1.	$A \to B$	Premise
2.	$\sim B$	Premise
3.	$\sim A$	Modus tollens (1,2)
4.	$\sim A \to (C \wedge D)$	Premise
5.	$C \wedge D$	Modus ponens (3,4)
6.	C	Decomposing a conjunction (5)

1.	$P \wedge Q$	Premise
2.	P	Decomposing a conjunction (1)
3.	Q	Decomposing a conjunction (1)
4.	$P \to \sim (Q \wedge R)$	Premise
5.	$\sim (Q \wedge R)$	Modus ponens (3,4)
6.	$\sim Q \vee \sim R$	DeMorgan (5)
7.	$\sim R$	Disjunctive syllogism (3,6)
8.	$S \to R$	Premise
9.	$\sim S$	Modus tollens (7,8) $\square$

Lecture 16:Inference

Michael Fourman

The 9 Elementary Valid Arg't Forms

1. Modus Ponens (MP)
$P \to Q$
$\underline{P}$
Q

2. Modus Tollens (MT)
$P \to Q$
$\underline{\sim Q}$
$\sim P$

3. Hypothetical Syllogism (HS)
$P \to Q$
$\underline{Q \to R}$
$P \to R$

4. Disjunctive Syllogism (DS)
$P \vee Q$
$\underline{\sim P}$
Q

5. Constructive Dilemma (CD)
$(P \to Q) \& (R \to S)$
$\underline{P \vee R}$
$Q \vee S$

6. Absorption (Abs)
$\underline{P \to Q}$
$P \to (P \& Q)$

7. Simplification (Simp)
$\underline{P \& Q}$
P

8. Conjunction (Conj)
P
$\underline{Q}$
$P \& Q$

9. Addition (Add)
$\underline{P}$
$P \vee Q$

10 Logically Equivalent Expressions

10. De Morgan's Theorums (DeM)
$\sim (P \& Q) \equiv (\sim P \vee \sim Q)$
$\sim (P \vee Q) \equiv (\sim P \& \sim Q)$

11. Commutation (Com)
$(P \vee Q) \equiv (Q \vee P)$
$(P \& Q) \equiv (Q \& P)$

12. Association (Assoc)
$[P \vee (Q \vee R)] \equiv [(P \vee Q) \vee R]$
$[P \& (Q \& R)] \equiv [(P \& Q) \& R]$

13. Distribution (Dist)
$[P \& (Q \vee R)] \equiv [(P \& Q) \vee (P \& R)]$
$[P \vee (Q \& R)] \equiv [(P \vee Q) \& (P \vee R)]$

14. Double Negation (DN)
$\sim\sim P \equiv P$

15. Transposition (Trans)
$(P \to Q) \equiv (\sim Q \to \sim P)$

16. Material Implication (Impl)
$(P \to Q) \equiv (\sim P \vee Q)$

17. Material Equivalence (Equiv)
$(P \equiv Q) \equiv [(P \to Q) \& (Q \to P)]$
$(P \equiv Q) \equiv [(P \& Q) \vee (\sim P \& \sim Q)]$

18. Exportation (Exp)
$[(P \& Q) \to R] \equiv [P \to (Q \to R)]$

19. Tautology (Taut)
$P \equiv (P \vee P)$
$P \equiv (P \& P)$

Is this a valid argument?

- Assumptions:
  If the races are fixed or the gambling houses are crooked, then the tourist trade will decline.
  If the tourist trade declines then the police force will be happy.
  The police force is never happy.

- Conclusion:
  The races are not fixed

2

Assumptions: If the races are fixed or the gambling houses are crooked, then the tourist trade will decline.
If the tourist trade declines then the police force will be happy.
The police force is never happy.

Conclusion: The races are not fixed.

$$\dfrac{\dfrac{\dfrac{(\mathrm{RF} \lor \mathrm{GC}) \to \mathrm{TT} \qquad \dfrac{\mathrm{TT} \to \mathrm{PH} \qquad \neg\mathrm{PH}}{\neg\mathrm{TT}}}{\neg(\mathrm{RF} \lor \mathrm{GC})}}{\neg\mathrm{RF} \land \neg\mathrm{GC}}}{\neg\mathrm{RF}}$$

we represent the argument by a deduction composed of sound deduction rules assumptions

$$\frac{X \to Y \quad \neg Y}{\neg X} \; \textit{modus tollendo tollens}$$

conclusion

*A deduction rule is **sound** if whenever its assumptions are true then its conclusion is true*

*If we can deduce some conclusion from a set of assumptions, using only sound rules, and the assumptions are true then the conclusion is true; the argument is **valid***

$$\frac{A \to B \quad \neg B}{\neg A}\ \textit{modus tollendo tollens} \qquad \frac{\neg A \quad A \vee B}{B}\ \textit{modus tollendo ponens}$$

$$\frac{A \quad \neg(A \wedge B)}{\neg B}\ \textit{modus ponendo tollens} \qquad \frac{A \quad A \to B}{B}\ \textit{modus ponendo ponens}$$

Can we find a finite set of sound rules sufficient to give a proof for any valid argument?

A set of deduction rules that is sufficient to give a proof for any valid argument is said to be ***complete***

Some sound deduction rules

$$\frac{A \to B \quad \neg B}{\neg A}\ \textit{modus tollendo tollens} \qquad \frac{\neg A \quad A \vee B}{B}\ \textit{modus tollendo ponens}$$

$$\frac{A \quad \neg(A \wedge B)}{\neg B}\ \textit{modus ponendo tollens} \qquad \frac{A \quad A \to B}{B}\ \textit{modus ponendo ponens}$$

$$\frac{\neg A \vee B \quad \neg B}{\neg A}\ \textit{modus tollendo tollens} \qquad \frac{\neg A \quad A \vee B}{B}\ \textit{modus tollendo ponens}$$

$$\frac{A \quad \neg A \vee \neg B}{\neg B}\ \textit{modus ponendo tollens} \qquad \frac{A \quad \neg A \vee B}{B}\ \textit{modus ponendo ponens}$$

these rules are all equivalent to special cases of resolution, so we should expect that the answer will be yes, but we also want to formalise more natural forms of argument

Some sound deduction rules

$$\frac{A \to B \quad \neg B}{\neg A}\ \textit{modus tollendo tollens} \qquad \frac{\neg A \quad A \vee B}{B}\ \textit{modus tollendo ponens}$$

$$\frac{A \quad \neg(A \wedge B)}{\neg B}\ \textit{modus ponendo tollens} \qquad \frac{A \quad A \to B}{B}\ \textit{modus ponendo ponens}$$

$$\frac{\neg A \vee B \quad \neg B}{\neg A}\ \textit{modus tollendo tollens} \qquad \frac{\neg A \quad A \vee B}{B}\ \textit{modus tollendo ponens}$$

$$\frac{A \quad \neg A \vee \neg B}{\neg B}\ \textit{modus ponendo tollens} \qquad \frac{A \quad \neg A \vee B}{B}\ \textit{modus ponendo ponens}$$

each rule corresponds to a valid entailment

$$A \to B, \neg B \vdash \neg A \qquad \neg A, A \vee B \vdash B$$

$$A, \neg(A \wedge B) \vdash \neg B \qquad A, A \to B \vdash B$$

$$\neg A \vee B, \neg B \vdash \neg A \qquad \neg A, A \vee B \vdash B$$

$$A, \neg A \vee \neg B \vdash \neg B \qquad A, \neg A \vee B \vdash B$$

Entailment

antecedents $\vdash$ *consequent*

$A \to B, \neg B \vdash \neg A$

$A, \neg(A \wedge B) \vdash \neg B$

$\neg A, A \vee B \vdash B$

$A, A \to B \vdash B$

$\neg A \vee B, \neg B \vdash \neg A$

$A, \neg A \vee \neg B \vdash \neg B$

$\neg A, A \vee B \vdash B$

$A, \neg A \vee B \vdash B$

an entailment is ***valid*** *if every valuation that makes all of its antecedents true makes its consequent true* we can use rules with entailments to formalise and study the ways we can build deductions

$$\frac{\Gamma \vdash A \quad \Delta, A \vdash B}{\Gamma, \Delta \vdash B}\ Cut \qquad\qquad \begin{array}{c}\Gamma\\ \vdots\\ A\end{array}\quad \begin{array}{c}\Delta \quad A\\ \vdots\\ B\end{array} \quad \Rightarrow \quad \begin{array}{c} \qquad \Gamma\\ \qquad \vdots\\ \Delta \quad \cancel{A}\\ \vdots\\ B\end{array}$$

*An **inference rule** is **sound** if whenever its assumptions are **valid** then its conclusion is **valid***

Another rule of inference

$$\frac{\Delta, A \vdash B}{\Delta \vdash A \to B}\ (\to^{+}) \qquad \begin{array}{cc} \mathrm{A} & \Delta \\ \vdots & \\ \mathrm{B} & \end{array} \quad \Rightarrow \quad \begin{array}{cc} \cancel{\mathrm{A}} & \Delta \\ \vdots & \\ \mathrm{A} \to \mathrm{B} & \end{array}$$

More rules

$$\overline{\mathcal{A}, X \vdash X}\ (I)$$

$$\frac{\mathcal{A} \vdash X \quad \mathcal{A} \vdash Y}{\mathcal{A} \vdash X \wedge Y}\ (\wedge) \quad \frac{\mathcal{A}, X \vdash Z \quad \mathcal{A}, Y \vdash Z}{\mathcal{A}, X \vee Y \vdash Z}\ (\vee) \quad \frac{\mathcal{A}, X \vdash Y}{\mathcal{A} \vdash X \to Y}\ (\to)$$

a double line means that the rule is sound in either direction, up as well as down

A simple proof

$$\dfrac{\dfrac{\dfrac{\dfrac{\overline{A \to (B \to C) \vdash A \to (B \to C)}\ (I)}{A \to (B \to C) A \vdash B \to C}\ (\to^-)}{A \to (B \to C), A, B \vdash C}\ (\to^-)}{A \to (B \to C), B \vdash A \to C}\ (\to^+)}{A \to (B \to C) \vdash B \to (A \to C)}\ (\to^+)$$

Since each ***inference rule*** *is* ***sound***
if the assumptions are ***valid***
then the conclusion is ***valid***

Here, ***we have no assumptions so the conclusion is valid.***

More rules

$$\frac{}{\mathcal{A}, X \vdash X}\ (I)$$

$$\frac{\mathcal{A} \vdash X \quad \mathcal{A} \vdash Y}{\mathcal{A} \vdash X \wedge Y}\ (\wedge) \quad \frac{\mathcal{A}, X \vdash Z \quad \mathcal{A}, Y \vdash Z}{\mathcal{A}, X \vee Y \vdash Z}\ (\vee) \quad \frac{\mathcal{A}, X \vdash Y}{\mathcal{A} \vdash X \rightarrow Y}\ (\rightarrow)$$

Can we prove $X \wedge Y \vdash X \vee Y$?

If each inference rule is sound, then,
if *we can prove some conclusion (without assumptions)*
then *the conclusion is* ***valid***

More rules

$$\overline{\mathcal{A}, X \vdash X} \;(I)$$

$$\frac{\mathcal{A} \vdash X \quad \mathcal{A} \vdash Y}{\mathcal{A} \vdash X \wedge Y} \;(\wedge) \qquad \frac{\mathcal{A}, X \vdash Z \quad \mathcal{A}, Y \vdash Z}{\mathcal{A}, X \vee Y \vdash Z} \;(\vee) \qquad \frac{\mathcal{A}, X \vdash Y}{\mathcal{A} \vdash X \to Y} \;(\to)$$

Can we prove $X \wedge Y \vdash X \vee Y$?

*we say a set of inference rules is **complete**, iff*

***if** a conclusion is valid **then** we can prove it*

(without assumptions)

Another Proof

$$\dfrac{\dfrac{}{A \wedge B \vdash A \wedge B}\,(I)}{A \wedge B \vdash A}\,(\wedge^-) \quad \dfrac{\dfrac{}{A \vee B \vdash A \vee B}\,(I)}{A \vdash A \vee B}\,(\vee^-)$$
$$\dfrac{A \wedge B \vdash A \qquad A \vdash A \vee B}{A \wedge B \vdash A \vee B}\,Cut$$

*a set of entailment rules is **complete** if every valid entailment has a proof*

¿can we find a complete set of sound rules?

If we just ask for a complete set of rules, without requiring that they are sound, what is the answer?

1924

Gentzen's Rules (I)

1945

$$\frac{}{\Gamma, A \vdash \Delta, A}\ (I)$$

$$\frac{\Gamma, A, B \vdash \Delta}{\Gamma, A \wedge B \vdash \Delta}\ (\wedge L) \qquad \frac{\Gamma \vdash A, B, \Delta}{\Gamma \vdash A \vee B, \Delta}\ (\vee R)$$

$$\frac{\Gamma, A \vdash \Delta \quad \Gamma, B \vdash \Delta}{\Gamma, A \vee B \vdash \Delta}\ (\vee L) \qquad \frac{\Gamma \vdash A, \Delta \quad \Gamma \vdash B, \Delta}{\Gamma \vdash A \wedge B, \Delta}\ (\wedge R)$$

a **sequent** Γ ⊢ Δ,

where Γ and Δ are finite sets of expressions

is **valid** iff

whenever every expression in Γ is true

some expression in Δ is true

Gerhard Karl Erich Gentzen (November 24, 1909 – August 4, 1945)

Gentzen's Rules (I)

$$\frac{}{\Gamma, A \vdash \Delta, A}\ (I)$$

$$\frac{\Gamma, A, B \vdash \Delta}{\Gamma, A \wedge B \vdash \Delta}\ (\wedge L) \qquad \frac{\Gamma \vdash A, B, \Delta}{\Gamma \vdash A \vee B, \Delta}\ (\vee R)$$

$$\frac{\Gamma, A \vdash \Delta \quad \Gamma, B \vdash \Delta}{\Gamma, A \vee B \vdash \Delta}\ (\vee L) \qquad \frac{\Gamma \vdash A, \Delta \quad \Gamma \vdash B, \Delta}{\Gamma \vdash A \wedge B, \Delta}\ (\wedge R)$$

a **counterexample** to the **sequent** $\Gamma \vdash \Delta$,
is a valuation that makes
every expression in Γ true
and
every expression in Δ false

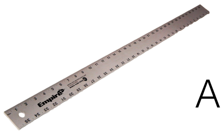
(a sequent is valid iff it has no counterexample)

$$\dfrac{\dfrac{\dfrac{}{A, B \vdash A, B}\ (I)}{A \wedge B \vdash A, B}\ (\wedge L)}{A \wedge B \vdash A \vee B}\ (\vee R)$$

A rule

$$\frac{\Gamma, A \vdash B, \Delta}{\Gamma \vdash A \to B, \Delta}\ (\to R)$$

A valuation is a counterexample to the top line
iff it is a counterexample to the bottom line



Another rule

$$\frac{\Gamma \vdash A, \Delta \quad \Gamma, B \vdash \Delta}{\Gamma, A \rightarrow B \vdash \Delta} \; (\rightarrow L)$$

A valuation is a counterexample to the bottom line
iff it is a counterexample to
at least one of the entailments on the top line

a valuation is a counterexample to the conclusion ***iff*** *it is a counterexample to at least one assumption*

$$\frac{}{\Gamma, A \vdash \Delta, A}\ (I)$$

$$\frac{\Gamma, A, B \vdash \Delta}{\Gamma, A \wedge B \vdash \Delta}\ (\wedge L) \qquad \frac{\Gamma \vdash A, B, \Delta}{\Gamma \vdash A \vee B, \Delta}\ (\vee R)$$

$$\frac{\Gamma, A \vdash \Delta \quad \Gamma, B \vdash \Delta}{\Gamma, A \vee B \vdash \Delta}\ (\vee L) \qquad \frac{\Gamma \vdash A, \Delta \quad \Gamma \vdash B, \Delta}{\Gamma \vdash A \wedge B, \Delta}\ (\wedge R)$$

$$\frac{\Gamma \vdash A, \Delta \quad \Gamma, B \vdash \Delta}{\Gamma, A \to B \vdash \Delta}\ (\to L) \qquad \frac{\Gamma, A \vdash B, \Delta}{\Gamma \vdash A \to B, \Delta}\ (\to R)$$

$$\frac{\Gamma \vdash A, \Delta}{\Gamma, \neg A \vdash \Delta}\ (\neg L) \qquad \frac{\Gamma, A \vdash \Delta}{\Gamma \vdash \neg A, \Delta}\ (\neg R)$$

$$\frac{}{\Gamma, A \vdash \Delta, A}\ (I)$$

$$\frac{\Gamma \vdash A, \Delta \quad \Gamma, B \vdash \Delta}{\Gamma, A \to B \vdash \Delta}\ (\to L) \qquad \frac{\Gamma, A \vdash B, \Delta}{\Gamma \vdash A \to B, \Delta}\ (\to R)$$

$$\frac{??}{A \to (B \to C) \vdash B \to (A \to C)}$$

$$\frac{}{\Gamma, A \vdash \Delta, A}\ (I)$$

$$\frac{\Gamma \vdash A, \Delta \quad \Gamma, B \vdash \Delta}{\Gamma, A \to B \vdash \Delta}\ (\to L) \qquad \frac{\Gamma, A \vdash B, \Delta}{\Gamma \vdash A \to B, \Delta}\ (\to R)$$

$$\dfrac{\dfrac{??}{A \to (B \to C), B \vdash A \to C}}{A \to (B \to C) \vdash B \to (A \to C)}\ (\to R)$$

$$\frac{}{\Gamma, A \vdash \Delta, A}\ (I)$$

$$\frac{\Gamma \vdash A, \Delta \quad \Gamma, B \vdash \Delta}{\Gamma, A \to B \vdash \Delta}\ (\to L) \qquad \frac{\Gamma, A \vdash B, \Delta}{\Gamma \vdash A \to B, \Delta}\ (\to R)$$

$$\cfrac{\cfrac{\cfrac{??}{A \to (B \to C), B, A \vdash C}}{A \to (B \to C), B \vdash A \to C}\ (\to R)}{A \to (B \to C) \vdash B \to (A \to C)}\ (\to R)$$

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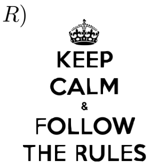
$$\frac{}{\Gamma, A \vdash \Delta, A}\ (I)$$

$$\frac{\Gamma \vdash A, \Delta \quad \Gamma, B \vdash \Delta}{\Gamma, A \to B \vdash \Delta}\ (\to L) \qquad \frac{\Gamma, A \vdash B, \Delta}{\Gamma \vdash A \to B, \Delta}\ (\to R)$$

$$\dfrac{\dfrac{\dfrac{\dfrac{}{B, A \vdash A, C}\ (I) \quad \dfrac{??}{B \to C, B, A \vdash C}}{A \to (B \to C), B, A \vdash C}\ (\to L)}{A \to (B \to C), B \vdash A \to C}\ (\to R)}{A \to (B \to C) \vdash B \to (A \to C)}\ (\to R)$$



$$\frac{}{\Gamma, A \vdash \Delta, A}\ (I)$$

$$\frac{\Gamma \vdash A, \Delta \quad \Gamma, B \vdash \Delta}{\Gamma, A \to B \vdash \Delta}\ (\to L) \qquad \frac{\Gamma, A \vdash B, \Delta}{\Gamma \vdash A \to B, \Delta}\ (\to R)$$

$$\dfrac{\dfrac{\dfrac{\dfrac{}{B, A \vdash A, C}\,(I) \quad \dfrac{\dfrac{}{B, A \vdash B, C}\,(I) \quad \dfrac{}{C, B, A \vdash C}\,(I)}{B \to C, B, A \vdash C}\,(\to L)}{A \to (B \to C), B, A \vdash C}\,(\to L)}{A \to (B \to C), B \vdash A \to C}\,(\to R)}{A \to (B \to C) \vdash B \to (A \to C)}\,(\to R)$$

$$\frac{??}{A \to (B \to C) \vdash B \to (C \to A)}$$

$$\dfrac{\dfrac{??}{A \to (B \to C), B \vdash C \to A}}{A \to (B \to C) \vdash B \to (C \to A)}\ (\to R)$$

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$$\dfrac{\dfrac{\dfrac{??}{A \to (B \to C), B, C \vdash A}}{A \to (B \to C), B \vdash C \to A}\ (\to R)}{A \to (B \to C) \vdash B \to (C \to A)}\ (\to R)$$

$$\dfrac{\dfrac{\dfrac{B, C \vdash A \qquad \dfrac{??}{B \to C, B, C \vdash A}}{A \to (B \to C), B, C \vdash A}\,(\to L)}{A \to (B \to C), B \vdash C \to A}\,(\to R)}{A \to (B \to C) \vdash B \to (C \to A)}\,(\to R)$$

$$\dfrac{\dfrac{\dfrac{B, C \vdash A \qquad \dfrac{\dfrac{}{B, C \vdash B, A}\,(I) \qquad B, C \vdash A}{B \to C, B, C \vdash A}\,(\to L)}{A \to (B \to C), B, C \vdash A}\,(\to L)}{A \to (B \to C), B \vdash C \to A}\,(\to R)}{A \to (B \to C) \vdash B \to (C \to A)}\,(\to R)$$

$$\frac{\Gamma \vdash A, \Delta \quad \Gamma, B \vdash \Delta}{\Gamma, A \to B \vdash \Delta} \ (\to L)$$

a **counterexample** to the **sequent** $\Gamma \vdash A, \Delta$
is a **counterexample** to $\Gamma, A \to B \vdash \Delta$
(since if A is false then $A \to B$ is true)

a **counterexample** to the **sequent** $\Gamma, B \vdash \Delta$
is a **counterexample** to $\Gamma, A \to B \vdash \Delta$
(since if B is true then $A \to B$ is true)

$$\frac{\Gamma, A \vdash B, \Delta}{\Gamma \vdash A \to B, \Delta}\ (\to R)$$

a **counterexample** to Γ, A ⊢ B, Δ
is a **counterexample** to Γ ⊢ A→B, Δ
(if A is true and B false then A→B is false)

$$\frac{\Gamma \vdash A, \Delta \quad \Gamma, B \vdash \Delta}{\Gamma, A \to B \vdash \Delta} \ (\to L) \qquad \frac{\Gamma, A \vdash B, \Delta}{\Gamma \vdash A \to B, \Delta} \ (\to R)$$

for these rules,
a **counterexample** to any assumption
is a **counterexample** to the conclusion

counterexample

$$B, C \nvdash A \qquad B = \top, C = \top, A = \bot$$

$$\dfrac{B, C \vdash A \qquad \dfrac{\dfrac{}{B, C \vdash B, A}\,(I) \qquad B, C \vdash A}{B \to C, B, C \vdash A}\,(\to L)}{\dfrac{A \to (B \to C), B, C \vdash A}{\dfrac{A \to (B \to C), B \vdash C \to A}{A \to (B \to C) \vdash B \to (C \to A)}\,(\to R)}\,(\to R)}\,(\to L)$$

$$A \to (B \to C) = \top \qquad B \vdash C \to A = \bot$$

$$A \to (B \to C) \nvdash B \to (C \to A)$$

$$\frac{}{\Gamma, A \vdash \Delta, A}\ (I)$$

$$\frac{\Gamma, A, B \vdash \Delta}{\Gamma, A \wedge B \vdash \Delta}\ (\wedge L) \qquad \frac{\Gamma \vdash A, B, \Delta}{\Gamma \vdash A \vee B, \Delta}\ (\vee R)$$

$$\frac{\Gamma, A \vdash \Delta \quad \Gamma, B \vdash \Delta}{\Gamma, A \vee B \vdash \Delta}\ (\vee L) \qquad \frac{\Gamma \vdash A, \Delta \quad \Gamma \vdash B, \Delta}{\Gamma \vdash A \wedge B, \Delta}\ (\wedge R)$$

$$\frac{\Gamma \vdash A, \Delta \quad \Gamma, B \vdash \Delta}{\Gamma, A \rightarrow B \vdash \Delta}\ (\rightarrow L) \qquad \frac{\Gamma, A \vdash B, \Delta}{\Gamma \vdash A \rightarrow B, \Delta}\ (\rightarrow R)$$

$$\frac{\Gamma \vdash A, \Delta}{\Gamma, \neg A \vdash \Delta}\ (\neg L) \qquad \frac{\Gamma, A \vdash \Delta}{\Gamma \vdash \neg A, \Delta}\ (\neg R)$$

for all these (sound) rules,
a **counterexample** to any assumption
is a **counterexample** to the conclusion

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$$\frac{}{\Gamma, A \vdash \Delta, A}\ (I)$$

$$\frac{\Gamma, A, B \vdash \Delta}{\Gamma, A \wedge B \vdash \Delta}\ (\wedge L) \qquad \frac{\Gamma \vdash A, B, \Delta}{\Gamma \vdash A \vee B, \Delta}\ (\vee R)$$

$$\frac{\Gamma, A \vdash \Delta \quad \Gamma, B \vdash \Delta}{\Gamma, A \vee B \vdash \Delta}\ (\vee L) \qquad \frac{\Gamma \vdash A, \Delta \quad \Gamma \vdash B, \Delta}{\Gamma \vdash A \wedge B, \Delta}\ (\wedge R)$$

$$\frac{\Gamma \vdash A, \Delta \quad \Gamma, B \vdash \Delta}{\Gamma, A \to B \vdash \Delta}\ (\to L) \qquad \frac{\Gamma, A \vdash B, \Delta}{\Gamma \vdash A \to B, \Delta}\ (\to R)$$

$$\frac{\Gamma \vdash A, \Delta}{\Gamma, \neg A \vdash \Delta}\ (\neg L) \qquad \frac{\Gamma, A \vdash \Delta}{\Gamma \vdash \neg A, \Delta}\ (\neg R)$$

Each of Gentzen's rules is sound:

$\therefore$ if a sequent can be proved using these rules it is valid

¿ if a sequent is valid can it be proved ?

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$$\frac{}{\Gamma, A \vdash \Delta, A}\ (I)$$

$$\frac{\Gamma, A, B \vdash \Delta}{\Gamma, A \wedge B \vdash \Delta}\ (\wedge L) \qquad \frac{\Gamma \vdash A, B, \Delta}{\Gamma \vdash A \vee B, \Delta}\ (\vee R)$$

$$\frac{\Gamma, A \vdash \Delta \quad \Gamma, B \vdash \Delta}{\Gamma, A \vee B \vdash \Delta}\ (\vee L) \qquad \frac{\Gamma \vdash A, \Delta \quad \Gamma \vdash B, \Delta}{\Gamma \vdash A \wedge B, \Delta}\ (\wedge R)$$

$$\frac{\Gamma \vdash A, \Delta \quad \Gamma, B \vdash \Delta}{\Gamma, A \to B \vdash \Delta}\ (\to L) \qquad \frac{\Gamma, A \vdash B, \Delta}{\Gamma \vdash A \to B, \Delta}\ (\to R)$$

$$\frac{\Gamma \vdash A, \Delta}{\Gamma, \neg A \vdash \Delta}\ (\neg L) \qquad \frac{\Gamma, A \vdash \Delta}{\Gamma \vdash \neg A, \Delta}\ (\neg R)$$

Each of Gentzen's rules has the property that:

a counterexample to any of its assumptions
is also
a counterexample to its conclusion

if the search for a proof fails,
we can use this property to provide a counterexample to the conclusion

Gentzen's rules are sound and complete

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we apply the rules, until we can do no more;
at each step there are fewer connectives
in the assumptions than in the conclusion

eventually we run out of connectives,
at which point, only atoms remain
either $\Gamma \cap \Delta = \varnothing$
in which case we can construct a counterexample
or some atom occurs in both Γ *and* Δ
so, we can apply rule I to discharge the assumption

if all assumptions are discharged we have a proof;
otherwise,
any counterexample can be pushed down the tree to
show that the conclusion is not valid

$$\frac{}{\Gamma, A \vdash \Delta, A}\ (I)$$

$$\frac{\Gamma, A, B \vdash \Delta}{\Gamma, A \wedge B \vdash \Delta}\ (\wedge L) \qquad \frac{\Gamma \vdash A, B, \Delta}{\Gamma \vdash A \vee B, \Delta}\ (\vee R)$$

$$\frac{\Gamma, A \vdash \Delta \quad \Gamma, B \vdash \Delta}{\Gamma, A \vee B \vdash \Delta}\ (\vee L) \qquad \frac{\Gamma \vdash A, \Delta \quad \Gamma \vdash B, \Delta}{\Gamma \vdash A \wedge B, \Delta}\ (\wedge R)$$

$$\frac{\Gamma \vdash A, \Delta \quad \Gamma, B \vdash \Delta}{\Gamma, A \to B \vdash \Delta}\ (\to L) \qquad \frac{\Gamma, A \vdash B, \Delta}{\Gamma \vdash A \to B, \Delta}\ (\to R)$$

$$\frac{\Gamma \vdash A, \Delta}{\Gamma, \neg A \vdash \Delta}\ (\neg L) \qquad \frac{\Gamma, A \vdash \Delta}{\Gamma \vdash \neg A, \Delta}\ (\neg R)$$

This shows that Gentzen's set of rules is ***complete,*** *that is to say:*

if a sequent is valid then it has a proof

(without assumptions)