Entailment

In algebra, we consider expressions with variables, and write equations to express relationships between different expressions.

\[ \text{LHS} = \text{RHS} \]

Boolean algebra, with equalities between expressions, gives us one way to express relationships between different logical expressions.

If we want to study logical arguments it is more natural to consider entailments.

\[ \text{LHS} \vdash \text{RHS} \]
Entailment

If we want to study logical arguments it is more natural to consider entailments.

LHS $\vdash$ RHS

The entailment is **valid** if any valuation that makes everything on the LHS true, makes the RHS true

$\vdash$ RHS

an entailment with empty LHS is **valid** iff RHS is a tautology

i.e. every valuation makes it true
Is this a valid argument?

• Assumptions:
  If the races are fixed or the gambling houses are crooked, then the tourist trade will decline. If the tourist trade declines then the police force will be happy. The police force is never happy.

• Conclusion:
  The races are not fixed
• Assumptions:
  If the races are fixed or the gambling houses are crooked, then the tourist trade will decline.
  If the tourist trade declines then the police force will be happy.
  The police force is never happy.

• Conclusion:
  The races are not fixed

\[
(RF \lor GC) \rightarrow TT, TT \rightarrow PH, \neg PH \vdash \neg RF
\]

the deduction is summarised in an entailment
Is this a valid argument?

• Assumptions:
  If I am clever then I will pass
  If I will pass then I am clever,
  Either I am clever or I will pass

• Conclusion:
  I am clever and I will pass

¿is this valid?

\[ C \rightarrow P, P \rightarrow C, C \lor P \vdash C \land P \]
C → P, P → C, C ∨ P ⊢ C ∧ P

Everything excluded by C ∧ P is already excluded by one of the assumptions
\( \text{C} \rightarrow \text{P}, \text{P} \rightarrow \text{C}, \text{C} \lor \text{P} \vdash \text{C} \land \text{P} \)

Everything excluded by \( \text{C} \land \text{P} \) is already excluded by one of the assumptions.

\[ \equiv \]

Nothing excluded by \( \text{C} \land \text{P} \) is allowed by all of the assumptions

States excluded by \( \text{C} \land \text{P} \) satisfy \( \neg (\text{C} \land \text{P}) \)

So we show that

\( \text{C} \rightarrow \text{P}, \text{P} \rightarrow \text{C}, \text{C} \lor \text{P}, \neg (\text{C} \land \text{P}) \vdash \) these constraints are inconsistent
Entailment

LHS ⊨ RHS

The entailment is valid if any valuation that makes everything on the LHS true, makes something on the RHS true.

⊨ RHS

an entailment with empty LHS is valid iff RHS is a tautology
an entailment with empty RHS is valid iff LHS is a contradiction
So we show that these constraints are inconsistent:

<table>
<thead>
<tr>
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<th>C</th>
<th>P</th>
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<tr>
<td>¬C ∨ P</td>
<td>¬P ∨ P</td>
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<td>¬C ∨ ¬P</td>
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Given:

C → P, P → C, C ∨ P ⊢ C ∧ P
So we show that these constraints are inconsistent:

\[
\begin{array}{c|c|c}
\neg C \lor P & \neg P \lor P & \{} \\
\neg P \lor C & P & {} \\
C \lor P & \neg P & {} \\
\neg C \lor \neg P & \neg P \lor P & {} \\
\end{array}
\]
$C \rightarrow P, P \rightarrow C$, $C \lor P \vdash C \land P$

So we show that

$C \rightarrow P, P \rightarrow C$, $C \lor P$, $\neg(C \land P)$

is inconsistent
• Assumptions:
  If the races are fixed or the gambling houses are crooked, then the tourist trade will decline.
  If the tourist trade declines then the police force will be happy.
  The police force is never happy.

• Conclusion:
  The races are not fixed

\[(\neg RF \land \neg GC) \iff TT\]

\[\neg TT\]

\[\neg (RF \lor GC)\]

\[\neg RF \land \neg GC\]

\[\neg RF\]

\[RF \lor GC \iff TT, \quad TT \iff PH, \quad \neg PH \iff \neg RF\]

the deduction is summarised in an entailment
\[ (RF \lor GC) \rightarrow TT \]

\[ \neg (RF \lor GC) \]

\[ \neg RF \land \neg GC \]

\[ \neg RF \]

\[ RF \lor GC \rightarrow TT, TT \rightarrow PH, \neg PH \models \neg RF \]

\[ RF \rightarrow TT, GC \rightarrow TT, TT \rightarrow PH, \neg PH \models RF \]

\[ \neg RF \lor TT, \neg GC \lor TT, \neg TT \lor PH, \neg PH \models RF \]
\neg RF \lor TT, \neg GC \lor TT, \neg TT \lor PH, \neg PH, RF
\[ \neg RF \lor TT, \neg GC \lor TT, \neg TT \lor PH, \neg PH, RF \]

<table>
<thead>
<tr>
<th>PH</th>
<th>RF</th>
<th>TT</th>
<th>TT</th>
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<tbody>
<tr>
<td>\neg RF \lor TT</td>
<td>\neg TT</td>
<td>\neg TT</td>
<td>PH</td>
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<tr>
<td>\neg GC \lor TT</td>
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<td>\neg TT \lor PH</td>
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<tr>
<td>RF</td>
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</table>
\[ \neg RF \lor TT, \neg GC \lor TT, \neg TT \lor PH, \neg PH \lor RF \]
\( \neg RF \lor TT, \neg GC \lor TT, \neg TT \lor PH, \neg PH, RF \)

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<thead>
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<th>PH</th>
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<tr>
<td>( \neg RF \lor TT )</td>
<td>( \neg TT )</td>
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<tr>
<td>( \neg GC \lor TT )</td>
<td>( \neg GC )</td>
</tr>
<tr>
<td>( \neg TT \lor PH )</td>
<td>( \neg PH )</td>
</tr>
</tbody>
</table>
\[ \neg RF \lor TT, \neg GC \lor TT, \neg TT \lor PH, \neg PH, RF \]
\(-RF \lor TT, -GC \lor TT, -TT \lor PH, -PH, RF\)

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<tr>
<th></th>
<th>PH</th>
<th>TT</th>
<th>RF</th>
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<tbody>
<tr>
<td>1</td>
<td>(-RF \lor TT)</td>
<td>(-TT)</td>
<td>(-RF)</td>
</tr>
<tr>
<td>2</td>
<td>(-GC \lor TT)</td>
<td>(-GC)</td>
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<tr>
<td>3</td>
<td>(-TT \lor PH)</td>
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<tr>
<td>4</td>
<td>(-PH)</td>
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<tr>
<td>5</td>
<td>(RF)</td>
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\{ \}  

A resolution proof shows that these constraints are inconsistent.
A cycle including an atom and its negation shows that the constraints are inconsistent.