A \quad \neg A \vee \neg B}{\neg B}\ \mathit{modus\ ponendo\ tollens} \qquad \frac{A \quad \neg A \vee B}{B}\ \mathit{modus\ ponendo\ ponens}$$

these rules are all equivalent to special cases of resolution, so we should expect that the answer will be yes, but we also want to formalise more natural forms of argument

Some sound deduction rules

$$\frac{A \to B \quad \neg B}{\neg A}\ \textit{modus tollendo tollens} \qquad \frac{\neg A \quad A \vee B}{B}\ \textit{modus tollendo ponens}$$

$$\frac{A \quad \neg(A \wedge B)}{\neg B}\ \textit{modus ponendo tollens} \qquad \frac{A \quad A \to B}{B}\ \textit{modus ponendo ponens}$$

$$\frac{\neg A \vee B \quad \neg B}{\neg A}\ \textit{modus tollendo tollens} \qquad \frac{\neg A \quad A \vee B}{B}\ \textit{modus tollendo ponens}$$

$$\frac{A \quad \neg A \vee \neg B}{\neg B}\ \textit{modus ponendo tollens} \qquad \frac{A \quad \neg A \vee B}{B}\ \textit{modus ponendo ponens}$$

each rule corresponds to a valid entailment

$$A \to B, \neg B \vdash \neg A \qquad \neg A, A \vee B \vdash B$$

$$A, \neg(A \wedge B) \vdash \neg B \qquad A, A \to B \vdash B$$

$$\neg A \vee B, \neg B \vdash \neg A \qquad \neg A, A \vee B \vdash B$$

$$A, \neg A \vee \neg B \vdash \neg B \qquad A, \neg A \vee B \vdash B$$

Entailment

antecedents $\vdash$ consequent

$$A \to B, \neg B \vdash \neg A \qquad \neg A, A \lor B \vdash B$$

$$A, \neg(A \land B) \vdash \neg B \qquad A, A \to B \vdash B$$

$$\neg A \lor B, \neg B \vdash \neg A \qquad \neg A, A \lor B \vdash B$$

$$A, \neg A \lor \neg B \vdash \neg B \qquad A, \neg A \lor B \vdash B$$

*an entailment is **valid** if every valuation that makes all of its antecedents true makes its consequent true* we can use rules with entailments to formalise and study the ways we can build deductions

$$\frac{\Gamma \vdash A \quad \Delta, A \vdash B}{\Gamma, \Delta \vdash B}\ Cut$$

$$\begin{array}{c}\Gamma\\ \vdots\\ A\end{array} \qquad \begin{array}{c}\Delta \quad A\\ \vdots\\ B\end{array} \qquad \Rightarrow \qquad \begin{array}{c} \qquad \Gamma \\ \qquad \vdots \\ \Delta \quad \cancel{A} \\ \vdots \\ B\end{array}$$

*An **inference rule** is **sound** if whenever its assumptions are **valid** then its conclusion is **valid***

Another rule of inference

$$\frac{\Delta, A \vdash B}{\Delta \vdash A \to B}\ (\to^{+})$$

$$\begin{array}{cc} \mathrm{A} & \Delta \\ \multicolumn{2}{c}{\vdots} \\ \multicolumn{2}{c}{\mathrm{B}} \end{array} \quad \Rightarrow \quad \begin{array}{cc} \cancel{\mathrm{A}} & \Delta \\ \multicolumn{2}{c}{\vdots} \\ \multicolumn{2}{c}{\mathrm{A} \to \mathrm{B}} \end{array}$$

More rules

$$\frac{}{\mathcal{A}, X \vdash X}\ (I)$$

$$\frac{\mathcal{A} \vdash X \quad \mathcal{A} \vdash Y}{\mathcal{A} \vdash X \wedge Y}\ (\wedge) \qquad \frac{\mathcal{A}, X \vdash Z \quad \mathcal{A}, Y \vdash Z}{\mathcal{A}, X \vee Y \vdash Z}\ (\vee) \qquad \frac{\mathcal{A}, X \vdash Y}{\mathcal{A} \vdash X \to Y}\ (\to)$$

a double line means that the rule is sound in either direction, up as well as down

going down (+) introduces the connective going up (-) eliminates the connective

A simple proof

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{A \to (B \to C) \vdash A \to (B \to C)}\,(I)}{A \to (B \to C) A \vdash B \to C}\,(\to^-)}{A \to (B \to C), A, B \vdash C}\,(\to^-)}{A \to (B \to C), B \vdash A \to C}\,(\to^+)}{A \to (B \to C) \vdash B \to (A \to C)}\,(\to^+)
$$

*Since each **inference rule** is **sound***
*if the assumptions are **valid***
*then the conclusion is **valid***

*Here, **we have no assumptions so the conclusion is valid.***

More rules

$$\frac{}{\mathcal{A}, X \vdash X}\ (I)$$

$$\frac{\mathcal{A} \vdash X \quad \mathcal{A} \vdash Y}{\mathcal{A} \vdash X \wedge Y}\ (\wedge) \qquad \frac{\mathcal{A}, X \vdash Z \quad \mathcal{A}, Y \vdash Z}{\mathcal{A}, X \vee Y \vdash Z}\ (\vee) \qquad \frac{\mathcal{A}, X \vdash Y}{\mathcal{A} \vdash X \rightarrow Y}\ (\rightarrow)$$

Can we prove $X \wedge Y \vdash X \vee Y$?

If each inference rule is sound, then,
if *we can prove some conclusion (without assumptions)*
then *the conclusion is* ***valid***

More rules

$$\frac{}{\mathcal{A}, X \vdash X}\ (I)$$

$$\dfrac{\mathcal{A} \vdash X \quad \mathcal{A} \vdash Y}{\mathcal{A} \vdash X \wedge Y}\ (\wedge) \qquad \dfrac{\mathcal{A}, X \vdash Z \quad \mathcal{A}, Y \vdash Z}{\mathcal{A}, X \vee Y \vdash Z}\ (\vee) \qquad \dfrac{\mathcal{A}, X \vdash Y}{\mathcal{A} \vdash X \to Y}\ (\to)$$

Can we prove $X \wedge Y \vdash X \vee Y$?

*we say a set of inference rules is **complete**, iff*

***if** a conclusion is valid **then** we can prove it*

(without assumptions)

Another Proof

$$\dfrac{\dfrac{\dfrac{}{A \wedge B \vdash A \wedge B}\ (I)}{A \wedge B \vdash A}\ (\wedge^-) \qquad \dfrac{\dfrac{}{A \vee B \vdash A \vee B}\ (I)}{A \vdash A \vee B}\ (\vee^-)}{A \wedge B \vdash A \vee B}\ Cut$$

a set of entailment rules is ***complete*** *if every valid entailment has a proof*

¿can we find a complete set of sound rules?

1924

Gentzen's Rules (I)

1945

$$\frac{}{\Gamma, A \vdash \Delta, A}\ (I)$$

$$\frac{\Gamma, A, B \vdash \Delta}{\Gamma, A \wedge B \vdash \Delta}\ (\wedge L) \qquad \frac{\Gamma \vdash A, B, \Delta}{\Gamma \vdash A \vee B, \Delta}\ (\vee R)$$

$$\frac{\Gamma, A \vdash \Delta \quad \Gamma, B \vdash \Delta}{\Gamma, A \vee B \vdash \Delta}\ (\vee L) \qquad \frac{\Gamma \vdash A, \Delta \quad \Gamma \vdash B, \Delta}{\Gamma \vdash A \wedge B, \Delta}\ (\wedge R)$$

a **sequent,** $\Gamma \vdash \Delta$

where Γ and Δ are finite sets of expressions

is **valid** iff

whenever every expression in Γ is true

some expression in Δ is true

Gerhard Karl Erich Gentzen (November 24, 1909 – August 4, 1945)

Gentzen's Rules (I)

$$\frac{}{\Gamma, A \vdash \Delta, A} \ (I)$$

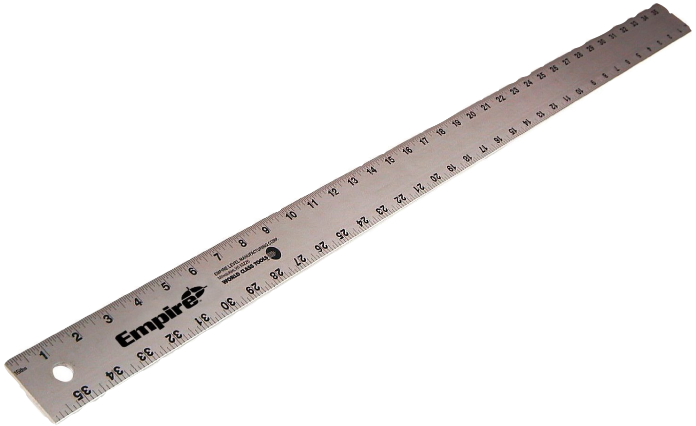
$$\frac{\Gamma, A, B \vdash \Delta}{\Gamma, A \wedge B \vdash \Delta} \ (\wedge L) \qquad \frac{\Gamma \vdash A, B, \Delta}{\Gamma \vdash A \vee B, \Delta} \ (\vee R)$$

$$\frac{\Gamma, A \vdash \Delta \quad \Gamma, B \vdash \Delta}{\Gamma, A \vee B \vdash \Delta} \ (\vee L) \qquad \frac{\Gamma \vdash A, \Delta \quad \Gamma \vdash B, \Delta}{\Gamma \vdash A \wedge B, \Delta} \ (\wedge R)$$

a **counterexample** to the **sequent** $\Gamma \vdash \Delta$,
is a valuation that makes
every expression in Γ true
and
every expression in Δ false

(a sequent is valid iff it has no counterexample)

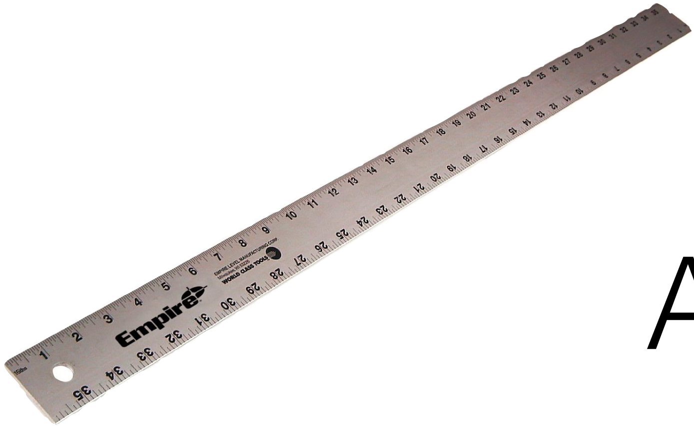
$$
\dfrac{\dfrac{\dfrac{}{A, B \vdash A, B}\ (I)}{A \wedge B \vdash A, B}\ (\wedge L)}{A \wedge B \vdash A \vee B}\ (\vee R)
$$



A rule

$$\frac{\Gamma, A \vdash B, \Delta}{\Gamma \vdash A \to B, \Delta} \; (\to R)$$

A valuation is a counterexample to the top line
iff it is a counterexample to the bottom line



Another rule

$$\frac{\Gamma \vdash A, \Delta \qquad \Gamma, B \vdash \Delta}{\Gamma, A \to B \vdash \Delta}\ (\to L)$$

A valuation is a counterexample to the bottom line
iff it is a counterexample to
at least one of the entailments on the top line

a valuation is a counterexample to the conclusion *iff* *it is a counterexample to at least one assumption*

$$\frac{}{\Gamma, A \vdash \Delta, A}\ (I)$$

$$\frac{\Gamma, A, B \vdash \Delta}{\Gamma, A \wedge B \vdash \Delta}\ (\wedge L) \qquad \frac{\Gamma \vdash A, B, \Delta}{\Gamma \vdash A \vee B, \Delta}\ (\vee R)$$

$$\frac{\Gamma, A \vdash \Delta \quad \Gamma, B \vdash \Delta}{\Gamma, A \vee B \vdash \Delta}\ (\vee L) \qquad \frac{\Gamma \vdash A, \Delta \quad \Gamma \vdash B, \Delta}{\Gamma \vdash A \wedge B, \Delta}\ (\wedge R)$$

$$\frac{\Gamma \vdash A, \Delta \quad \Gamma, B \vdash \Delta}{\Gamma, A \to B \vdash \Delta}\ (\to L) \qquad \frac{\Gamma, A \vdash B, \Delta}{\Gamma \vdash A \to B, \Delta}\ (\to R)$$

$$\frac{\Gamma \vdash A, \Delta}{\Gamma, \neg A \vdash \Delta}\ (\neg L) \qquad \frac{\Gamma, A \vdash \Delta}{\Gamma \vdash \neg A, \Delta}\ (\neg R)$$