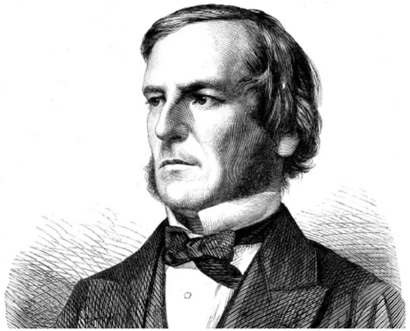
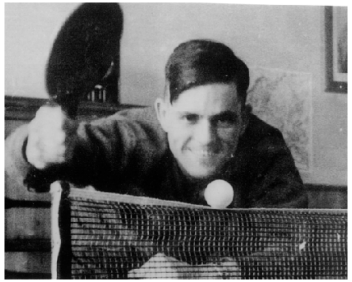
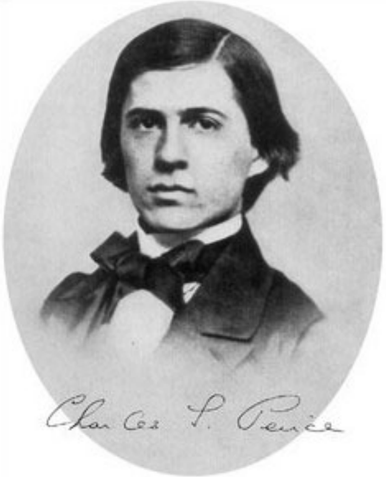
Sequents



George Boole 1815—1864



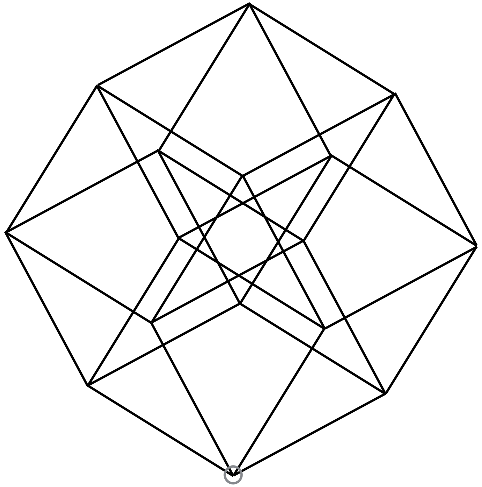
Gerhard Gentzen 1909—1945

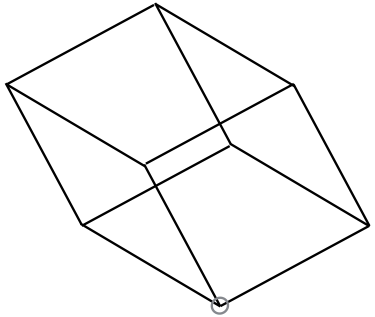


Charles Peirce 1839—1914

https://en.wikipedia.org/wiki/Predicate_(mathematical_logic)

inf1a-cl
Michael Fourman





$$\begin{array}{c} c \models a \quad c \models b \\ \hline\hline c \models a \wedge b \end{array} \qquad\qquad \begin{array}{c} a \models c \quad b \models c \\ \hline\hline a \vee b \models c \end{array}$$

$$\begin{array}{c} \begin{array}{c} c \models \neg a \\ \hline\hline a \models \neg c \end{array} \quad \begin{array}{c} c \models \neg b \\ \hline\hline b \models \neg c \end{array} \\ \hline\hline \begin{array}{c} a \vee b \models \neg c \\ \hline\hline c \models \neg(a \vee b) \end{array} \end{array} \quad \text{and} \quad \begin{array}{c} c \models \neg a \quad c \models \neg b \\ \hline\hline c \models \neg a \wedge \neg b \end{array} \quad \text{so,} \ \begin{array}{c} c \models \neg(a \vee b) \\ \hline\hline c \models \neg a \wedge \neg b \end{array}$$

Substituting $\neg a \wedge \neg b$ and $\neg(a \vee b)$ for c gives,

$$\begin{array}{c} \hline \neg a \wedge \neg b \models \neg a \wedge \neg b \\ \hline \neg a \wedge \neg b \models \neg(a \vee b) \end{array} \qquad\qquad \begin{array}{c} \hline \neg(a \vee b) \models \neg(a \vee b) \\ \hline \neg(a \vee b) \models \neg a \vee \neg b \end{array}$$

$$\neg(a \vee b) = \neg a \wedge \neg b \qquad\qquad \text{(de Morgan)}$$

How should we treat $\vee$ on the right?

$$\dfrac{\dfrac{\dfrac{\dfrac{c \models a \vee b}{\neg(a \vee b) \models \neg c}}{\neg a \wedge \neg b \models \neg c}\ \text{de Morgan}}{\neg a, \neg b \models \neg c}}{c \models a, b}\ ??$$

$$\frac{a, b \vDash c}{\overline{a \wedge b \vDash c}} \qquad \frac{c \vDash a \quad c \vDash b}{\overline{c \vDash a \wedge b}}$$

$$\frac{a \vDash c \quad b \vDash c}{\overline{a \vee b \vDash c}} \qquad \frac{c \vDash a, b}{\overline{c \vDash a \vee b}}$$

$$\frac{c \vDash a, b}{c \vDash a \vee b}$$

Gerhard Gentzen
1909—1945

Every thing in c is in either a or b, or both.

A *sequent* is satisfied

$$a_0, \ldots, a_{n-1} \vDash s_0, \ldots, s_{m-1}$$

iff every thing that
is in **every** *antecedent*, a_i,
is in **some** *succedent*, s_j.
(a_i and s_j are predicates in some universe.)

Sequents

Gerhard Gentzen
1909—1945

A sequent is valid in a given universe
iff
whenever every antecedent holds then
some succedent holds

```
gamma |= delta =
  and[ or[ d x | d <- delta ]
     | x <- things, and[ g x | g <- gamma ]]
```

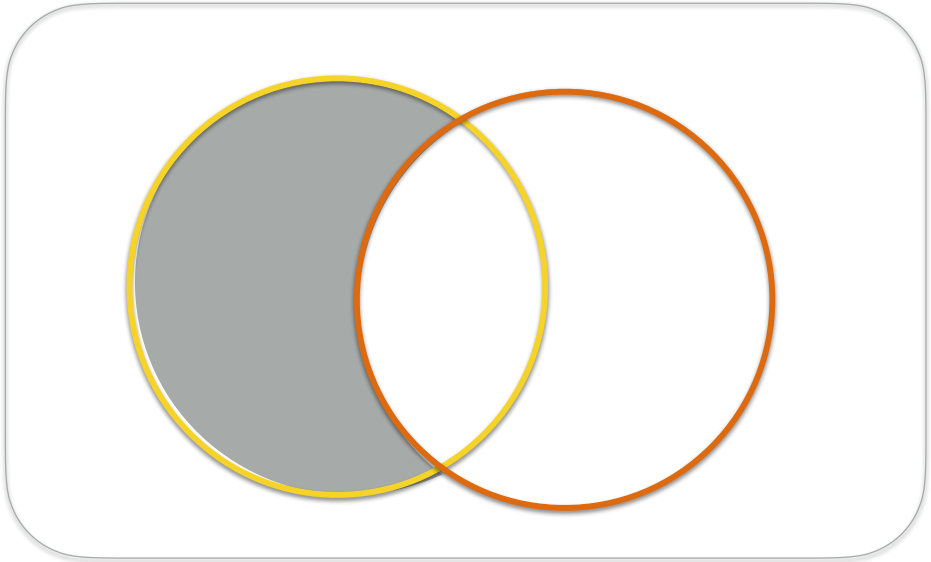
Here, Γ and Δ are finite sets of predicates.

$$\Gamma \vDash \Delta \textbf{ iff } \bigwedge \Gamma \subseteq \bigvee \Delta$$

The operations, $\bigwedge$, $\bigvee$, on predicates correspond to intersection, $\bigcap$, and union, $\bigcup$, of sets.

all **a** is **b**

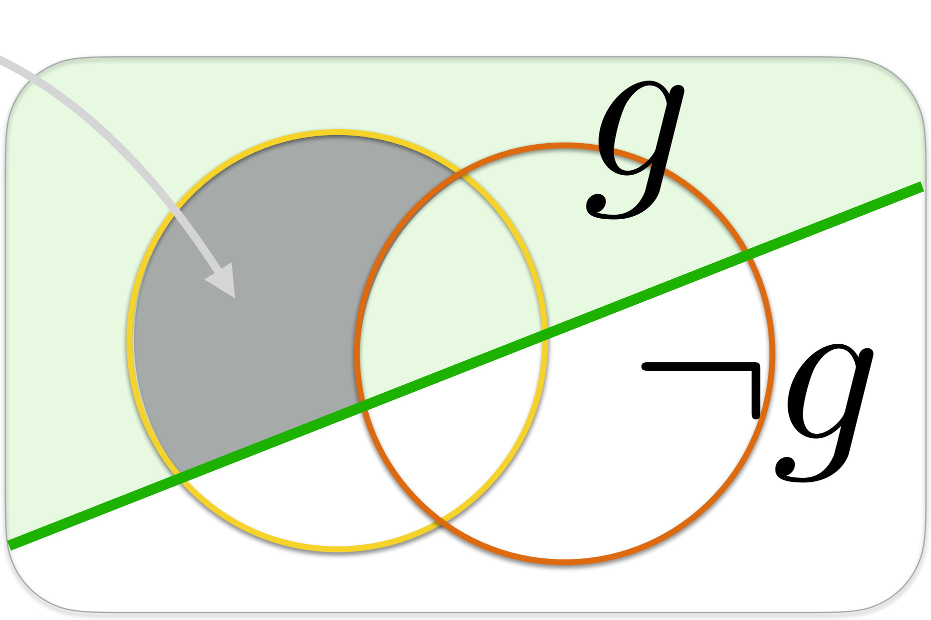
$$a \vDash b$$



What does this mean?

$$g, a \vDash b$$

$$g, a \vDash b$$

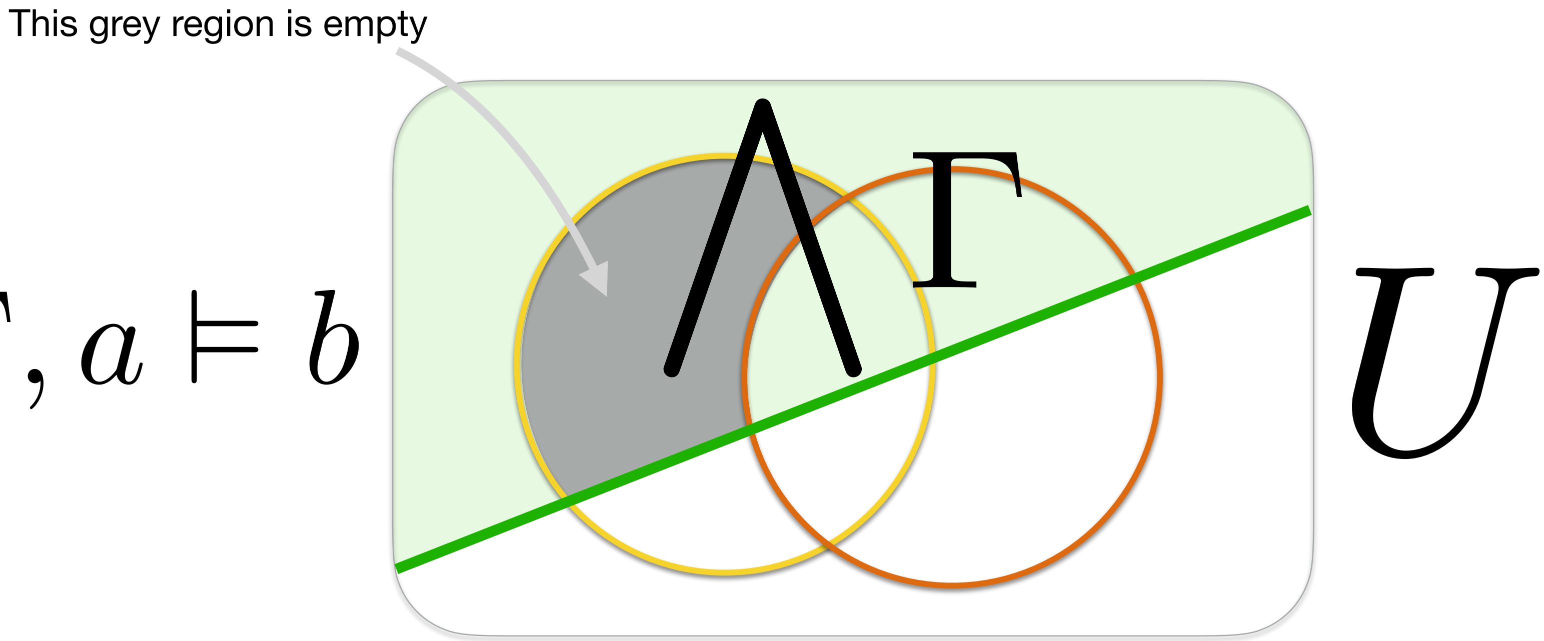


$g, a \vDash b$ is valid in U

iff

$a \vDash b$ is valid in $\{x \in U \mid g\, x\}$

$$\Gamma, a \vDash b$$

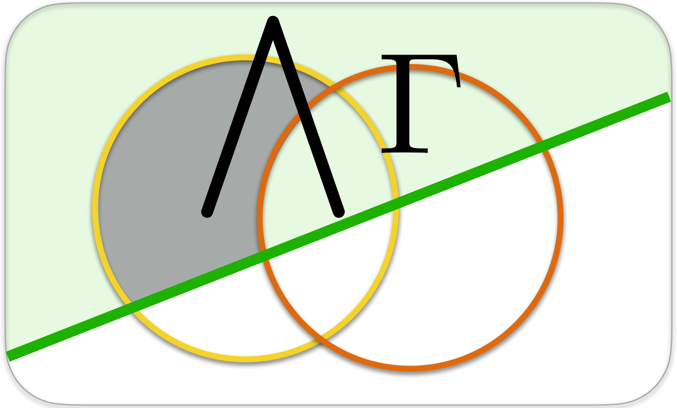


$\Gamma, a \vDash b$ is valid in U

iff

$a \vDash b$ is valid in $\left\{ x \in U \mid \bigwedge \Gamma x \right\}$

$$\Gamma, a \vDash b$$



$\Gamma, a \vDash b$ is valid in U

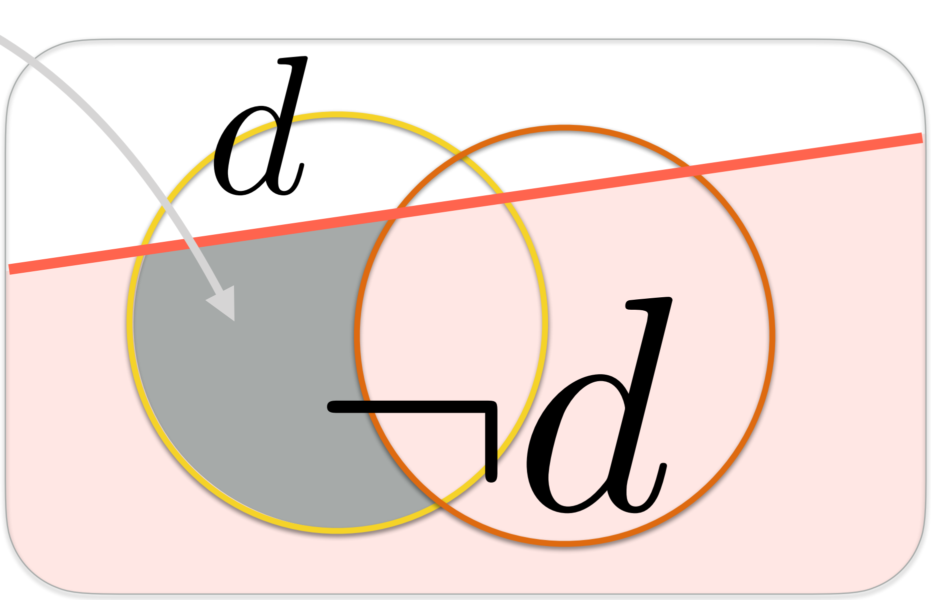
iff

$a \vDash b$ is valid in $\left\{ x \in U \mid \bigwedge \Gamma x \right\}$

$=$ the universe of those things for which every $g \in \Gamma$ is true

$$\frac{a \vDash b, d}{\neg d, a \vDash b}$$

This grey region is empty



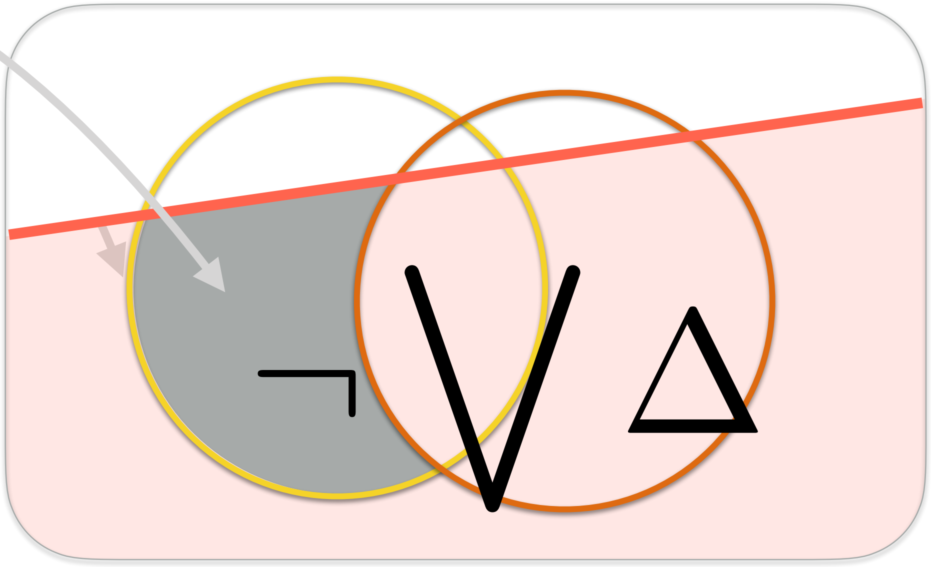
$a \vDash b, d$ is valid in U

iff

$a \vDash b$ is valid in $\{x \in U \mid \neg d\, x\}$

This grey region is empty

$$\frac{a \vDash b, \Delta}{\neg \bigvee \Delta, a \vDash b}$$

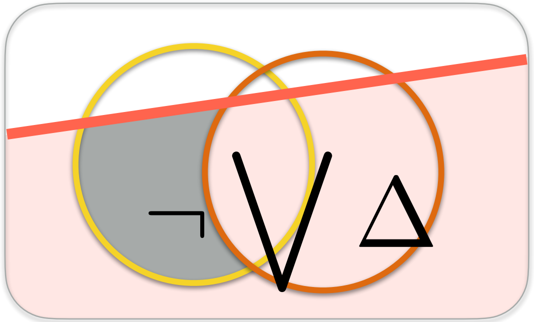


$a \vDash b, \Delta$ is valid in U

iff

$a \vDash b$ is valid in $\{x \in U \mid \neg \bigvee \Delta\, x\}$

$$a \vDash b, \Delta$$

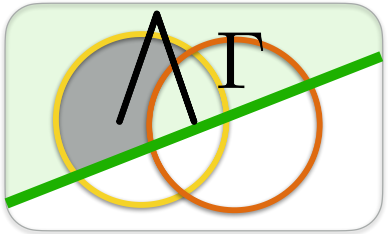


$a \vDash b, \Delta$ is valid in U

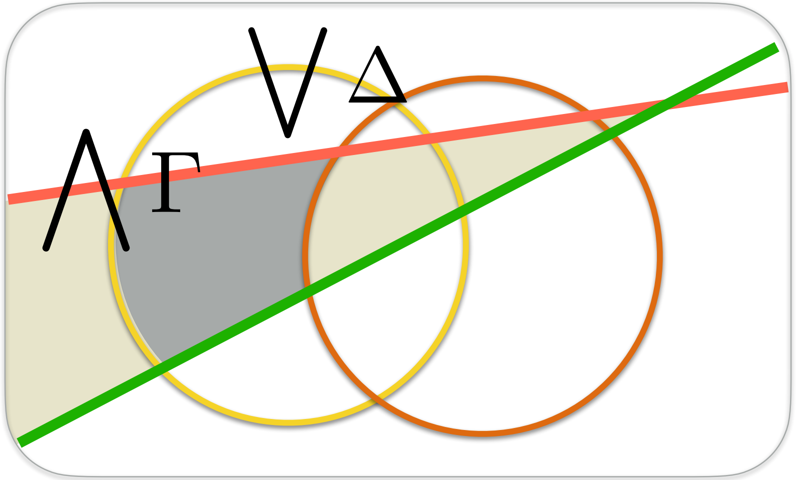
iff

$a \vDash b$ is valid in $\left\{ x \in U \mid \neg \bigvee \Delta \; x \right\}$

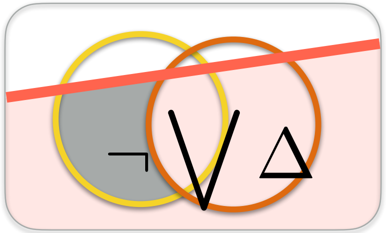
$=$ the universe of those things for which every $d \in \Delta$ is false where every $g \in \Gamma$ is true



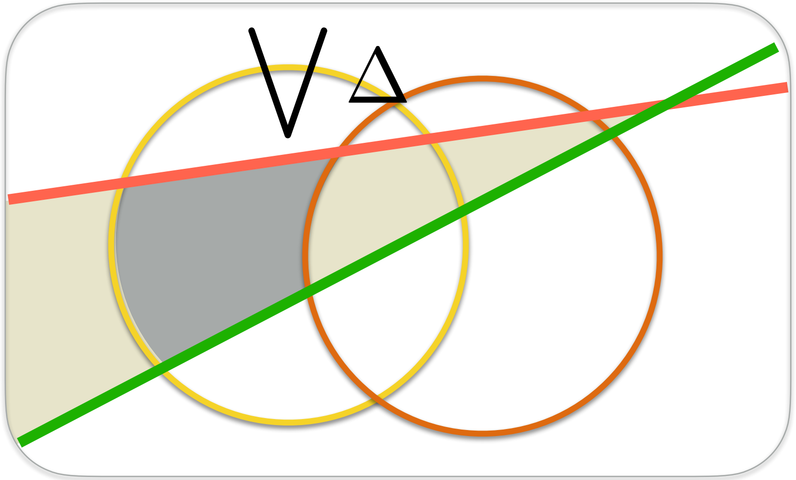
where every $g \in \Gamma$ is true
and every $d \in \Delta$ is false



where every $d \in \Delta$ is false



$$\Gamma, a \vDash b, \Delta$$



$\Gamma, a \vDash b, \Delta$ is valid in U

iff

$a \vDash b$ is valid in $\left\{ x \in U \mid \bigwedge \Gamma\ x \wedge \neg \bigvee \Delta\ x \right\}$

= the universe of those things
for which every $g \in \Gamma$ is true
and, every $d \in \Delta$ is false

$$\frac{a, b \models c}{\overline{\overline{a \wedge b \models c}}}$$

$$\frac{c \models a \quad c \models b}{\overline{\overline{c \models a \wedge b}}}$$

$$\frac{a \models c \quad b \models c}{\overline{\overline{a \vee b \models c}}}$$

$$\frac{c \models a, b}{\overline{\overline{c \models a \vee b}}}$$

$$\frac{\Gamma, a, b \vDash c, \Delta}{\Gamma, a \wedge b \vDash c, \Delta} \qquad \frac{\Gamma, c \vDash a, \Delta \quad \Gamma, c \vDash b, \Delta}{\Gamma, c \vDash a \wedge b, \Delta}$$

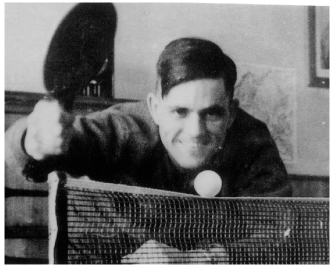
$$\frac{\Gamma, a \vDash c \quad b \vDash c, \Delta}{\Gamma, a \vee b \vDash c, \Delta} \qquad \frac{\Gamma, c \vDash a, b, \Delta}{\Gamma, c \vDash a \vee b, \Delta}$$

$$\frac{}{\Gamma, a \vDash \Delta, a}\ (I)$$

$$\frac{\Gamma, a, b \vDash \Delta}{\Gamma, a \wedge b \vDash \Delta}\ (\wedge L) \qquad \frac{\Gamma \vDash a, b, \Delta}{\Gamma \vDash a \vee b, \Delta}\ (\vee R)$$

$$\frac{\Gamma, a \vDash \Delta \quad \Gamma, b \vDash \Delta}{\Gamma, a \vee b \vDash \Delta}\ (\vee L) \qquad \frac{\Gamma \vDash a, \Delta \quad \Gamma \vDash b, \Delta}{\Gamma \vDash a \wedge b, \Delta}\ (\wedge R)$$

$$\frac{\Gamma \vDash a, \Delta}{\Gamma, \neg a \vDash \Delta}\ (\neg L) \qquad \frac{\Gamma, a \vDash \Delta}{\Gamma \vDash \neg a, \Delta}\ (\neg R)$$



- a and b are predicates from some universe,
- Γ, Δ are finite sets of predicates from some universe,
- Γ, a refers to $\Gamma \cup \{a\}$, and a, Δ refers to $\{a\} \cup \Delta$.

$$\frac{}{\Gamma, a \vDash \Delta, a}\ (I)$$

$$\frac{\Gamma, a, b \vDash \Delta}{\Gamma, a \wedge b \vDash \Delta}\ (\wedge L) \qquad \frac{\Gamma \vDash a, b, \Delta}{\Gamma \vDash a \vee b, \Delta}\ (\vee R)$$

$$\frac{\Gamma, a \vDash \Delta \quad \Gamma, b \vDash \Delta}{\Gamma, a \vee b \vDash \Delta}\ (\vee L) \qquad \frac{\Gamma \vDash a, \Delta \quad \Gamma \vDash b, \Delta}{\Gamma \vDash a \wedge b, \Delta}\ (\wedge R)$$

$$\frac{\Gamma \vDash a, \Delta}{\Gamma, \neg a \vDash \Delta}\ (\neg L) \qquad \frac{\Gamma, a \vDash \Delta}{\Gamma \vDash \neg a, \Delta}\ (\neg R)$$

$$\vDash ((\neg p \vee q) \wedge \neg p) \vee p$$

$$\frac{}{\Gamma, a \models \Delta, a}\ (I)$$

$$\frac{\Gamma, a, b \models \Delta}{\Gamma, a \wedge b \models \Delta}\ (\wedge L) \qquad \frac{\Gamma \models a, b, \Delta}{\Gamma \models a \vee b, \Delta}\ (\vee R)$$

$$\frac{\Gamma, a \models \Delta \quad \Gamma, b \models \Delta}{\Gamma, a \vee b \models \Delta}\ (\vee L) \qquad \frac{\Gamma \models a, \Delta \quad \Gamma \models b, \Delta}{\Gamma \models a \wedge b, \Delta}\ (\wedge R)$$

$$\frac{\Gamma \models a, \Delta}{\Gamma, \neg a \models \Delta}\ (\neg L) \qquad \frac{\Gamma, a \models \Delta}{\Gamma \models \neg a, \Delta}\ (\neg R)$$

$$\frac{\dfrac{\dfrac{\dfrac{\dfrac{}{p \models q, p}\ I}{\models \neg p, q, p}\ \neg R}{\models \neg p \vee q, p}\ \vee R \qquad \dfrac{\dfrac{}{p \models p}\ I}{\models \neg p, p}\ \neg R}{\models (\neg p \vee q) \wedge \neg p, p}\ \wedge R}{\models ((\neg p \vee q) \wedge \neg p) \vee p}\ \vee R$$

$$\frac{}{\Gamma, a \vDash \Delta, a}\ (I)$$

$$\frac{\Gamma, a, b \vDash \Delta}{\Gamma, a \land b \vDash \Delta}\ (\land L) \qquad \frac{\Gamma \vDash a, b, \Delta}{\Gamma \vDash a \lor b, \Delta}\ (\lor R)$$

$$\frac{\Gamma, a \vDash \Delta \quad \Gamma, b \vDash \Delta}{\Gamma, a \lor b \vDash \Delta}\ (\lor L) \qquad \frac{\Gamma \vDash a, \Delta \quad \Gamma \vDash b, \Delta}{\Gamma \vDash a \land b, \Delta}\ (\land R)$$

$$\frac{\Gamma \vDash a, \Delta}{\Gamma, \neg a \vDash \Delta}\ (\neg L) \qquad \frac{\Gamma, a \vDash \Delta}{\Gamma \vDash \neg a, \Delta}\ (\neg R)$$

$$\frac{\vDash (\neg p \lor q) \land \neg p, p}{\vDash ((\neg p \lor q) \land \neg p) \lor p}\ \lor R$$

$$\frac{}{\Gamma, a \vDash \Delta, a}\ (I)$$

$$\frac{\Gamma, a, b \vDash \Delta}{\Gamma, a \wedge b \vDash \Delta}\ (\wedge L) \qquad \frac{\Gamma \vDash a, b, \Delta}{\Gamma \vDash a \vee b, \Delta}\ (\vee R)$$

$$\frac{\Gamma, a \vDash \Delta \quad \Gamma, b \vDash \Delta}{\Gamma, a \vee b \vDash \Delta}\ (\vee L) \qquad \frac{\Gamma \vDash a, \Delta \quad \Gamma \vDash b, \Delta}{\Gamma \vDash a \wedge b, \Delta}\ (\wedge R)$$

$$\frac{\Gamma \vDash a, \Delta}{\Gamma, \neg a \vDash \Delta}\ (\neg L) \qquad \frac{\Gamma, a \vDash \Delta}{\Gamma \vDash \neg a, \Delta}\ (\neg R)$$

$$\dfrac{\dfrac{\dfrac{\dfrac{}{p \vDash q, p}\ I}{\vDash \neg p, q, p}\ \neg R}{\vDash \neg p \vee q, p}\ \vee R \qquad \dfrac{\dfrac{}{p \vDash p}\ I}{\vDash \neg p, p}\ \neg R}{\dfrac{\vDash (\neg p \vee q) \wedge \neg p, p}{\vDash ((\neg p \vee q) \wedge \neg p) \vee p}\ \vee R}\ \wedge R$$

$$\frac{}{\Gamma, a \vDash \Delta, a}\ (I)$$

$$\frac{\Gamma, a, b \vDash \Delta}{\Gamma, a \wedge b \vDash \Delta}\ (\wedge L) \qquad \frac{\Gamma \vDash a, b, \Delta}{\Gamma \vDash a \vee b, \Delta}\ (\vee R)$$

$$\frac{\Gamma, a \vDash \Delta \quad \Gamma, b \vDash \Delta}{\Gamma, a \vee b \vDash \Delta}\ (\vee L) \qquad \frac{\Gamma \vDash a, \Delta \quad \Gamma \vDash b, \Delta}{\Gamma \vDash a \wedge b, \Delta}\ (\wedge R)$$

$$\frac{\Gamma \vDash a, \Delta}{\Gamma, \neg a \vDash \Delta}\ (\neg L) \qquad \frac{\Gamma, a \vDash \Delta}{\Gamma \vDash \neg a, \Delta}\ (\neg R)$$

$$\frac{\dfrac{\dfrac{\dfrac{}{p \vDash q, p}\ I}{\vDash \neg p, q, p}\ \neg R}{\vDash \neg p \vee q, p}\ \vee R \qquad \dfrac{\dfrac{}{p \vDash p}\ I}{\vDash \neg p, p}\ \neg R}{\dfrac{\vDash (\neg p \vee q) \wedge \neg p, p}{\vDash ((\neg p \vee q) \wedge \neg p) \vee p}\ \vee R}\ \wedge R$$

$$\frac{}{\Gamma, a \vDash \Delta, a}\ (I)$$

$$\frac{\Gamma, a, b \vDash \Delta}{\Gamma, a \land b \vDash \Delta}\ (\land L) \qquad \frac{\Gamma \vDash a, b, \Delta}{\Gamma \vDash a \lor b, \Delta}\ (\lor R)$$

$$\frac{\Gamma, a \vDash \Delta \quad \Gamma, b \vDash \Delta}{\Gamma, a \lor b \vDash \Delta}\ (\lor L) \qquad \frac{\Gamma \vDash a, \Delta \quad \Gamma \vDash b, \Delta}{\Gamma \vDash a \land b, \Delta}\ (\land R)$$

$$\frac{\Gamma \vDash a, \Delta}{\Gamma, \neg a \vDash \Delta}\ (\neg L) \qquad \frac{\Gamma, a \vDash \Delta}{\Gamma \vDash \neg a, \Delta}\ (\neg R)$$

$$\dfrac{\dfrac{\dfrac{\dfrac{}{p \vDash q, p}\ I}{\vDash \neg p, q, p}\ \neg R}{\vDash \neg p \lor q, p}\ \lor R \qquad \dfrac{\dfrac{}{p \vDash p}\ I}{\vDash \neg p, p}\ \neg R}{\dfrac{\vDash (\neg p \lor q) \land \neg p, p}{\vDash ((\neg p \lor q) \land \neg p) \lor p}\ \lor R}\ \land R$$

$$\frac{}{\Gamma, a \vDash \Delta, a}\ (I)$$

$$\frac{\Gamma, a, b \vDash \Delta}{\Gamma, a \wedge b \vDash \Delta}\ (\wedge L) \qquad \frac{\Gamma \vDash a, b, \Delta}{\Gamma \vDash a \vee b, \Delta}\ (\vee R)$$

$$\frac{\Gamma, a \vDash \Delta \quad \Gamma, b \vDash \Delta}{\Gamma, a \vee b \vDash \Delta}\ (\vee L) \qquad \frac{\Gamma \vDash a, \Delta \quad \Gamma \vDash b, \Delta}{\Gamma \vDash a \wedge b, \Delta}\ (\wedge R)$$

$$\frac{\Gamma \vDash a, \Delta}{\Gamma, \neg a \vDash \Delta}\ (\neg L) \qquad \frac{\Gamma, a \vDash \Delta}{\Gamma \vDash \neg a, \Delta}\ (\neg R)$$

$$\dfrac{\dfrac{\dfrac{\dfrac{}{p \vDash q, p}\ I}{\vDash \neg p, q, p}\ \neg R}{\vDash \neg p \vee q, p}\ \vee R \qquad \dfrac{\dfrac{}{p \vDash p}\ I}{\vDash \neg p, p}\ \neg R}{\dfrac{\vDash (\neg p \vee q) \wedge \neg p, p}{\vDash ((\neg p \vee q) \wedge \neg p) \vee p}\ \vee R}\ \wedge R$$

$$\frac{}{\Gamma, a \vDash \Delta, a}\ (I)$$

$$\frac{\Gamma, a, b \vDash \Delta}{\Gamma, a \wedge b \vDash \Delta}\ (\wedge L) \qquad \frac{\Gamma \vDash a, b, \Delta}{\Gamma \vDash a \vee b, \Delta}\ (\vee R)$$

$$\frac{\Gamma, a \vDash \Delta \quad \Gamma, b \vDash \Delta}{\Gamma, a \vee b \vDash \Delta}\ (\vee L) \qquad \frac{\Gamma \vDash a, \Delta \quad \Gamma \vDash b, \Delta}{\Gamma \vDash a \wedge b, \Delta}\ (\wedge R)$$

$$\frac{\Gamma \vDash a, \Delta}{\Gamma, \neg a \vDash \Delta}\ (\neg L) \qquad \frac{\Gamma, a \vDash \Delta}{\Gamma \vDash \neg a, \Delta}\ (\neg R)$$

$$\dfrac{\dfrac{\dfrac{\dfrac{}{p \vDash q, p}\ I}{\vDash \neg p, q, p}\ \neg R}{\vDash \neg p \vee q, p}\ \vee R \qquad \dfrac{\dfrac{}{p \vDash p}\ I}{\vDash \neg p, p}\ \neg R}{\dfrac{\vDash (\neg p \vee q) \wedge \neg p, p}{\vDash ((\neg p \vee q) \wedge \neg p) \vee p}\ \vee R}\ \wedge R$$

$$\frac{}{\Gamma, a \vDash \Delta, a}\ (I)$$

$$\frac{\Gamma, a, b \vDash \Delta}{\Gamma, a \wedge b \vDash \Delta}\ (\wedge L) \qquad \frac{\Gamma \vDash a, b, \Delta}{\Gamma \vDash a \vee b, \Delta}\ (\vee R)$$

$$\frac{\Gamma, a \vDash \Delta \quad \Gamma, b \vDash \Delta}{\Gamma, a \vee b \vDash \Delta}\ (\vee L) \qquad \frac{\Gamma \vDash a, \Delta \quad \Gamma \vDash b, \Delta}{\Gamma \vDash a \wedge b, \Delta}\ (\wedge R)$$

$$\frac{\Gamma \vDash a, \Delta}{\Gamma, \neg a \vDash \Delta}\ (\neg L) \qquad \frac{\Gamma, a \vDash \Delta}{\Gamma \vDash \neg a, \Delta}\ (\neg R)$$

$$\frac{\dfrac{}{p \vDash q, p}\ I}{\vDash \neg p, q, p}\ \neg R$$

$$\frac{\dfrac{\vDash \neg p, q, p}{\vDash \neg p \vee q, p}\ \vee R \qquad \dfrac{\dfrac{}{p \vDash p}\ I}{\vDash \neg p, p}\ \neg R}{\dfrac{\vDash (\neg p \vee q) \wedge \neg p, p}{\vDash ((\neg p \vee q) \wedge \neg p) \vee p}\ \vee R}\ \wedge R$$

$$\frac{}{\Gamma, a \vDash \Delta, a}\ (I)$$

$$\frac{\Gamma, a, b \vDash \Delta}{\Gamma, a \wedge b \vDash \Delta}\ (\wedge L) \qquad \frac{\Gamma \vDash a, b, \Delta}{\Gamma \vDash a \vee b, \Delta}\ (\vee R)$$

$$\frac{\Gamma, a \vDash \Delta \quad \Gamma, b \vDash \Delta}{\Gamma, a \vee b \vDash \Delta}\ (\vee L) \qquad \frac{\Gamma \vDash a, \Delta \quad \Gamma \vDash b, \Delta}{\Gamma \vDash a \wedge b, \Delta}\ (\wedge R)$$

$$\frac{\Gamma \vDash a, \Delta}{\Gamma, \neg a \vDash \Delta}\ (\neg L) \qquad \frac{\Gamma, a \vDash \Delta}{\Gamma \vDash \neg a, \Delta}\ (\neg R)$$

$$\cfrac{\cfrac{\cfrac{\vDash \neg p, q, p}{\vDash \neg p \vee q, p}\ \vee R \qquad \cfrac{\cfrac{}{p \vDash p}\ I}{\vDash \neg p, p}\ \neg R}{\vDash (\neg p \vee q) \wedge \neg p, p}\ \wedge R}{\vDash ((\neg p \vee q) \wedge \neg p) \vee p}\ \vee R$$

$$\frac{}{\Gamma, a \vDash \Delta, a}\ (I)$$

$$\frac{\Gamma, a, b \vDash \Delta}{\Gamma, a \wedge b \vDash \Delta}\ (\wedge L) \qquad \frac{\Gamma \vDash a, b, \Delta}{\Gamma \vDash a \vee b, \Delta}\ (\vee R)$$

$$\frac{\Gamma, a \vDash \Delta \quad \Gamma, b \vDash \Delta}{\Gamma, a \vee b \vDash \Delta}\ (\vee L) \qquad \frac{\Gamma \vDash a, \Delta \quad \Gamma \vDash b, \Delta}{\Gamma \vDash a \wedge b, \Delta}\ (\wedge R)$$

$$\frac{\Gamma \vDash a, \Delta}{\Gamma, \neg a \vDash \Delta}\ (\neg L) \qquad \frac{\Gamma, a \vDash \Delta}{\Gamma \vDash \neg a, \Delta}\ (\neg R)$$

$$\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{}{p \vDash q, p}\ I}{\vDash \neg p, q, p}\ \neg R}{\vDash \neg p \vee q, p}\ \vee R \quad \cfrac{\cfrac{}{p \vDash p}\ I}{\vDash \neg p, p}\ \neg R}{\vDash (\neg p \vee q) \wedge \neg p, p}\ \wedge R}{\vDash ((\neg p \vee q) \wedge \neg p) \vee p}\ \vee R$$

$$\frac{}{\Gamma, a \vDash \Delta, a}\ (I)$$

$$\frac{\Gamma, a, b \vDash \Delta}{\Gamma, a \wedge b \vDash \Delta}\ (\wedge L) \qquad \frac{\Gamma \vDash a, b, \Delta}{\Gamma \vDash a \vee b, \Delta}\ (\vee R)$$

$$\frac{\Gamma, a \vDash \Delta \quad \Gamma, b \vDash \Delta}{\Gamma, a \vee b \vDash \Delta}\ (\vee L) \qquad \frac{\Gamma \vDash a, \Delta \quad \Gamma \vDash b, \Delta}{\Gamma \vDash a \wedge b, \Delta}\ (\wedge R)$$

$$\frac{\Gamma \vDash a, \Delta}{\Gamma, \neg a \vDash \Delta}\ (\neg L) \qquad \frac{\Gamma, a \vDash \Delta}{\Gamma \vDash \neg a, \Delta}\ (\neg R)$$

$$\cfrac{\cfrac{\cfrac{\cfrac{}{\vDash \neg p, q, p}\ \neg R}{\vDash \neg p \vee q, p}\ \vee R \qquad \cfrac{\cfrac{}{p \vDash p}\ I}{\vDash \neg p, p}\ \neg R}{\vDash (\neg p \vee q) \wedge \neg p, p}\ \wedge R}{\vDash ((\neg p \vee q) \wedge \neg p) \vee p}\ \vee R$$

$$\frac{}{\Gamma, a \vDash \Delta, a}\ (I)$$

$$\frac{\Gamma, a, b \vDash \Delta}{\Gamma, a \wedge b \vDash \Delta}\ (\wedge L) \qquad \frac{\Gamma \vDash a, b, \Delta}{\Gamma \vDash a \vee b, \Delta}\ (\vee R)$$

$$\frac{\Gamma, a \vDash \Delta \quad \Gamma, b \vDash \Delta}{\Gamma, a \vee b \vDash \Delta}\ (\vee L) \qquad \frac{\Gamma \vDash a, \Delta \quad \Gamma \vDash b, \Delta}{\Gamma \vDash a \wedge b, \Delta}\ (\wedge R)$$

$$\frac{\Gamma \vDash a, \Delta}{\Gamma, \neg a \vDash \Delta}\ (\neg L) \qquad \frac{\Gamma, a \vDash \Delta}{\Gamma \vDash \neg a, \Delta}\ (\neg R)$$

$$\dfrac{\dfrac{\dfrac{\dfrac{p \vDash q, p}{\vDash \neg p, q, p}\ \neg R}{\vDash \neg p \vee q, p}\ \vee R \qquad \dfrac{\dfrac{}{p \vDash p}\ I}{\vDash \neg p, p}\ \neg R}{\vDash (\neg p \vee q) \wedge \neg p, p}\ \wedge R}{\vDash ((\neg p \vee q) \wedge \neg p) \vee p}\ \vee R$$

$$\frac{}{\Gamma, a \vDash \Delta, a}\ (I)$$

$$\frac{\Gamma, a, b \vDash \Delta}{\Gamma, a \wedge b \vDash \Delta}\ (\wedge L) \qquad \frac{\Gamma \vDash a, b, \Delta}{\Gamma \vDash a \vee b, \Delta}\ (\vee R)$$

$$\frac{\Gamma, a \vDash \Delta \quad \Gamma, b \vDash \Delta}{\Gamma, a \vee b \vDash \Delta}\ (\vee L) \qquad \frac{\Gamma \vDash a, \Delta \quad \Gamma \vDash b, \Delta}{\Gamma \vDash a \wedge b, \Delta}\ (\wedge R)$$

$$\frac{\Gamma \vDash a, \Delta}{\Gamma, \neg a \vDash \Delta}\ (\neg L) \qquad \frac{\Gamma, a \vDash \Delta}{\Gamma \vDash \neg a, \Delta}\ (\neg R)$$

$$\dfrac{\dfrac{\dfrac{\dfrac{}{p \vDash q, p}\ I}{\vDash \neg p, q, p}\ \neg R}{\vDash \neg p \vee q, p}\ \vee R \qquad \dfrac{\dfrac{}{p \vDash p}\ I}{\vDash \neg p, p}\ \neg R}{\dfrac{\vDash (\neg p \vee q) \wedge \neg p, p}{\vDash ((\neg p \vee q) \wedge \neg p) \vee p}\ \vee R}\ \wedge R$$

$$\frac{}{\Gamma, a \vDash \Delta, a}\ (I)$$

$$\frac{\Gamma, a, b \vDash \Delta}{\Gamma, a \wedge b \vDash \Delta}\ (\wedge L) \qquad \frac{\Gamma \vDash a, b, \Delta}{\Gamma \vDash a \vee b, \Delta}\ (\vee R)$$

$$\frac{\Gamma, a \vDash \Delta \quad \Gamma, b \vDash \Delta}{\Gamma, a \vee b \vDash \Delta}\ (\vee L) \qquad \frac{\Gamma \vDash a, \Delta \quad \Gamma \vDash b, \Delta}{\Gamma \vDash a \wedge b, \Delta}\ (\wedge R)$$

$$\frac{\Gamma \vDash a, \Delta}{\Gamma, \neg a \vDash \Delta}\ (\neg L) \qquad \frac{\Gamma, a \vDash \Delta}{\Gamma \vDash \neg a, \Delta}\ (\neg R)$$

these are valid in every universe

$$\cfrac{\cfrac{\cfrac{\cfrac{}{p \vDash q, p}\ I}{\vDash \neg p, q, p}\ \neg R}{\vDash \neg p \vee q, p}\ \vee R \qquad \cfrac{\cfrac{}{p \vDash p}\ I}{\vDash \neg p, p}\ \neg R}{\cfrac{\vDash (\neg p \vee q) \wedge \neg p, p}{\vDash ((\neg p \vee q) \wedge \neg p) \vee p}\ \vee R}\ \wedge R$$

so this is valid in every universe

$$\frac{}{\Gamma, a \vDash \Delta, a}\ (I)$$

$$\frac{\Gamma, a, b \vDash \Delta}{\Gamma, a \wedge b \vDash \Delta}\ (\wedge L) \qquad \frac{\Gamma \vDash a, b, \Delta}{\Gamma \vDash a \vee b, \Delta}\ (\vee R)$$

$$\frac{\Gamma, a \vDash \Delta \quad \Gamma, b \vDash \Delta}{\Gamma, a \vee b \vDash \Delta}\ (\vee L) \qquad \frac{\Gamma \vDash a, \Delta \quad \Gamma \vDash b, \Delta}{\Gamma \vDash a \wedge b, \Delta}\ (\wedge R)$$

$$\frac{\Gamma \vDash a, \Delta}{\Gamma, \neg a \vDash \Delta}\ (\neg L) \qquad \frac{\Gamma, a \vDash \Delta}{\Gamma \vDash \neg a, \Delta}\ (\neg R)$$

$$\frac{\vDash \neg((\neg a \vee b) \wedge (\neg c \vee b)), (\neg a \vee c)}{\vDash \neg((\neg a \vee b) \wedge (\neg c \vee b)) \vee (\neg a \vee c)}$$

$$\frac{}{\Gamma, a \models \Delta, a}\ (I)$$

$$\frac{\Gamma, a, b \models \Delta}{\Gamma, a \wedge b \models \Delta}\ (\wedge L) \qquad \frac{\Gamma \models a, b, \Delta}{\Gamma \models a \vee b, \Delta}\ (\vee R)$$

$$\frac{\Gamma, a \models \Delta \quad \Gamma, b \models \Delta}{\Gamma, a \vee b \models \Delta}\ (\vee L) \qquad \frac{\Gamma \models a, \Delta \quad \Gamma \models b, \Delta}{\Gamma \models a \wedge b, \Delta}\ (\wedge R)$$

$$\frac{\Gamma \models a, \Delta}{\Gamma, \neg a \models \Delta}\ (\neg L) \qquad \frac{\Gamma, a \models \Delta}{\Gamma \models \neg a, \Delta}\ (\neg R)$$

$$\frac{\models \neg((\neg a \vee b) \wedge (\neg c \vee b)), (\neg a \vee c)}{\models \neg((\neg a \vee b) \wedge (\neg c \vee b)) \vee (\neg a \vee c)}$$

$$\frac{}{\Gamma, a \vDash \Delta, a}\ (I)$$

$$\frac{\Gamma, a, b \vDash \Delta}{\Gamma, a \wedge b \vDash \Delta}\ (\wedge L) \qquad \frac{\Gamma \vDash a, b, \Delta}{\Gamma \vDash a \vee b, \Delta}\ (\vee R)$$

$$\frac{\Gamma, a \vDash \Delta \quad \Gamma, b \vDash \Delta}{\Gamma, a \vee b \vDash \Delta}\ (\vee L) \qquad \frac{\Gamma \vDash a, \Delta \quad \Gamma \vDash b, \Delta}{\Gamma \vDash a \wedge b, \Delta}\ (\wedge R)$$

$$\frac{\Gamma \vDash a, \Delta}{\Gamma, \neg a \vDash \Delta}\ (\neg L) \qquad \frac{\Gamma, a \vDash \Delta}{\Gamma \vDash \neg a, \Delta}\ (\neg R)$$

$$(\neg a \vee b) \wedge (\neg c \vee b) \vDash \neg a \vee c$$

$$\vDash \neg((\neg a \vee b) \wedge (\neg c \vee b)), (\neg a \vee c)$$

$$\vDash \neg((\neg a \vee b) \wedge (\neg c \vee b)) \vee (\neg a \vee c)$$

$$\frac{}{\Gamma, a \vDash \Delta, a}\ (I)$$

$$\frac{\Gamma, a, b \vDash \Delta}{\Gamma, a \wedge b \vDash \Delta}\ (\wedge L) \qquad \frac{\Gamma \vDash a, b, \Delta}{\Gamma \vDash a \vee b, \Delta}\ (\vee R)$$

$$\frac{\Gamma, a \vDash \Delta \quad \Gamma, b \vDash \Delta}{\Gamma, a \vee b \vDash \Delta}\ (\vee L) \qquad \frac{\Gamma \vDash a, \Delta \quad \Gamma \vDash b, \Delta}{\Gamma \vDash a \wedge b, \Delta}\ (\wedge R)$$

$$\frac{\Gamma \vDash a, \Delta}{\Gamma, \neg a \vDash \Delta}\ (\neg L) \qquad \frac{\Gamma, a \vDash \Delta}{\Gamma \vDash \neg a, \Delta}\ (\neg R)$$

$$\neg a, \neg c \vee b \vDash \neg a, c \qquad b, \neg c \vee b \vDash \neg a, c$$

$$\neg a \vee b, \neg c \vee b \vDash \neg a, c$$

$$(\neg a \vee b) \wedge (\neg c \vee b) \vDash \neg a \vee c$$

$$\vDash \neg((\neg a \vee b) \wedge (\neg c \vee b)), (\neg a \vee c)$$

$$\vDash \neg((\neg a \vee b) \wedge (\neg c \vee b)) \vee (\neg a \vee c)$$

$$\frac{}{\Gamma, a \vDash \Delta, a}\ (I)$$

$$\frac{\Gamma, a, b \vDash \Delta}{\Gamma, a \wedge b \vDash \Delta}\ (\wedge L) \qquad \frac{\Gamma \vDash a, b, \Delta}{\Gamma \vDash a \vee b, \Delta}\ (\vee R)$$

$$\frac{\Gamma, a \vDash \Delta \quad \Gamma, b \vDash \Delta}{\Gamma, a \vee b \vDash \Delta}\ (\vee L) \qquad \frac{\Gamma \vDash a, \Delta \quad \Gamma \vDash b, \Delta}{\Gamma \vDash a \wedge b, \Delta}\ (\wedge R)$$

$$\frac{\Gamma \vDash a, \Delta}{\Gamma, \neg a \vDash \Delta}\ (\neg L) \qquad \frac{\Gamma, a \vDash \Delta}{\Gamma \vDash \neg a, \Delta}\ (\neg R)$$

$$\neg a, \neg c \vee b \vDash \neg a, c \qquad\qquad b, \neg c \vee b \vDash \neg a, c$$

$$\neg a \vee b, \neg c \vee b \vDash \neg a, c$$

$$(\neg a \vee b) \wedge (\neg c \vee b) \vDash \neg a \vee c$$

$$\vDash \neg((\neg a \vee b) \wedge (\neg c \vee b)), (\neg a \vee c)$$

$$\vDash \neg((\neg a \vee b) \wedge (\neg c \vee b)) \vee (\neg a \vee c)$$

$$\frac{}{\Gamma, a \vDash \Delta, a}\ (I)$$

$$\frac{\Gamma, a, b \vDash \Delta}{\Gamma, a \wedge b \vDash \Delta}\ (\wedge L) \qquad \frac{\Gamma \vDash a, b, \Delta}{\Gamma \vDash a \vee b, \Delta}\ (\vee R)$$

$$\frac{\Gamma, a \vDash \Delta \quad \Gamma, b \vDash \Delta}{\Gamma, a \vee b \vDash \Delta}\ (\vee L) \qquad \frac{\Gamma \vDash a, \Delta \quad \Gamma \vDash b, \Delta}{\Gamma \vDash a \wedge b, \Delta}\ (\wedge R)$$

$$\frac{\Gamma \vDash a, \Delta}{\Gamma, \neg a \vDash \Delta}\ (\neg L) \qquad \frac{\Gamma, a \vDash \Delta}{\Gamma \vDash \neg a, \Delta}\ (\neg R)$$

$$b, \neg c \vDash \neg a, c \qquad b, b \vDash \neg a, c$$

$$\frac{\dfrac{}{\neg a, \neg c \vee b \vDash \neg a, c} \qquad b, \neg c \vee b \vDash \neg a, c}{\neg a \vee b, \neg c \vee b \vDash \neg a, c}$$

$$\frac{\neg a \vee b, \neg c \vee b \vDash \neg a, c}{(\neg a \vee b) \wedge (\neg c \vee b) \vDash \neg a \vee c}$$

$$\frac{(\neg a \vee b) \wedge (\neg c \vee b) \vDash \neg a \vee c}{\vDash \neg((\neg a \vee b) \wedge (\neg c \vee b)), (\neg a \vee c)}$$

$$\frac{\vDash \neg((\neg a \vee b) \wedge (\neg c \vee b)), (\neg a \vee c)}{\vDash \neg((\neg a \vee b) \wedge (\neg c \vee b)) \vee (\neg a \vee c)}$$

$$\frac{}{\Gamma, a \vDash \Delta, a}\ (I)$$

$$\frac{\Gamma, a, b \vDash \Delta}{\Gamma, a \wedge b \vDash \Delta}\ (\wedge L) \qquad \frac{\Gamma \vDash a, b, \Delta}{\Gamma \vDash a \vee b, \Delta}\ (\vee R)$$

$$\frac{\Gamma, a \vDash \Delta \quad \Gamma, b \vDash \Delta}{\Gamma, a \vee b \vDash \Delta}\ (\vee L) \qquad \frac{\Gamma \vDash a, \Delta \quad \Gamma \vDash b, \Delta}{\Gamma \vDash a \wedge b, \Delta}\ (\wedge R)$$

$$\frac{\Gamma \vDash a, \Delta}{\Gamma, \neg a \vDash \Delta}\ (\neg L) \qquad \frac{\Gamma, a \vDash \Delta}{\Gamma \vDash \neg a, \Delta}\ (\neg R)$$

$$\frac{b, \neg c \vDash \neg a, c \qquad b, b \vDash \neg a, c}{b, \neg c \vee b \vDash \neg a, c}$$

$$\frac{\overline{\neg a, \neg c \vee b \vDash \neg a, c} \qquad b, \neg c \vee b \vDash \neg a, c}{\neg a \vee b, \neg c \vee b \vDash \neg a, c}$$

$$\frac{\neg a \vee b, \neg c \vee b \vDash \neg a, c}{(\neg a \vee b) \wedge (\neg c \vee b) \vDash \neg a \vee c}$$

$$\frac{(\neg a \vee b) \wedge (\neg c \vee b) \vDash \neg a \vee c}{\vDash \neg((\neg a \vee b) \wedge (\neg c \vee b)), (\neg a \vee c)}$$

$$\frac{\vDash \neg((\neg a \vee b) \wedge (\neg c \vee b)), (\neg a \vee c)}{\vDash \neg((\neg a \vee b) \wedge (\neg c \vee b)) \vee (\neg a \vee c)}$$

$$\frac{}{\Gamma, a \vDash \Delta, a}\ (I)$$

$$\frac{\Gamma, a, b \vDash \Delta}{\Gamma, a \wedge b \vDash \Delta}\ (\wedge L) \qquad \frac{\Gamma \vDash a, b, \Delta}{\Gamma \vDash a \vee b, \Delta}\ (\vee R)$$

$$\frac{\Gamma, a \vDash \Delta \quad \Gamma, b \vDash \Delta}{\Gamma, a \vee b \vDash \Delta}\ (\vee L) \qquad \frac{\Gamma \vDash a, \Delta \quad \Gamma \vDash b, \Delta}{\Gamma \vDash a \wedge b, \Delta}\ (\wedge R)$$

$$\frac{\Gamma \vDash a, \Delta}{\Gamma, \neg a \vDash \Delta}\ (\neg L) \qquad \frac{\Gamma, a \vDash \Delta}{\Gamma \vDash \neg a, \Delta}\ (\neg R)$$

$$a, b \vDash c$$

$$b, \vDash \neg a, c \qquad a, b \vDash c$$

$$b, \neg c \vDash \neg a, c \qquad b, b \vDash \neg a, c$$

$$\neg a, \neg c \vee b \vDash \neg a, c \qquad b, \neg c \vee b \vDash \neg a, c$$

$$\neg a \vee b, \neg c \vee b \vDash \neg a, c$$

$$(\neg a \vee b) \wedge (\neg c \vee b) \vDash \neg a \vee c$$

$$\vDash \neg((\neg a \vee b) \wedge (\neg c \vee b)), (\neg a \vee c)$$

$$\vDash \neg((\neg a \vee b) \wedge (\neg c \vee b)) \vee (\neg a \vee c)$$

$$\frac{}{\Gamma, a \vDash \Delta, a}\ (I)$$

$$\frac{\Gamma, a, b \vDash \Delta}{\Gamma, a \wedge b \vDash \Delta}\ (\wedge L) \qquad \frac{\Gamma \vDash a, b, \Delta}{\Gamma \vDash a \vee b, \Delta}\ (\vee R)$$

$$\frac{\Gamma, a \vDash \Delta \quad \Gamma, b \vDash \Delta}{\Gamma, a \vee b \vDash \Delta}\ (\vee L) \qquad \frac{\Gamma \vDash a, \Delta \quad \Gamma \vDash b, \Delta}{\Gamma \vDash a \wedge b, \Delta}\ (\wedge R)$$

$$\frac{\Gamma \vDash a, \Delta}{\Gamma, \neg a \vDash \Delta}\ (\neg L) \qquad \frac{\Gamma, a \vDash \Delta}{\Gamma \vDash \neg a, \Delta}\ (\neg R)$$

$$a, b \vDash c$$

$$b, \vDash \neg a, c$$

$$b, \neg c \vDash \neg a, c \qquad a, b \vDash c \qquad b, b \vDash \neg a, c$$

$$\neg a, \neg c \vee b \vDash \neg a, c \qquad b, \neg c \vee b \vDash \neg a, c$$

$$\neg a \vee b, \neg c \vee b \vDash \neg a, c$$

$$(\neg a \vee b) \wedge (\neg c \vee b) \vDash \neg a \vee c$$

$$\vDash \neg((\neg a \vee b) \wedge (\neg c \vee b)), (\neg a \vee c)$$

$$\vDash \neg((\neg a \vee b) \wedge (\neg c \vee b)) \vee (\neg a \vee c)$$

$$\frac{}{\Gamma, a \vDash \Delta, a}\ (I)$$

$$\frac{\Gamma, a, b \vDash \Delta}{\Gamma, a \wedge b \vDash \Delta}\ (\wedge L) \qquad \frac{\Gamma \vDash a, b, \Delta}{\Gamma \vDash a \vee b, \Delta}\ (\vee R)$$

$$\frac{\Gamma, a \vDash \Delta \quad \Gamma, b \vDash \Delta}{\Gamma, a \vee b \vDash \Delta}\ (\vee L) \qquad \frac{\Gamma \vDash a, \Delta \quad \Gamma \vDash b, \Delta}{\Gamma \vDash a \wedge b, \Delta}\ (\wedge R)$$

$$\frac{\Gamma \vDash a, \Delta}{\Gamma, \neg a \vDash \Delta}\ (\neg L) \qquad \frac{\Gamma, a \vDash \Delta}{\Gamma \vDash \neg a, \Delta}\ (\neg R)$$

$$\frac{a, b \vDash c}{b, \vDash \neg a, c}$$

$$\frac{b, \vDash \neg a, c}{b, \neg c \vDash \neg a, c} \qquad \frac{a, b \vDash c}{b, b \vDash \neg a, c}$$

$$\frac{b, \neg c \vDash \neg a, c \quad b, b \vDash \neg a, c}{b, \neg c \vee b \vDash \neg a, c}$$

$$\frac{}{\neg a, \neg c \vee b \vDash \neg a, c}$$

$$\frac{\neg a, \neg c \vee b \vDash \neg a, c \quad b, \neg c \vee b \vDash \neg a, c}{\neg a \vee b, \neg c \vee b \vDash \neg a, c}$$

$$\frac{\neg a \vee b, \neg c \vee b \vDash \neg a, c}{(\neg a \vee b) \wedge (\neg c \vee b) \vDash \neg a \vee c}$$

$$\frac{(\neg a \vee b) \wedge (\neg c \vee b) \vDash \neg a \vee c}{\vDash \neg((\neg a \vee b) \wedge (\neg c \vee b)), (\neg a \vee c)}$$

$$\frac{\vDash \neg((\neg a \vee b) \wedge (\neg c \vee b)), (\neg a \vee c)}{\vDash \neg((\neg a \vee b) \wedge (\neg c \vee b)) \vee (\neg a \vee c)}$$

$$\frac{}{\Gamma, a \models \Delta, a}\ (I)$$

$$\frac{\Gamma, a, b \models \Delta}{\Gamma, a \wedge b \models \Delta}\ (\wedge L) \qquad \frac{\Gamma \models a, b, \Delta}{\Gamma \models a \vee b, \Delta}\ (\vee R)$$

$$\frac{\Gamma, a \models \Delta \quad \Gamma, b \models \Delta}{\Gamma, a \vee b \models \Delta}\ (\vee L) \qquad \frac{\Gamma \models a, \Delta \quad \Gamma \models b, \Delta}{\Gamma \models a \wedge b, \Delta}\ (\wedge R)$$

$$\frac{\Gamma \models a, \Delta}{\Gamma, \neg a \models \Delta}\ (\neg L) \qquad \frac{\Gamma, a \models \Delta}{\Gamma \models \neg a, \Delta}\ (\neg R)$$

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\neg a, \neg c \vee b \models \neg a, c}\ I \qquad \dfrac{\dfrac{\dfrac{a, b \models c}{b, \models \neg a, c}\ \neg R}{b, \neg c \models \neg a, c}\ \neg L \qquad \dfrac{a, b \models c}{b, b \models \neg a, c}\ \neg R}{b, \neg c \vee b \models \neg a, c}\ \vee L}{\neg a \vee b, \neg c \vee b \models \neg a, c}\ \vee L}{(\neg a \vee b) \wedge (\neg c \vee b) \models \neg a \vee c}\ \wedge L, \vee R$$

$$\dfrac{\dfrac{(\neg a \vee b) \wedge (\neg c \vee b) \models \neg a \vee c}{\models \neg((\neg a \vee b) \wedge (\neg c \vee b)), (\neg a \vee c)}\ \neg R}{\models \neg((\neg a \vee b) \wedge (\neg c \vee b)) \vee (\neg a \vee c)}\ \vee R$$

these are valid
in some universe, U

iff this is valid in U

$$a, b \vDash c$$

$$b, \vDash \neg a, c \qquad a, b \vDash c$$

$$b, \neg c \vDash \neg a, c \qquad b, b \vDash \neg a, c$$

$$\neg a, \neg c \vee b \vDash \neg a, c \qquad b, \neg c \vee b \vDash \neg a, c$$

$$\neg a \vee b, \neg c \vee b \vDash \neg a, c$$

$$(\neg a \vee b) \wedge (\neg c \vee b) \vDash \neg a \vee c$$

$$\vDash \neg((\neg a \vee b) \wedge (\neg c \vee b)), (\neg a \vee c)$$

$$\vDash \neg((\neg a \vee b) \wedge (\neg c \vee b)) \vee (\neg a \vee c)$$

$$a, b \vDash c \quad a, b \vDash c$$

$$\vDash \neg((\neg a \vee b) \wedge (\neg c \vee b)) \vee (\neg a \vee c)$$

$$\dfrac{\dfrac{}{\neg a,\ \neg c \vee b \vDash \neg a,\ c} \qquad \dfrac{\dfrac{\dfrac{a, b \vDash c}{b, \vDash \neg a,\ c}}{b,\ \neg c \vDash \neg a,\ c} \quad \dfrac{a, b \vDash c}{b,\ b \vDash \neg a,\ c}}{b,\ \neg c \vee b \vDash \neg a,\ c}}{\dfrac{\neg a \vee b,\ \neg c \vee b \vDash \neg a,\ c}{\dfrac{(\neg a \vee b) \wedge (\neg c \vee b) \vDash \neg a \vee c}{\dfrac{\vDash \neg((\neg a \vee b) \wedge (\neg c \vee b)),\ (\neg a \vee c)}{\vDash \neg((\neg a \vee b) \wedge (\neg c \vee b)) \vee (\neg a \vee c)}}}}$$

in this example,
these are the same

$$\dfrac{a, b \vDash c \quad a, b \vDash c}{\vDash \neg((\neg a \vee b) \wedge (\neg c \vee b)) \vee (\neg a \vee c)}$$

so we can just say

$$\dfrac{a, b \vDash c}{\vDash \neg((\neg a \vee b) \wedge (\neg c \vee b)) \vee (\neg a \vee c)}$$

$$a, b \vDash c$$

$$b, \vDash \neg a, c$$

$$b, \neg c \vDash \neg a, c \qquad b, b \vDash \neg a, c$$

$$\neg a, \neg c \vee b \vDash \neg a, c \qquad b, \neg c \vee b \vDash \neg a, c$$

$$\neg a \vee b, \neg c \vee b \vDash \neg a, c$$

$$(\neg a \vee b) \wedge (\neg c \vee b) \vDash \neg a \vee c$$

$$\vDash \neg((\neg a \vee b) \wedge (\neg c \vee b)), (\neg a \vee c)$$

$$\vDash \neg((\neg a \vee b) \wedge (\neg c \vee b)) \vee (\neg a \vee c)$$

$$a, b \vDash c$$

$$\vDash \neg((\neg a \vee b) \wedge (\neg c \vee b)) \vee (\neg a \vee c)$$

A *counterexample* to $a, b \vDash c$

makes *every antecedent true*

and *every succedent false.*

An individual, x, such that $a\,x$ and $b\,x$ and $\neg c\,x$

is a **counterexample**

Our two inference trees tell two different stories …

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{p \vDash q,\, p}}{\vDash \neg p,\, q,\, p}}{\vDash \neg p \lor q,\, p} \qquad \dfrac{\dfrac{}{p \vDash p}}{\vDash \neg p,\, p}}{\vDash (\neg p \lor q) \land \neg p,\, p}}{\vDash ((\neg p \lor q) \land \neg p) \lor p}$$

Every branch is terminated by an immediate rule.

The sequent we started from is valid in every universe!

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\neg a,\, \neg c \lor b \vDash \neg a,\, c} \qquad \dfrac{\dfrac{\dfrac{a, b \vDash c}{b, \vDash \neg a,\, c}}{b,\, \neg c \vDash \neg a,\, c} \quad \dfrac{a, b \vDash c}{b,\, b \vDash \neg a,\, c}}{b,\, \neg c \lor b \vDash \neg a,\, c}}{\neg a \lor b,\, \neg c \lor b \vDash \neg a,\, c}}{(\neg a \lor b) \land (\neg c \lor b) \vDash \neg a \lor c}}{\vDash \neg((\neg a \lor b) \land (\neg c \lor b)),\, (\neg a \lor c)}}{\vDash \neg((\neg a \lor b) \land (\neg c \lor b)) \lor (\neg a \lor c)}$$

Some branches lead to *leaves*, sequences with only atoms, in which no atom occurs on both sides of the turnstile.

Our starting sequent is valid in some universe U iff each of these leaves is valid.

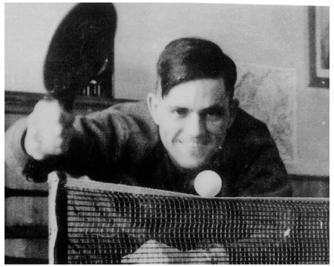
It is easy to construct a counterexample to any one of these leaves.

$$\frac{}{\Gamma, a \vDash \Delta, a}\ (I)$$

$$\frac{\Gamma, a, b \vDash \Delta}{\Gamma, a \wedge b \vDash \Delta}\ (\wedge L) \qquad \frac{\Gamma \vDash a, b, \Delta}{\Gamma \vDash a \vee b, \Delta}\ (\vee R)$$

$$\frac{\Gamma, a \vDash \Delta \quad \Gamma, b \vDash \Delta}{\Gamma, a \vee b \vDash \Delta}\ (\vee L) \qquad \frac{\Gamma \vDash a, \Delta \quad \Gamma \vDash b, \Delta}{\Gamma \vDash a \wedge b, \Delta}\ (\wedge R)$$

$$\frac{\Gamma \vDash a, \Delta}{\Gamma, \neg a \vDash \Delta}\ (\neg L) \qquad \frac{\Gamma, a \vDash \Delta}{\Gamma \vDash \neg a, \Delta}\ (\neg R)$$



- a and b are predicates from some universe,
- Γ, Δ are finite sets of predicates from some universe,
- Γ, a refers to $\Gamma \cup \{a\}$, and a, Δ refers to $\{a\} \cup \Delta$.

$$3 + 5 = 8 \qquad\qquad ax^2 - bx + c$$

$$ax^2 - bx + c = 0 \qquad \text{iff} \qquad x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

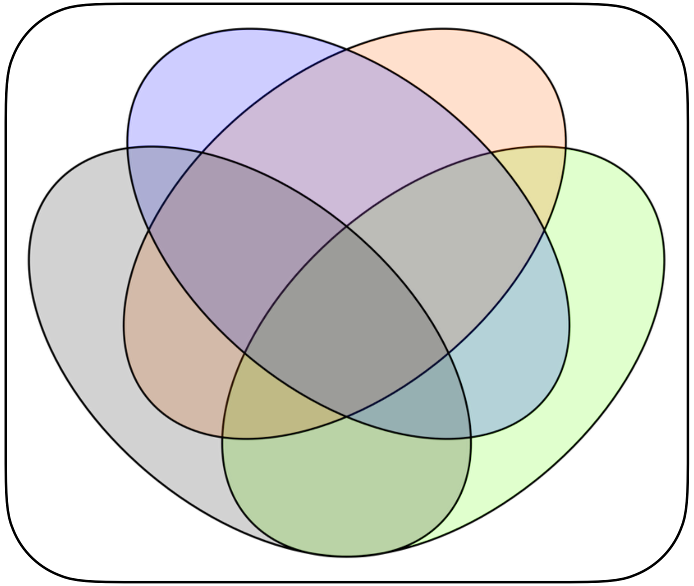
In general, What is a predicates?
What operations are there on predicates?

We consider a finite universe U
— a collection of things.

Any subset $a \subseteq U$ can be treated as a predicate:
$a\ x$ is true iff $x \in a$
$a \models b$ iff $a \subseteq b$ and barbara holds
$x \in \neg a$ iff $x \notin a$ $a \models a$ iff $\neg b \models \neg a$

propositions are just functions :: U -> Bool
our first operation on propositions is negation

in Haskell :   (neg a) x = not (a x)



every subset has a
characteristic function
we will represent propositions
as functions, as in lecture 1

a	$\neg a$
$\bot$	$\top$
$\top$	$\bot$

Gottfried Wilhelm Leibniz

1646–1716

- Identity of Indiscernables:
  "No two distinct things exactly resemble one another."
  — Leibniz

  That is, two objects are identical if and only if they satisfy the same properties.

- "A difference that makes no difference is no difference." — Spock

- "Equals may be substituted for equals."

```
(&:&) a b x = a x && b x
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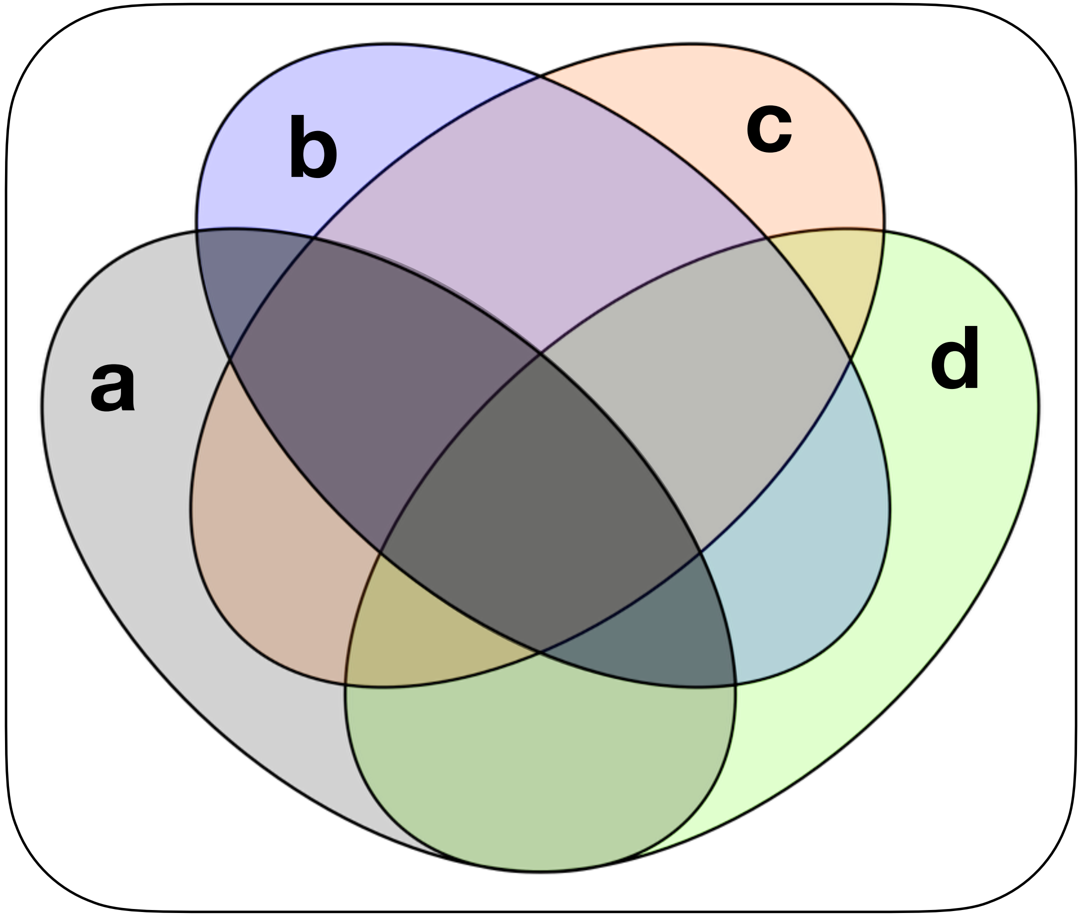
∧	⊥	⊤
⊥	⊥	⊥
⊤	⊥	⊤



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(|:|) a b x = a x || b x
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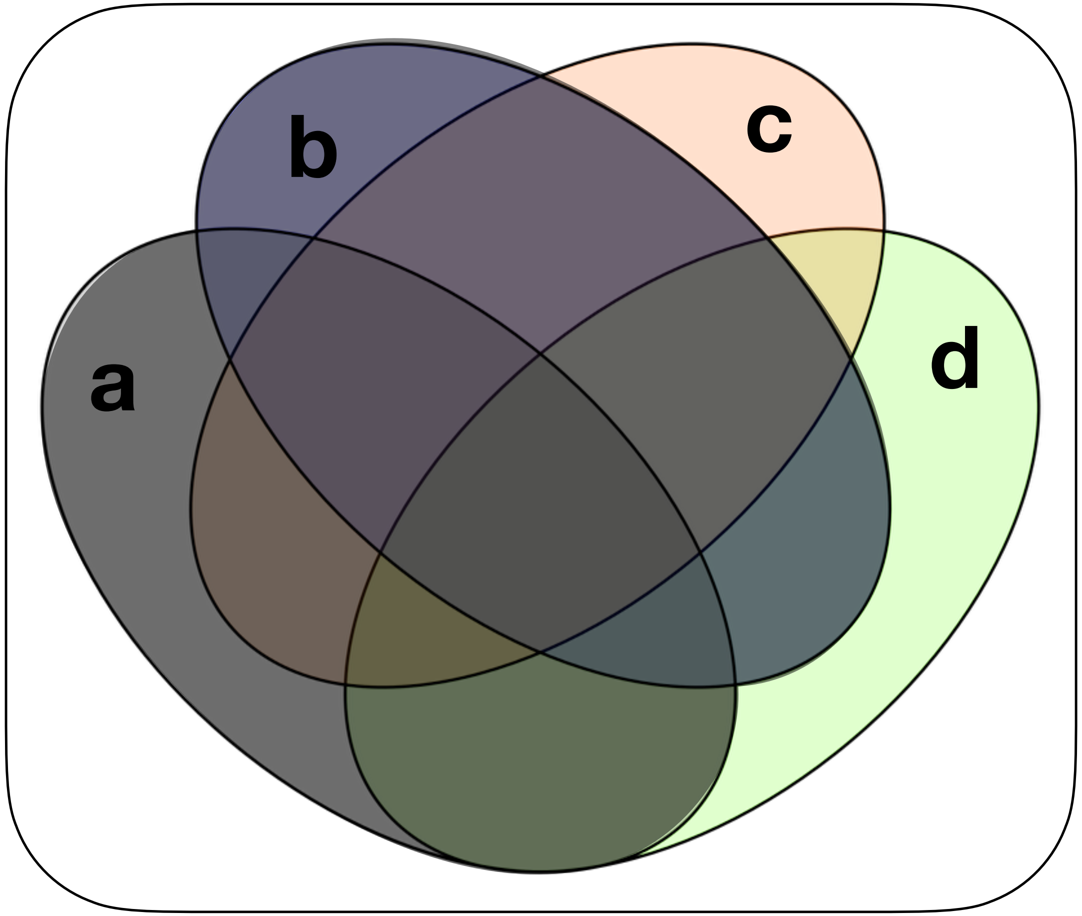
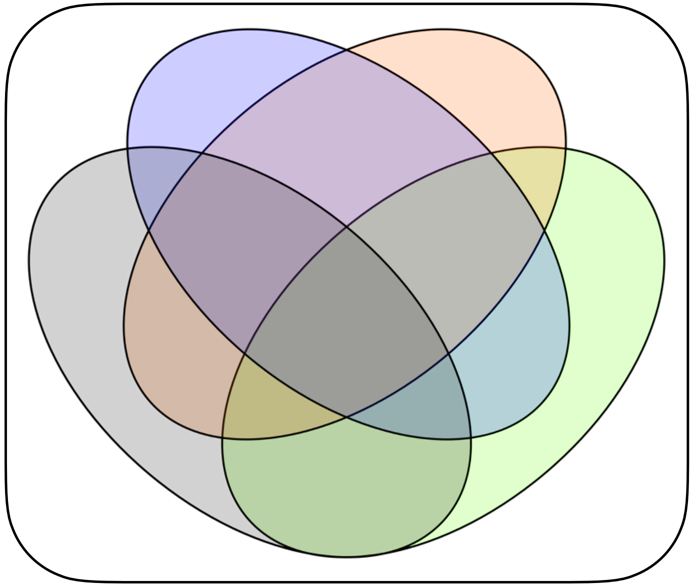
∨	⊥	⊤
⊥	⊥	⊤
⊤	⊤	⊤



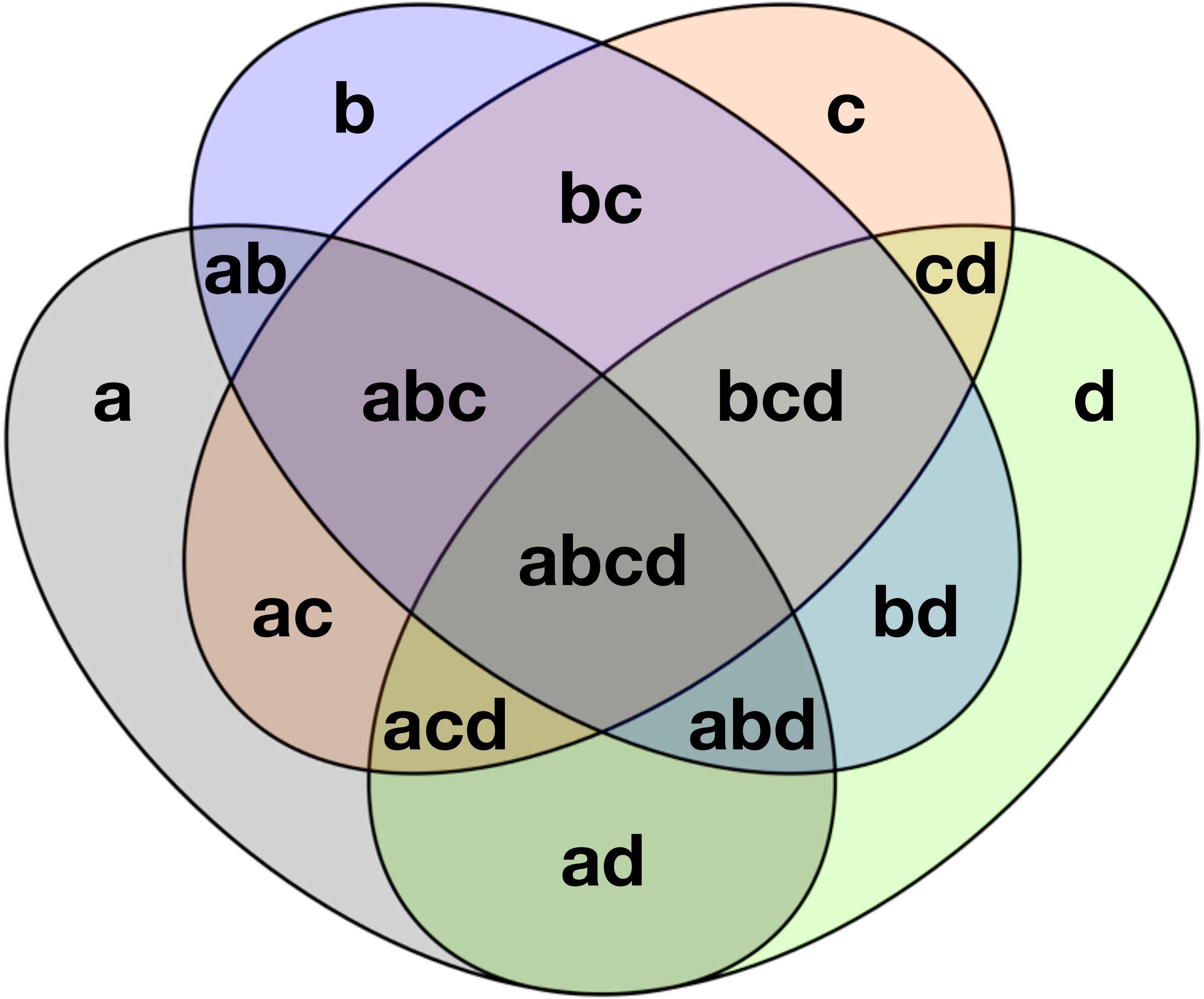
In general, we consider a finite universe U — a collection of things with a finite number of predicates. We consider two things to be *indistinguishable*, $x \equiv y$, iff they satisfy the same predicates.

If we have n predicates then we can distinguish 2^n different kinds of thing. In this example we have 4 predicates, so $2^4 = 16$ different kinds of thing. Each kind is represented by a region in the diagram.



We look at subsets X such that $$\frac{a \in X \quad a \equiv b}{b \in X}$$

Each such subset can be pictured by shading some regions of the diagram – those with things of a kind in X.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