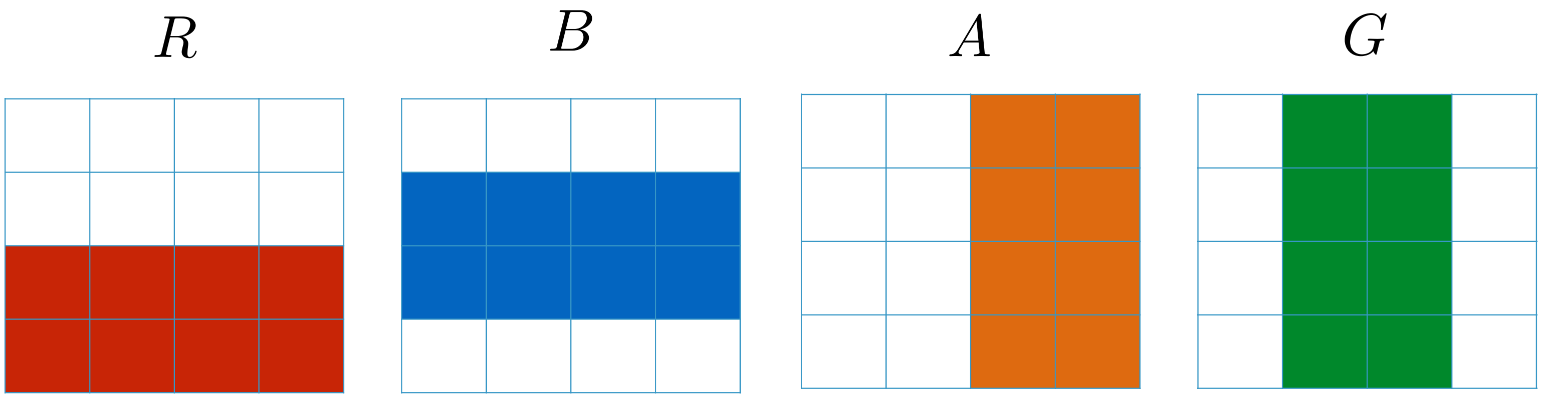
# INF1a-CL

CNF KM SequentCalculus Tseytin




| RB \ AG | 00 | 01 | 11 | 10 |
|---|---|---|---|---|
| 00 | 0 | 0 | 0 | 0 |
| 01 | 0 | 0 | 0 | 0 |
| 11 | 1 | 1 | 1 | 1 |
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$$r \leftrightarrow (a \vee b)$$

$$(\neg r \vee a \vee b) \wedge (r \vee \neg a) \wedge (r \vee \neg b)$$

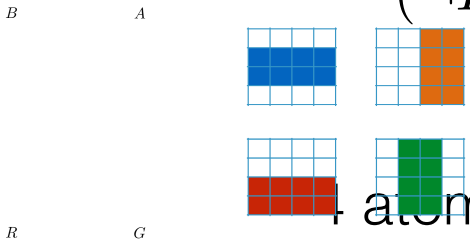
# Karnaugh Maps

to produce a CNF / POS
identify blocks of 0s
and write a sum for each

| RB \ AG | 00 | 01 | 11 | 10 |
|---|---|---|---|---|
| 00 | 1 | 0 | 0 | 0 |
| 01 | 1 | 1 | 0 | 0 |
| 11 | 0 | 1 | 1 | 1 |
| 10 | 0 | 0 | 1 | 1 |

(R′+A+G)(B+A+G′)(R+A′)

$$(\neg R \vee A \vee G) \wedge (B \vee A \vee \neg G) \wedge (R \vee \neg A)$$




4 atoms:16 states: 64Ki subsets

| a | 00 | 01 | 11 | 10 |
|---|---|---|---|---|
| 00 | 0 | 0 | 0 | 0 |
| 01 | 1 | 1 | 0 | 1 |
| 11 | 1 | 1 | 1 | 0 |
| 10 | 1 | 0 | 0 | 0 |

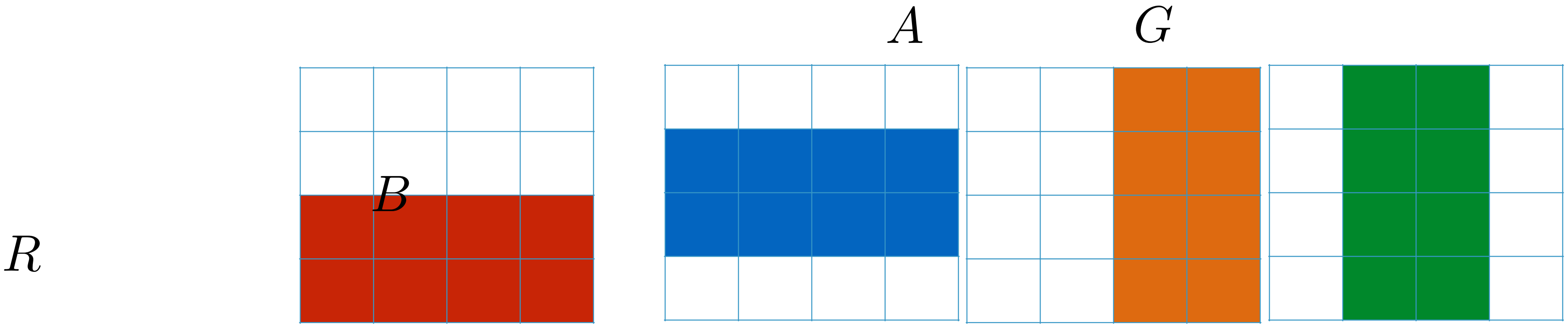
| b | 00 | 01 | 11 | 10 |
|---|---|---|---|---|
| 00 | 1 | 0 | 0 | 0 |
| 01 | 0 | 1 | 1 | 1 |
| 11 | 0 | 1 | 0 | 1 |
| 10 | 1 | 0 | 1 | 0 |

| c | 00 | 01 | 11 | 10 |
|---|---|---|---|---|
| 00 | 0 | 0 | 0 | 1 |
| 01 | 0 | 0 | 1 | 1 |
| 11 | 0 | 0 | 1 | 1 |
| 10 | 1 | 0 | 1 | 1 |

| d | 00 | 01 | 11 | 10 |
|---|---|---|---|---|
| 00 | 1 | 1 | 1 | 1 |
| 01 | 1 | 1 | 1 | 1 |
| 11 | 1 | 1 | 0 | 0 |
| 10 | 0 | 0 | 0 | 0 |

| e | 00 | 01 | 11 | 10 |
|---|---|---|---|---|
| 00 | 1 | 0 | 1 | 1 |
| 01 | 1 | 0 | 0 | 1 |
| 11 | 0 | 0 | 0 | 1 |
| 10 | 0 | 0 | 0 | 0 |

| f | 00 | 01 | 11 | 10 |
|---|---|---|---|---|
| 00 | 0 | 0 | 0 | 0 |
| 01 | 0 | 0 | 1 | 1 |
| 11 | 1 | 0 | 0 | 1 |
| 10 | 0 | 0 | 1 | 0 |



$$\frac{}{\Gamma, a \vDash a, \Delta} \ (I)$$

$$\frac{\Gamma \vDash a, \Delta \quad \Gamma \vDash b, \Delta}{\Gamma, a \to b \vDash \Delta} \ (\to L) \qquad \frac{\Gamma a, \vDash b, \Delta}{\Gamma \vDash a \to b, \Delta} \ (\to R)$$

$$\frac{\Gamma, a, b \vDash \Delta}{\Gamma, a \wedge b \vDash \Delta} \ (\wedge L) \qquad \frac{\Gamma \vDash a, \Delta \quad \Gamma \vDash b, \Delta}{\Gamma \vDash a \wedge b, \Delta} \ (\wedge R)$$

$$\frac{\Gamma, a \vDash \Delta \quad \Gamma, b \vDash \Delta}{a \vee b \vDash \Delta} \ (\vee L) \qquad \frac{\Gamma \vDash a, b, \Delta}{\Gamma \vDash a \vee b, \Delta} \ (\vee R)$$

$$\frac{\Gamma \vDash a, \Delta}{\Gamma, \neg a \vDash \Delta} \ (\neg L) \qquad \frac{\Gamma, a \vDash \Delta}{\Gamma \vDash \neg a, \Delta} \ (\neg R)$$

$$\frac{}{\Gamma, a \vDash \Delta, a}\ (I)$$

$$\frac{\Gamma, a, b \vDash \Delta}{\Gamma, a \wedge b \vDash \Delta}\ (\wedge L) \qquad \frac{\Gamma \vDash a, b, \Delta}{\Gamma \vDash a \vee b, \Delta}\ (\vee R)$$

$$\frac{\Gamma, a \vDash \Delta \quad \Gamma, b \vDash \Delta}{\Gamma, a \vee b \vDash \Delta}\ (\vee L) \qquad \frac{\Gamma \vDash a, \Delta \quad \Gamma \vDash b, \Delta}{\Gamma \vDash a \wedge b, \Delta}\ (\wedge R)$$

$$\frac{\Gamma \vDash a, \Delta}{\Gamma, \neg a \vDash \Delta}\ (\neg L) \qquad \frac{\Gamma, a \vDash \Delta}{\Gamma \vDash \neg a, \Delta}\ (\neg R)$$

$$\cfrac{\cfrac{\cfrac{\cfrac{}{\neg a, \neg c \vee b \vDash \neg a, c}\ I \qquad \cfrac{\cfrac{\cfrac{a, b \vDash c}{b, \vDash \neg a, c}\ \neg R}{b, \neg c \vDash \neg a, c}\ \neg L \qquad \cfrac{a, b \vDash c}{b, b \vDash \neg a, c}\ \neg R}{b, \neg c \vee b \vDash \neg a, c}\ \vee L}{\neg a \vee b, \neg c \vee b \vDash \neg a, c}\ \vee L}{(\neg a \vee b) \wedge (\neg c \vee b) \vDash \neg a \vee c}\ \wedge L; \vee R$$

$$\cfrac{\cfrac{(\neg a \vee b) \wedge (\neg c \vee b) \vDash \neg a \vee c}{\vDash \neg((\neg a \vee b) \wedge (\neg c \vee b)), (\neg a \vee c)}\ \neg R}{\vDash \neg((\neg a \vee b) \wedge (\neg c \vee b)) \vee (\neg a \vee c)}\ \vee R$$

these are valid
in some universe, U

iff this is valid in U

$$
\frac{\dfrac{\dfrac{\neg a,\ \neg c \vee b \vDash \neg a,\ c \qquad \dfrac{\dfrac{\dfrac{a, b \vDash c}{b, \vDash \neg a,\ c}}{b,\ \neg c \vDash \neg a,\ c} \qquad \dfrac{a, b \vDash c}{b,\ b \vDash \neg a,\ c}}{b,\ \neg c \vee b \vDash \neg a,\ c}}{\neg a \vee b,\ \neg c \vee b \vDash \neg a,\ c}}{(\neg a \vee b) \wedge (\neg c \vee b) \vDash \neg a \vee c}}{\dfrac{\vDash \neg((\neg a \vee b) \wedge (\neg c \vee b)),\ (\neg a \vee c)}{\vDash \neg((\neg a \vee b) \wedge (\neg c \vee b)) \vee (\neg a \vee c)}}
$$

$$
\frac{a, b \vDash c \qquad a, b \vDash c}{\vDash \neg((\neg a \vee b) \wedge (\neg c \vee b)) \vee (\neg a \vee c)}
$$

Our two inference trees tell two different stories …

$$\overline{p \vDash q, p}$$
$$\vDash \neg p, q, p \qquad \overline{p \vDash p}$$
$$\vDash \neg p \vee q, p \qquad \vDash \neg p, p$$
$$\vDash (\neg p \vee q) \wedge \neg p, p$$
$$\vDash ((\neg p \vee q) \wedge \neg p) \vee p$$

$$a, b \vDash c$$
$$b, \vDash \neg a, c \qquad a, b \vDash c$$
$$b, \neg c \vDash \neg a, c \qquad b, b \vDash \neg a, c$$
$$\overline{\neg a, \neg c \vee b \vDash \neg a, c} \qquad b, \neg c \vee b \vDash \neg a, c$$
$$\neg a \vee b, \neg c \vee b \vDash \neg a, c$$
$$(\neg a \vee b) \wedge (\neg c \vee b) \vDash \neg a \vee c$$
$$\vDash \neg((\neg a \vee b) \wedge (\neg c \vee b)), (\neg a \vee c)$$
$$\vDash \neg((\neg a \vee b) \wedge (\neg c \vee b)) \vee (\neg a \vee c)$$

Every branch is terminated by an immediate rule.

The sequent we started from is valid in every universe!

Some branches lead to *leaves*, sequences with only atoms, in which no atom occurs on both sides of the turnstile.

Our starting sequent is valid in some universe U iff each of these leaves is valid.

It is easy to construct a counterexample to any one of these leaves.

$$\frac{}{\Gamma, a \vdash a, \Delta} \ (I)$$

$$\frac{\Gamma \vdash a, \Delta \quad \Gamma, b \vdash \Delta}{\Gamma, a \to b \vdash \Delta} \ (\to L) \qquad \frac{\Gamma, a \vdash b, \Delta}{\Gamma \vdash a \to b, \Delta} \ (\to R)$$

$$\frac{\Gamma, a, b \vdash \Delta}{\Gamma, a \wedge b \vdash \Delta} \ (\wedge L) \qquad \frac{\Gamma \vdash a, \Delta \quad \Gamma \vdash b, \Delta}{\Gamma \vdash a \wedge b, \Delta} \ (\wedge R)$$

$$\frac{\Gamma, a \vdash \Delta \quad \Gamma, b \vdash \Delta}{a \vee b \vdash \Delta} \ (\vee L) \qquad \frac{\Gamma \vdash a, b, \Delta}{\Gamma \vdash a \vee b, \Delta} \ (\vee R)$$

$$\frac{\Gamma \vdash a, \Delta}{\Gamma, \neg a \vdash \Delta} \ (\neg L) \qquad \frac{\Gamma, a \vdash \Delta}{\Gamma \vdash \neg a, \Delta} \ (\neg R)$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{p \vDash q, p}}{\vDash \neg p, q, p}}{\vDash \neg p \vee q, p} \qquad \dfrac{\dfrac{}{p \vDash p}}{\vDash \neg p, p}}{\vDash (\neg p \vee q) \wedge \neg p, p}}{\vDash ((\neg p \vee q) \wedge \neg p) \vee p}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{P \vdash Q, P}}{\vdash \neg P, Q, P}}{\vdash \neg P \vee Q, P} \qquad \dfrac{\dfrac{}{P \vdash P}}{\vdash \neg P, P}}{\vdash (\neg P \vee Q) \wedge \neg P, P}}{\vdash ((\neg P \vee Q) \wedge \neg P) \vee P}$$

A proof is a tree of inferences, starting with immediate rules.

Prove the following entailment or if it not provable provide a counterexample

$$P \to (Q \vee R), (Q \wedge R) \to S \vdash P \to S$$

$$\frac{}{\Gamma, a \vdash a, \Delta}\ (I)$$

$$\frac{\Gamma, A[y/x] \vdash \Delta}{\Gamma, \exists x.\ \vdash \Delta}\ (\exists L) \qquad \frac{\Gamma \vdash A[t/x], \Delta}{\Gamma \vdash \exists x.\, A, \Delta}\ (\exists R)$$

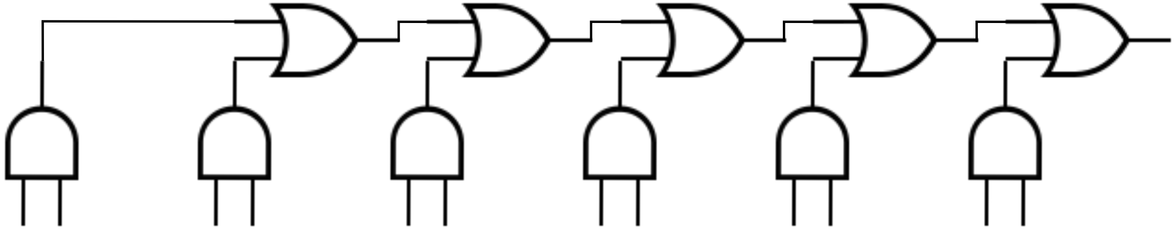
$$\frac{\Gamma, A[t/x] \vdash \Delta}{\Gamma, \forall x.\ \vdash \Delta}\ (\forall L) \qquad \frac{\Gamma \vdash A[y/x], \Delta}{\Gamma \vdash \forall x.\, A, \Delta}\ (\forall R)$$

$$\frac{\Gamma \vdash a, \Delta \quad \Gamma, b \vdash \Delta}{\Gamma, a \to b \vdash \Delta}\ (\to L) \qquad \frac{\Gamma, a \vdash b, \Delta}{\Gamma \vdash a \to b, \Delta}\ (\to R)$$

$$\frac{\Gamma, a, b \vdash \Delta}{\Gamma, a \wedge b \vdash \Delta}\ (\wedge L) \qquad \frac{\Gamma \vdash a, \Delta \quad \Gamma \vdash b, \Delta}{\Gamma \vdash a \wedge b, \Delta}\ (\wedge R)$$

$$\frac{\Gamma, a \vdash \Delta \quad \Gamma, b \vdash \Delta}{a \vee b \vdash \Delta}\ (\vee L) \qquad \frac{\Gamma \vdash a, b, \Delta}{\Gamma \vdash a \vee b, \Delta}\ (\vee R)$$

$$\frac{\Gamma \vdash a, \Delta}{\Gamma, \neg a \vdash \Delta}\ (\neg L) \qquad \frac{\Gamma, a \vdash \Delta}{\Gamma \vdash \neg a, \Delta}\ (\neg R)$$



$$a \wedge b \vee c \wedge d \vee e \wedge f \vee g \wedge h \vee j \wedge k \vee m \wedge n$$

$$(a \wedge b) \vee (c \wedge d) \vee (e \wedge f) \vee (g \wedge h) \vee (j \wedge k) \vee (m \wedge n)$$

How many clauses in the CNF?

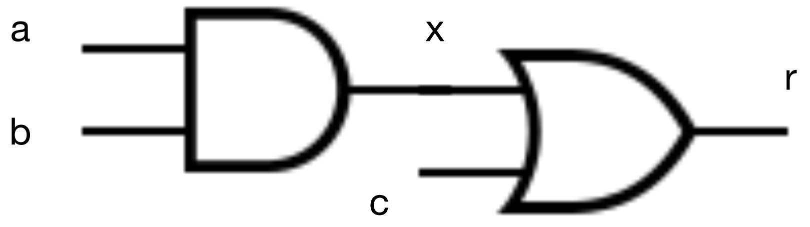
$2^6 = 64$

How many clauses to describe the circuit?

If we start from an expression then
we can draw an equivalent circuit with:

$$r = (a \wedge b) \vee c$$

a wire for each subexpression,
a logic gate for each operator,
and an input for each variable.



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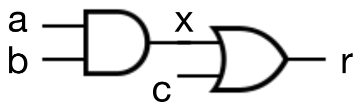


$$r \leftrightarrow (a \wedge b)$$

$$(r \vee \neg a \vee \neg b) \wedge (\neg r \vee a) \wedge (\neg r \vee b)$$

$$r \leftrightarrow (a \vee b)$$

$$(\neg r \vee a \vee b) \wedge (r \vee \neg a) \wedge (r \vee \neg b)$$

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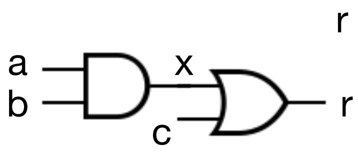



from km

$$r \leftrightarrow (a \wedge b)$$

$$(r \vee \neg a \vee \neg b) \wedge (\neg r \vee a) \wedge (\neg r \vee b)$$

from km

$$r \leftrightarrow (a \vee b)$$

$$(\neg r \vee a \vee b) \wedge (r \vee \neg a) \wedge (r \vee \neg b)$$

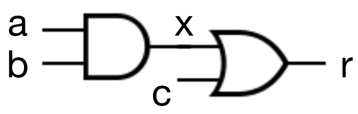
$$x \leftrightarrow (a \wedge b)$$

If we start from an expression then
we can draw an equivalent circuit with:

a wire for each subexpression,
a logic gate for each operator,
and an input for each variable.

$$r = (a \wedge b) \vee c$$

r




from km

$$r \leftrightarrow (a \wedge b)$$
$$(r \vee \neg a \vee \neg b) \wedge (\neg r \vee a) \wedge (\neg r \vee b)$$

from km

$$r \leftrightarrow (a \vee b)$$
$$(\neg r \vee a \vee b) \wedge (r \vee \neg a) \wedge (r \vee \neg b)$$

$$r \leftrightarrow (a \wedge b)$$

substitute

$$r := x \quad a := a \quad b := b$$

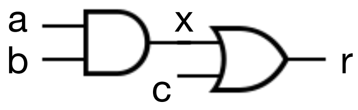
to give:

$$x \leftrightarrow (a \wedge b)$$

If we start from an expression then
we can draw an equivalent circuit with:

a wire for each subexpression,
a logic gate for each operator,
and an input for each variable.

$$r = (a \wedge b) \vee c$$




from km

$$r \leftrightarrow (a \wedge b)$$
$$(r \vee \neg a \vee \neg b) \wedge (\neg r \vee a) \wedge (\neg r \vee b)$$

substitute

$$r := x \quad a := a \quad b := b$$

to give:

$$x \leftrightarrow (a \wedge b)$$
$$(x \vee \neg a \vee \neg b) \wedge (\neg x \vee a) \wedge (\neg x \vee b)$$

from km

$$r \leftrightarrow (a \vee b)$$
$$(\neg r \vee a \vee b) \wedge (r \vee \neg a) \wedge (r \vee \neg b)$$

substitute

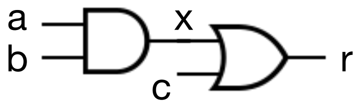
$$r := r \quad a := x \quad b := c$$

to give:

$$r \leftrightarrow (x \vee c)$$

If we start from an expression then
we can draw an equivalent circuit with:

$$r = (a \wedge b) \vee c$$




a wire for each subexpression,
a logic gate for each operator,
and an input for each variable.

from km

$$r \leftrightarrow (a \wedge b)$$
$$(r \vee \neg a \vee \neg b) \wedge (\neg r \vee a) \wedge (\neg r \vee b)$$

substitute

$$r := x \quad a := a \quad b := b$$

to give:

$$x \leftrightarrow (a \wedge b)$$
$$(x \vee \neg a \vee \neg b) \wedge (\neg x \vee a) \wedge (\neg x \vee b)$$

from km

$$r \leftrightarrow (a \vee b)$$
$$(\neg r \vee a \vee b) \wedge (r \vee \neg a) \wedge (r \vee \neg b)$$

substitute

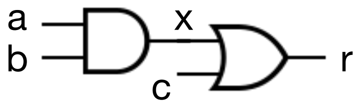
$$r := r \quad a := x \quad b := c$$

to give:

$$r \leftrightarrow (x \vee c)$$
$$(\neg r \vee x \vee c) \wedge (r \vee \neg x) \wedge (r \vee \neg c)$$

If we start from an expression then we can draw an equivalent circuit with:

$$r = (a \wedge b) \vee c$$




a wire for each subexpression,
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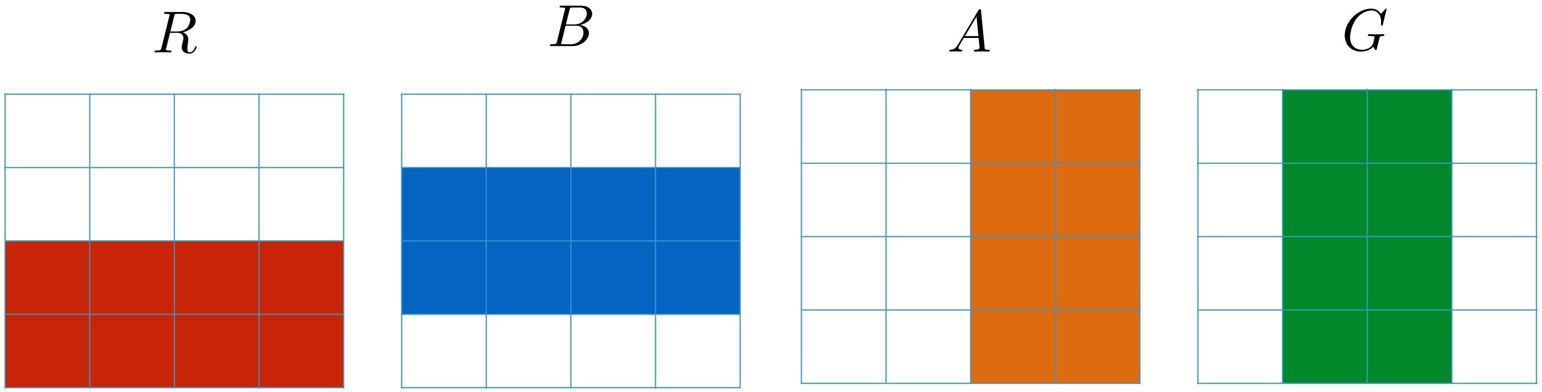
$$r \leftrightarrow (a \vee b)$$
$$(\neg r \vee a \vee b) \wedge (r \vee \neg a) \wedge (r \vee \neg b)$$

$$x \leftrightarrow (a \wedge b)$$
$$(x \vee \neg a \vee \neg b) \wedge (\neg x \vee a) \wedge (\neg x \vee b)$$

$$r \leftrightarrow (x \vee c)$$
$$(\neg r \vee x \vee c) \wedge (r \vee \neg x) \wedge (r \vee \neg c)$$

Combine the two CNF, with R = True

$$(x \vee \neg a \vee \neg b) \wedge (\neg x \vee a) \wedge (\neg x \vee b) \wedge (x \vee c)$$




| RB \ AG | 00 | 01 | 11 | 10 |
|---|---|---|---|---|
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|---|---|---|---|---|
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|---|---|---|---|---|
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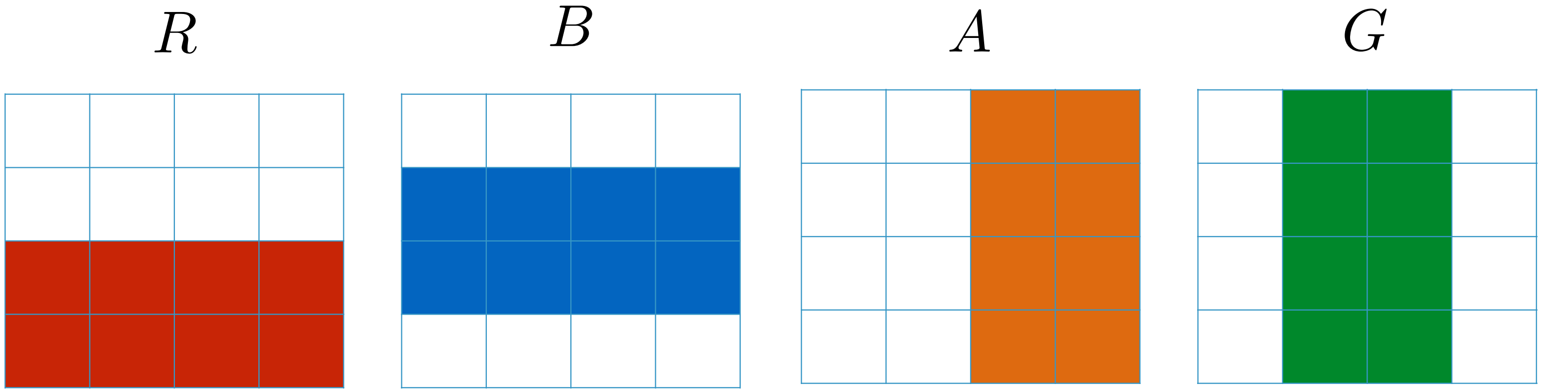
$$r \leftrightarrow (a \vee b)$$

$$(\neg r \vee a \vee b) \wedge (r \vee \neg a) \wedge (r \vee \neg b)$$

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|---|---|---|---|---|
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|---|---|---|---|---|
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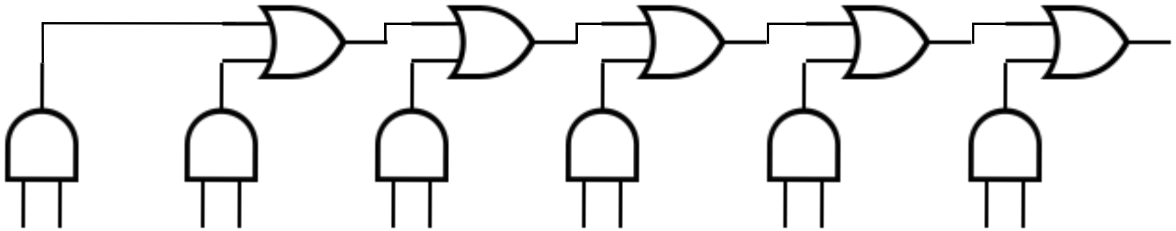
$$r \leftrightarrow (a \wedge b)$$

$$(r \vee \neg a \vee \neg b) \wedge (\neg r \vee a) \wedge (\neg r \vee b)$$



$$a \wedge b \vee c \wedge d \vee e \wedge f \vee g \wedge h \vee j \wedge k \vee m \wedge n$$

$$(a \wedge b) \vee (c \wedge d) \vee (e \wedge f) \vee (g \wedge h) \vee (j \wedge k) \vee (m \wedge n)$$

How many clauses in the CNF?

$2^6 = 64$

How many clauses to describe the circuit?

11 x 3 = 33 (before simplification)