Chapter 4
Propositional Proofs

§4.1 Introduction

Semantic methods for checking logical entailment have the merit of being conceptually simple; they directly manipulate interpretations of sentences. Unfortunately, the number of interpretations of a language grows exponentially with the number of logical constants. When the number of logical constants in a propositional language is large, the number of interpretations may be impossible to manipulate.

Proof methods provide an alternative way of checking and communicating logical entailment that addresses this problem. In many cases, it is possible to create a ``proof'' of a conclusion from a set of premises that is much smaller than the truth table for the language; moreover, it is often possible to find such proofs with less work than is necessary to check the entire truth table.

§4.2 Schemata

An important component in our treatment of proofs is the notion of a schema. A *schema* is an expression satisfying the grammatical rules of our language except for the occurrence of *metavariables* in place of various subparts of the expression. For example, the following expression is a pattern with metavariables φ and ψ.

$$\varphi \Rightarrow (\psi \Rightarrow \varphi)$$

An instance of a sentence schema is the expression obtained by substituting expressions of the appropriate sort for the metavariables in the pattern so that the result is a legal expression. For example, the following is an instance of the preceding pattern.

$$p \Rightarrow (q \Rightarrow p)$$

Note that, if a metavariable occurs more than once, the same expression must be used for every occurrence. In the preceding example, it would not be acceptable to replace one occurrence of φ with one expression and the other occurrence with a different expression.

Note also that the replacement can be an arbitrary expression so long as the result is a legal expression. For example, in the following instance, we have replaced the variables by complex sentences.

$$(p \Rightarrow r) \Rightarrow ((p \Rightarrow q) \Rightarrow (p \Rightarrow r))$$

§4.3 Rules of Inference

The basis for proof methods is the use of correct rules of inference' that can be applied directly to sentences to derive conclusions that are guaranteed to be correct under all interpretations. Since the interpretations are not enumerated, time and space can often be saved.

A *rule of inference* is a pattern of reasoning consisting of one set of sentence schemata, called *premises*, and a second set of sentence schemata, called *conclusions*. A rule of inference is *sound* if and only if, for every instance, the premises logically entail the conclusions.

The following is a rule of inference called *Modus Ponens*.

$$\frac{\begin{array}{l}\varphi \Rightarrow \psi \\ \varphi\end{array}}{\psi}$$

An instance of a rule of inference is the rule obtained by consistently substituting expressions for the metavariables in the rule so that all premises and conclusions are legal sentences. The following are all legal instances of Modus Ponens.

$$\frac{\begin{array}{l}\mathit{raining} \Rightarrow \mathit{wet} \\ \mathit{raining}\end{array}}{\mathit{wet}} \qquad \frac{\begin{array}{l}\mathit{wet} \Rightarrow \mathit{slippery} \\ \mathit{wet}\end{array}}{\mathit{slippery}}$$

$$\frac{\begin{array}{l}p \Rightarrow (q \Rightarrow r) \\ p\end{array}}{q \Rightarrow r} \qquad \frac{\begin{array}{l}(p \Rightarrow q) \Rightarrow r \\ p \Rightarrow q\end{array}}{r}$$

In applying rules of inference, it is important to remember that they apply only to top-level sentences, not to components of sentences. While this sometimes works, it can also lead to incorrect results.

As an example of such a problem, consider the incorrect application of Modus Ponens shown below. If it is raining, it is cloudy. If it is raining, then the ground is wet. We might try to apply Modus Ponens here, taking the first premise as the implication and taking the occurrence of *raining* in the second premise as the matching condition, leading us to conclude that, if it is cloudy, the ground is wet.

$$\frac{\begin{array}{l}\mathit{raining} \Rightarrow \mathit{cloudy} \\ \mathit{raining} \Rightarrow \mathit{wet}\end{array}}{\mathit{cloudy} \Rightarrow \mathit{wet}}$$

Unfortunately, this is not a proper logical conclusion from the premises, as we all know from experience and as we can quickly determine by looking at the associated truth table. It is important to remember that rules of inference apply only to top-level sentences.

§4.4 Axiom Schemata

If φ is a valid sentence, then it is logically entailed by any set of premises. In fact, it should be a proof of φ without assuming anything. Unfortunately, if there are no premises, there is no place to apply our rules of inference. The significance of this observation is that we need some rules of inference without premises to get started. Such rules are called *axiom schemata*.

The *implication introduction schema* (II), together with Modus Ponens, allows us to infer implications.

$$\varphi \Rightarrow (\psi \Rightarrow \varphi)$$

The *implication distribution schema* (ID) allows us to distribute one implication over another. If a sentence φ implies that ψ implies χ, then, if φ implies ψ, φ implies χ.

$$(\varphi \Rightarrow (\psi \Rightarrow \chi)) \Rightarrow ((\varphi \Rightarrow \psi) \Rightarrow (\varphi \Rightarrow \chi))$$

The *contradiction realization schemata* (CR) permit us to infer a sentence if the negation of that sentence implies some sentence and its negation.

$$(\psi \Rightarrow \neg\varphi) \Rightarrow ((\psi \Rightarrow \varphi) \Rightarrow \neg\psi)$$
$$(\neg\psi \Rightarrow \neg\varphi) \Rightarrow ((\neg\psi \Rightarrow \varphi) \Rightarrow \psi)$$

The *equivalence schemata* (EQ) capture the meaning of the ⇔ operator. If we know that two sentences are equivalent, then the first implies the second, and the second implies the first. If we know that two sentences imply each other, then we know that they are equivalent.

$$(\varphi \Leftrightarrow \psi) \Rightarrow (\varphi \Rightarrow \psi)$$
$$(\varphi \Leftrightarrow \psi) \Rightarrow (\psi \Rightarrow \varphi)$$
$$(\varphi \Rightarrow \psi) \Rightarrow ((\psi \Rightarrow \varphi) \Rightarrow (\psi \Leftrightarrow \varphi))$$

The meaning of the other operators in propositional logic is captured in the following axiom schemata.

$$(\varphi \Leftarrow \psi) \Leftrightarrow (\psi \Rightarrow \varphi)$$
$$(\varphi \vee \psi) \Leftrightarrow (\neg\varphi \Rightarrow \psi)$$
$$(\varphi \wedge \psi) \Leftrightarrow \neg(\neg\varphi \vee \neg\psi)$$

The axiom schemata in this section are jointly called the *standard axiom schemata* for Propositional Logic. The interesting thing about the standard axiom schemata is that, together with Modus Ponens, they alone are sufficient to prove all logical consequences from any set of premises.

§4.5 Proofs

Note that, by stringing together applications of rules of inference, it is possible to derive conclusions that cannot be derived in a single step. Suppose, for example, we are given the following premises. When it is raining, the ground is wet. When the ground is wet, it is slippery. It is raining. From these premises, we can prove that it is slippery. Using Modus Ponens on the first and third premise, we can derive the fact that it is wet. Applying Modus Ponens to the second premise and the conclusion just derived, we arrive at the desired conclusion. This idea of stringing together rule applications leads to the notion of a proof.

A *proof* of a conclusion from a set of premises is a sequence of sentences terminating in the conclusion in which each item is either (1) a premise, (2) an instance of an axiom schema, or (3) the result of applying a rule of inference to earlier items in sequence.

Here is an example. Whenever p is true, q is true. Whenever q is true, r is true. With these as premises, we can prove that, whenever p is true, r is true.

1.	$p \Rightarrow q$	Premise
2.	$q \Rightarrow r$	Premise
3.	$(q \Rightarrow r) \Rightarrow (p \Rightarrow (q \Rightarrow r))$	II
4.	$p \Rightarrow (q \Rightarrow r)$	MP : 3,2
5.	$(p \Rightarrow (q \Rightarrow r)) \Rightarrow ((p \Rightarrow q) \Rightarrow (p \Rightarrow r))$	ID
6.	$(p \Rightarrow q) \Rightarrow (p \Rightarrow r)$	MP : 5,4
7.	$p \Rightarrow r$	MP : 6,1

If there exists a proof of a sentence φ from a set Δ of premises and the standard axiom schemata using Modus Ponens, then φ is said to be *provable* from Δ (written as Δ |- φ) and is called a *theorem* of Δ.

Earlier, it was suggested that there is a close connection between provability and logical entailment. In fact, they are equivalent. A set of sentences Δ logically entails a sentence φ if and only if φ is provable from Δ.

Soundness Theorem: If φ is provable from Δ, then Δ logically entails φ.

Completeness Theorem: If Δ logically entails φ, then φ is provable from Δ.

The concept of provability is important because it suggests how we can automate the determination of logical entailment. Starting from a set of premises Δ, we enumerate conclusions from this set. If a sentence φ appears, then it is provable from Δ and is, therefore, a logical consequence. If the negation of φ appears, then ¬φ is a logical consequence of Δ and φ is not logically entailed (unless Δ is inconsistent). Note that it is possible that neither φ nor $\neg\varphi$ will appear.

Exercises

1. *Propositional Proof.* Give a proof of $((p \Rightarrow \neg r) \Rightarrow \neg p)$ from the premise $(p \Rightarrow q)$ and $(q \Rightarrow r)$ using Modus Ponens and the standard axiom schemata.

2. *Propositional Proof.* Give a formal proof of the sentence p from the single premise $\neg\neg p$ using only Modus Ponens and the standard axiom schemata. You should be able to do this using only Implication Introduction, Implication Distribution, and Contradiction Realization. Warning: This is surprisingly difficult. Though it takes no more than about ten steps, the proof is non-obvious. This problem illustrates the difficulties of working with proof methods devised more for minimality than ease of use.

3. *Distinctions and Confusions.* Distinguish the following three statements.
   (a) $p \Rightarrow q$
   (b) $p \models q$
   (c) $p \vdash q$