# NFA to DFA

cl

- the subset construction
- ε-transitions

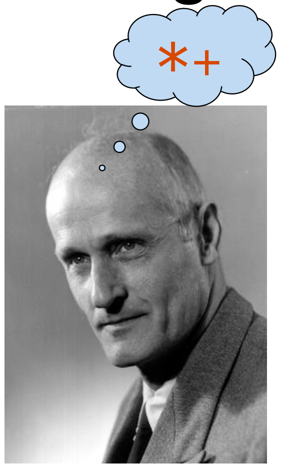
# regular expressions

## each regex is a pattern that matches a set of strings

- any character is a regex
  - matches itself
- if `R` and `S` are regex, so is `RS`
  - matches a match for `R` followed by a match for `S`
- if `R` and `S` are regex, so is `R|S`
  - matches any match for `R` or `S` (or both)
- if `R` is a regex, so is `R*`

  matches any sequence of 0 or more matches for `R`

Kleene *




Stephen Cole Kleene

1909-1994

- The algebra of regular expressions also includes elements `0` and `1`
  - `0 = ∅` matches nothing; `1 = Σ*` matches everything
  - `ε = ∅*` matches the empty string

```
0|R = R|0 = R     1|R = R|1 = 1       (S|T)R = SR|TR
 0R = R0 = 0       εR = Rε = R         R(S|T)= RS|RT
R|S = S|R           ε = 0*       A* = ε|AA* = ε|A*A
```

the language of strings that match a regex, R, is recognised by some ε-FSM

# A mathematical definition of a Finite State Machine.

$M = (Q, \Sigma, \Delta, S, F)$

Q: the set of states,

Σ: the alphabet of the machine

- the tokens the machine can process,

Δ: the set of **transitions**

is a set of (state, symbol, state) triples

$$\Delta \subseteq Q \times \Sigma \times Q.$$

S: the set of beginning or **start** states of the machine

F: the set of the machine's accepting of **finish** states.

A *trace* for $s = <x_0,\ldots x_{k-1}> \in \Sigma^*$ (a string of length **k**)

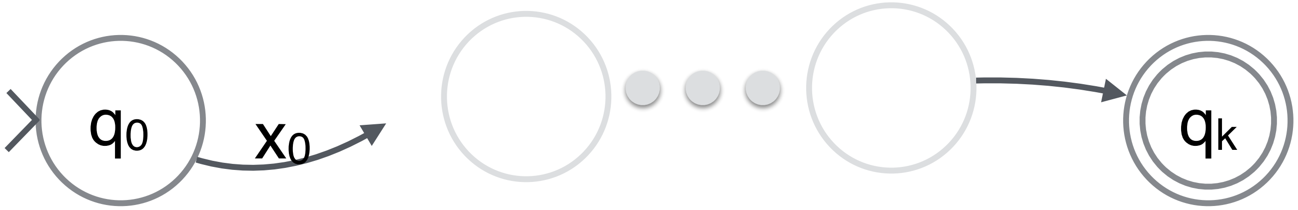
is a sequence of **k+1** states $<q_0,\ldots q_k>$

such that $(q_i, x_i, q_{i+1}) \in \Delta$ for each $i < k$

$M = (Q, \Sigma, \Delta, B, A, )$

A ***trace*** for $\mathbf{s} = \langle x_0, \ldots, x_{k-1} \rangle \in \Sigma^*$ (a string of length $\mathbf{k}$)
is a sequence of $\mathbf{k+1}$ states $\langle q_0, \ldots q_k \rangle$
such that $(q_i, x_i, q_{i+1}) \in \Delta$ for each $\mathbf{i < k}$

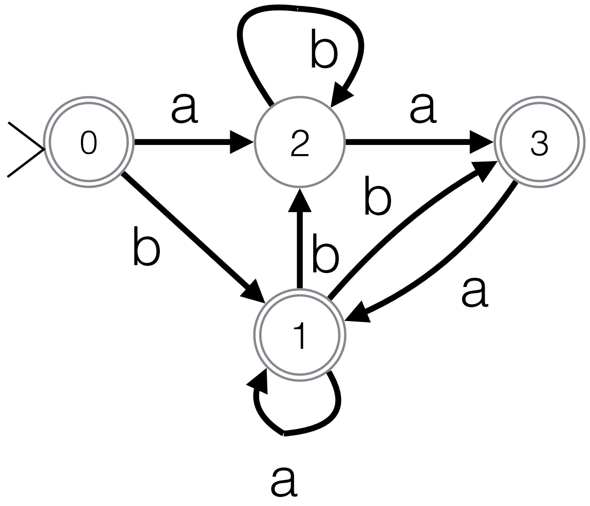
We say $\mathbf{s}$ is ***accepted*** by $\mathbf{M}$
iff there is
a trace $\langle q_0, \ldots q_k \rangle$ for $\mathbf{s}$
such that $q_0 \in B$ and $q_k \in A$



# Definition FSM

non-deterministic finite state automaton FSM

states — a set of states
sigma — a set of symbols
delta ⊆ (states × sigma × states)
start ⊆ states — starting states
accept ⊆ states — accepting states

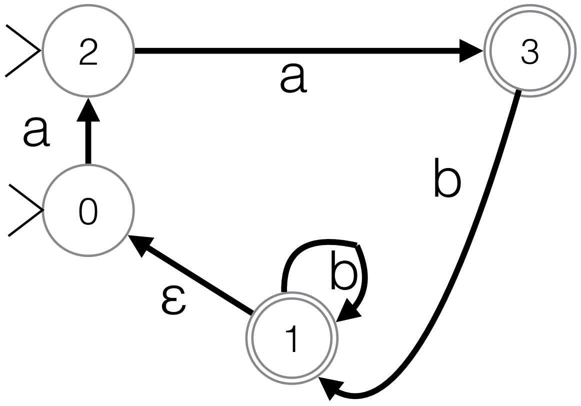



```
FSM qs as ts es ss fs where
  qs = [0..3]
  as = "ab"
  ts = [(0,'b',1),(0,'a',2),(1,'a',1),(1,'b',2)
       ,(1,'b',3),(2,'a',3),(2,'b',2),(3,'a',1)]
  ss = [0]
  fs = [0,1,3]
```

# Definition ε-FSM

finite state automaton FSM
with ε-transitions

qs states — a set of states
as sigma — a set of symbols
ts delta ⊆ (states × sigma × states)
es epsilon ⊆ (states × states)
ss start ⊆ states — starting states
fs final ⊆ states — accepting states




```
EPS qs as ts es ss fs where
  qs = [0..3]
  as = "ab"
  ts = [(0,'a',2), (1,'b',1), (2,'a',3),(3,'b',1)]
  es = [(1,0)]
  ss = [0,2]
  fs = [1,3]
```

# Definition DFA

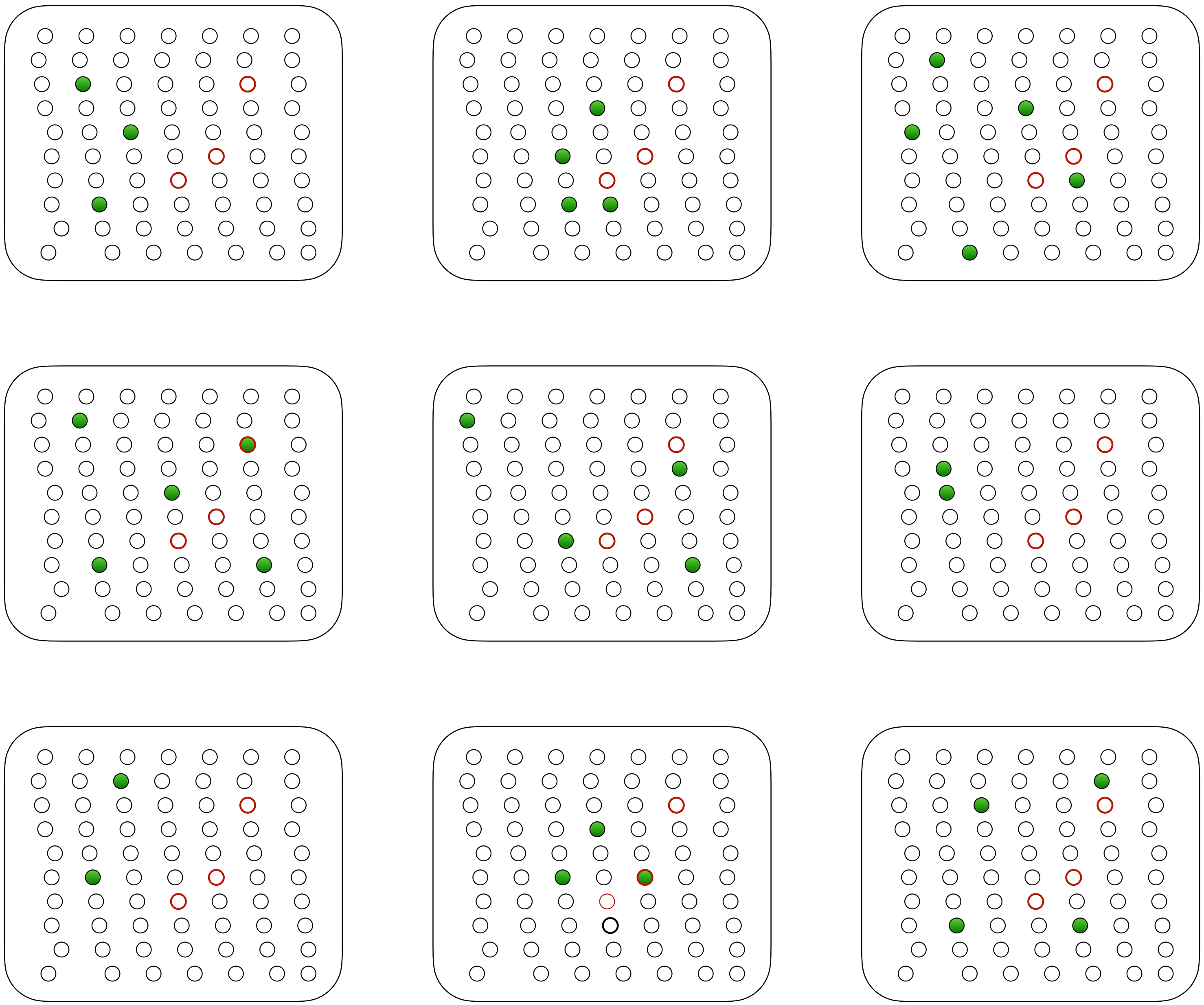
is a finite state automaton
(FSA, without ε)

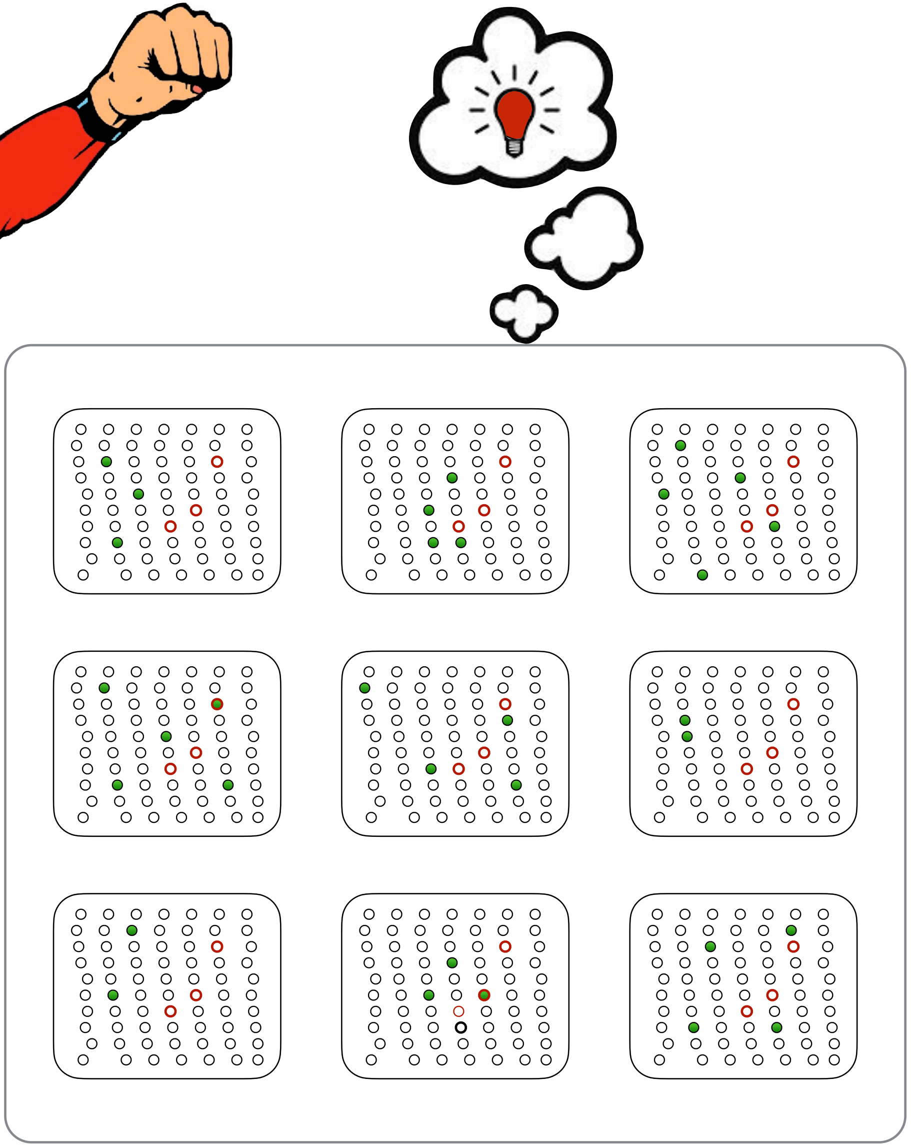
states — a set of states
sigma — a set of symbols
delta ⊆ (states × sigma × states)
start ⊆ states — starting states
accept ⊆ states — accepting states

A deterministic machine has

- no ε-transitions
- exactly one starting state
- for each (state, symbol) pair, (q, s)
  exactly one transition of the form (q, s, q')

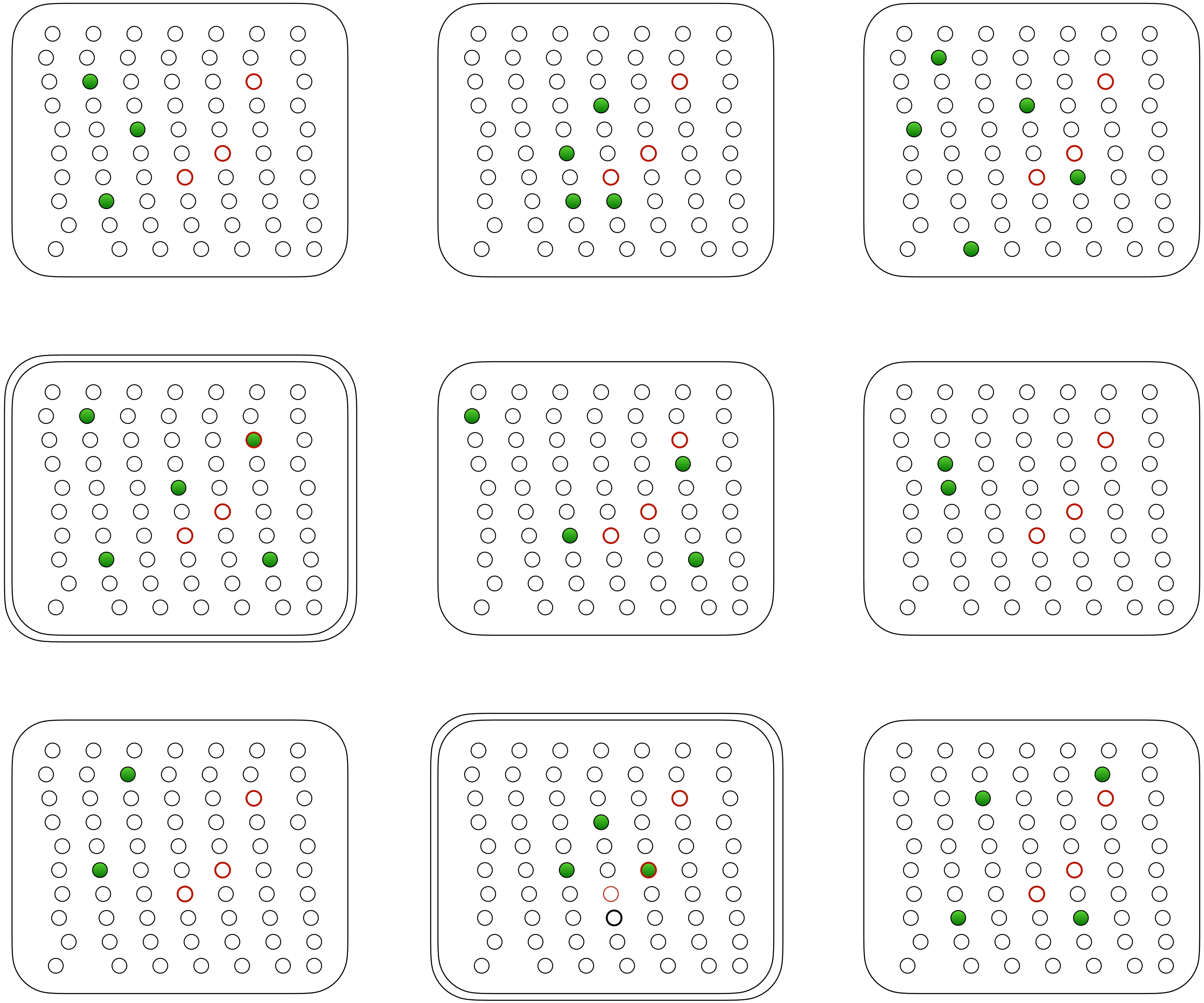
a DFA can be efficiently implemented
in software or hardware





# superstates

a **superstate** is a set of states



# superstates

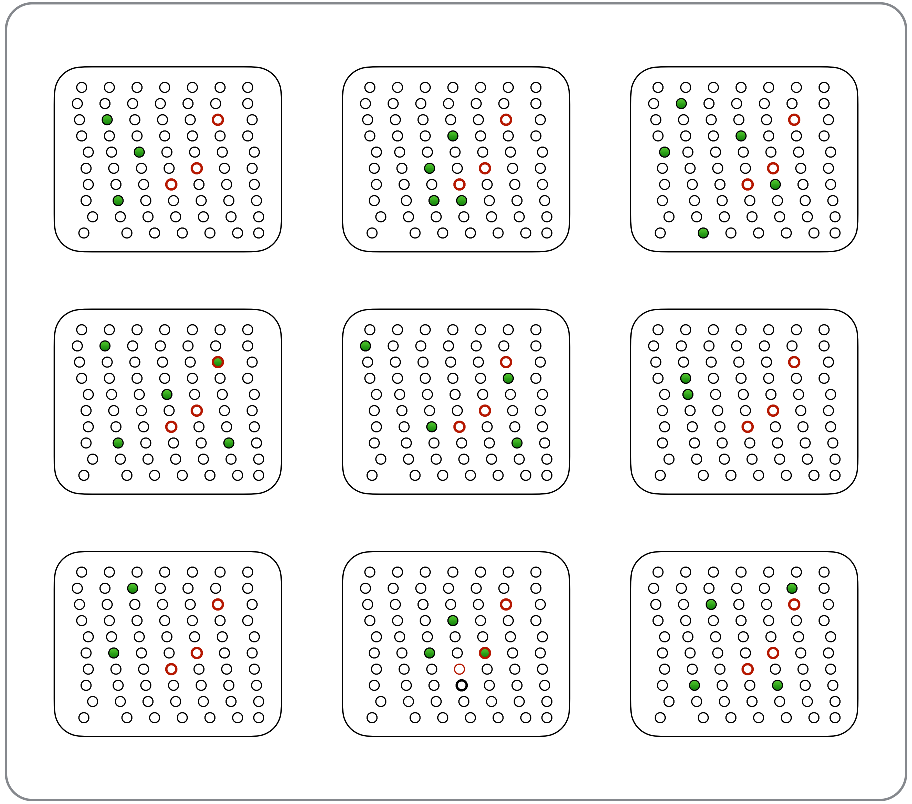
a **superstate** is a set of states

superstates are the states of DFA

The set of start states
is the unique start superstate

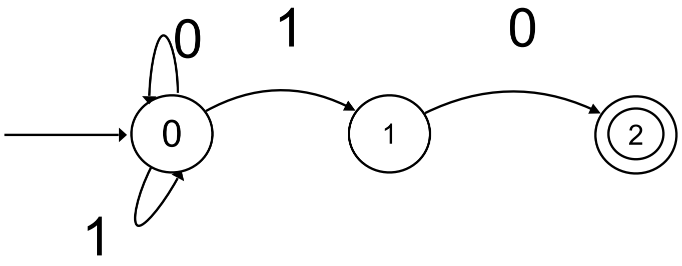
A finish superstate is
any superstate
that includes a finish state

A transition is the move from
one set of lit lights to the next

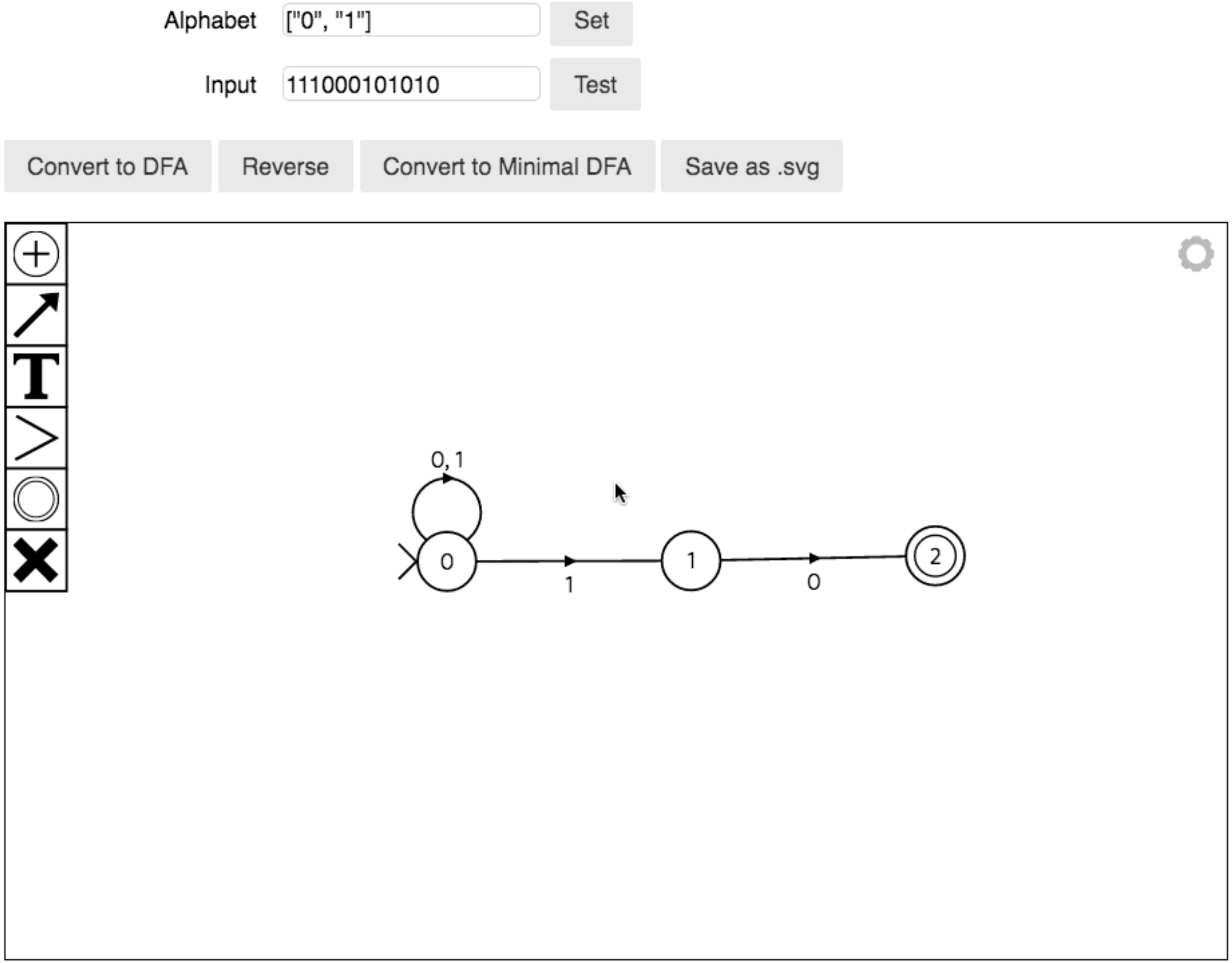


# Non Determinism

In a non-deterministic machine (NFA), each state may have any number of transitions with the same input symbol, leaving to different successor states.




| | 0 | 1 |
|---|---|---|
| **0** | 0 | 0,1 |
| **1** | 2 | |
| **2** | | |


Alphabet
["0", "1"]
Set
Input
111000101010
Test
Convert to DFA
Reverse
Convert to Minimal DFA
Save as .svg
0, 1
0
1
1
0
2

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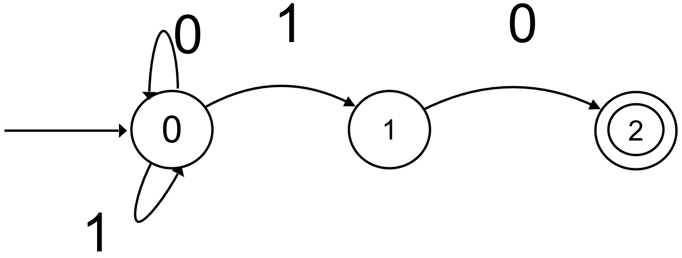


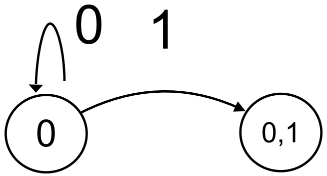


| | 0 | 1 |
|---|---|---|
| **0** | 0 | 0,1 |
| **1** | 2 | |
| **2** | | |
| | | |
| **0,1** | 0,2 | 0,1 |
| | | |

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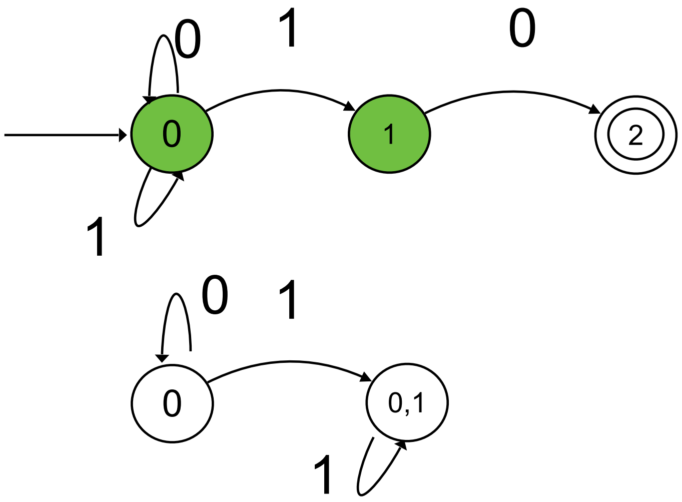


| | 0 | 1 |
|---|---|---|
| **0** | 0 | 0,1 |
| **1** | 2 | |
| **2** | | |
| | | |
| **0,1** | | 0,1 |
| | | |

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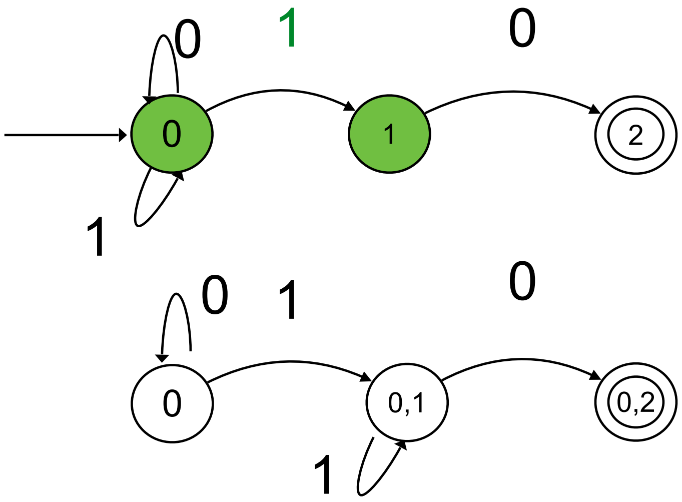


| | 0 | 1 |
|---|---|---|
| **0** | 0 | 0,1 |
| **1** | 2 | |
| **2** | | |
| | | |
| **0,1** | 0,2 | 0,1 |
| | | |

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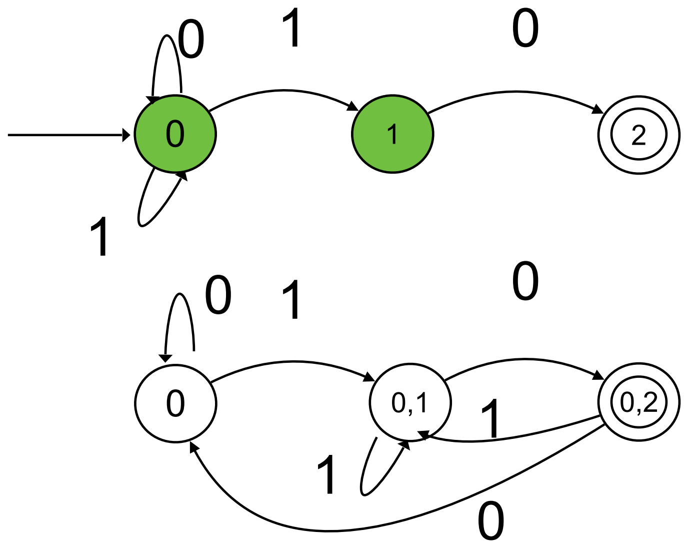
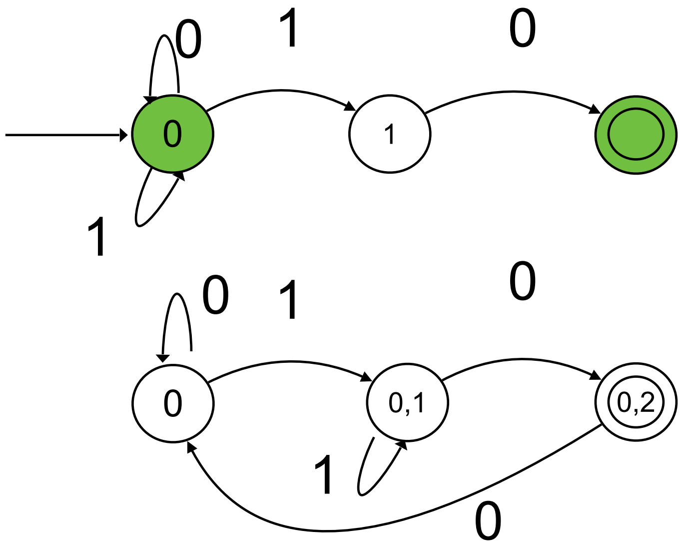



| | 0 | 1 |
|---|---|---|
| **0** | 0 | 0,1 |
| **1** | 2 | |
| **2** | | |
| | | |
| **0,1** | 0,2 | 0,1 |
| **0,2** | 0 | 0,1 |

# Non Determinism

We can simulate a non-deterministic machine using a deterministic machine – by keeping track of the set of states the NFA could possibly be in.




| | 0 | 1 |
|---|---|---|
| **0** | 0 | 0,1 |
| **1** | 2 | |
| **2** | | |
| | | |
| **0,1** | 0,2 | 0,1 |
| **0,2** | 0 | |

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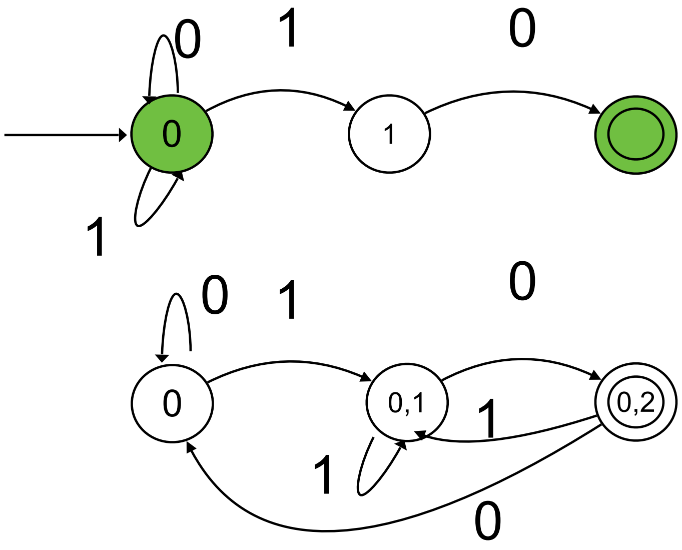
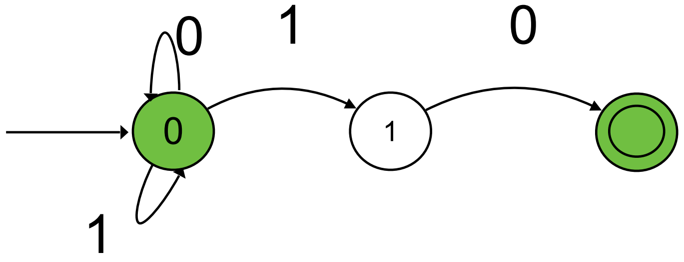
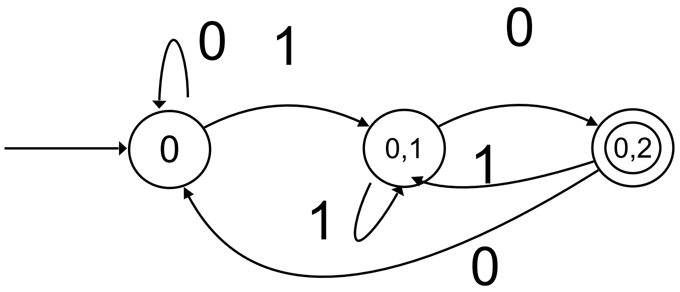



| | 0 | 1 |
|---|---|---|
| **0** | 0 | 0,1 |
| **1** | 2 | |
| **2** | | |
| | | |
| **0,1** | 0,2 | 0,1 |
| **0,2** | 0 | 0,1 |

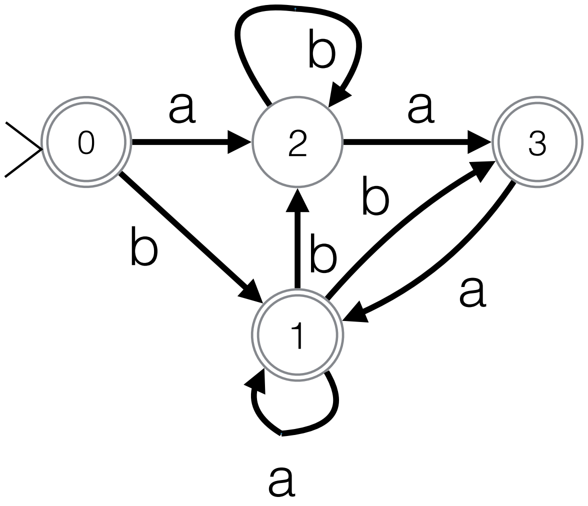



```
FSM qs as ts es ss fs where
  qs = [0..2]
  as = "01"
  ts = [(0,'0',0),(0,'1',0)
       ,(0,'1',1),(1,'0',2)]
  ss = [0]
  fs = [2]
```

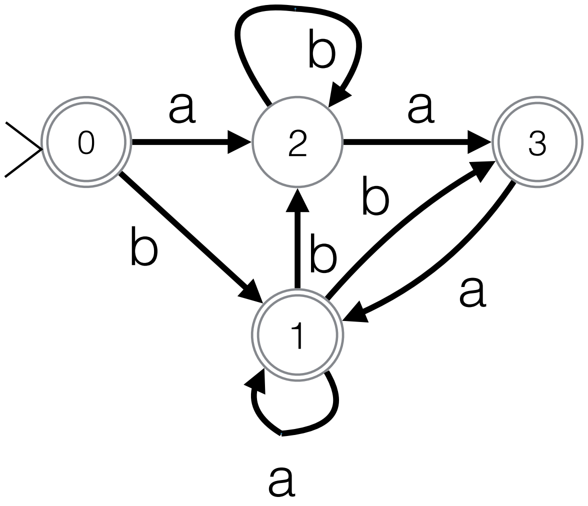



```
FSM qs as ts es ss fs where
  qs = [[0],[0,1],[0,2]]
  as = "01"
  ts = [([0],'0',[0])
       ,([0],'1',[0,1])
       ,([0,1],'0',[0,2])
       ,([0,1],'1',[0,1])
       ,([0,2],'0',[0])
       ,([0,2],'1',[0,1])]
  ss = [[0]]
  fs = [[0,2]]
```

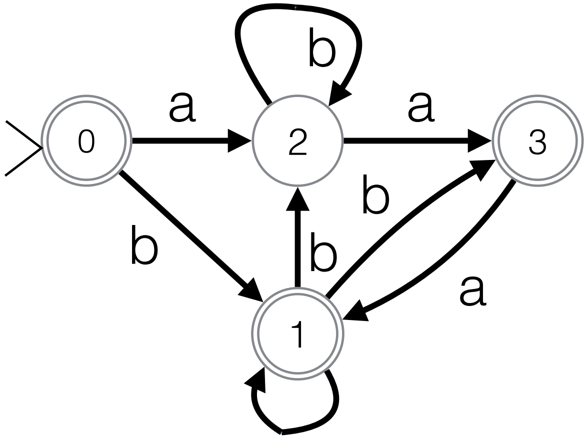
| | a | b |
|---|---|---|
| 0 | 2 | 1 |
| 1 | 1 | 2,3 |
| 2 | 3 | 2 |
| 3 | 1 | |
| | | |
| 2,3 | | |
| | | |

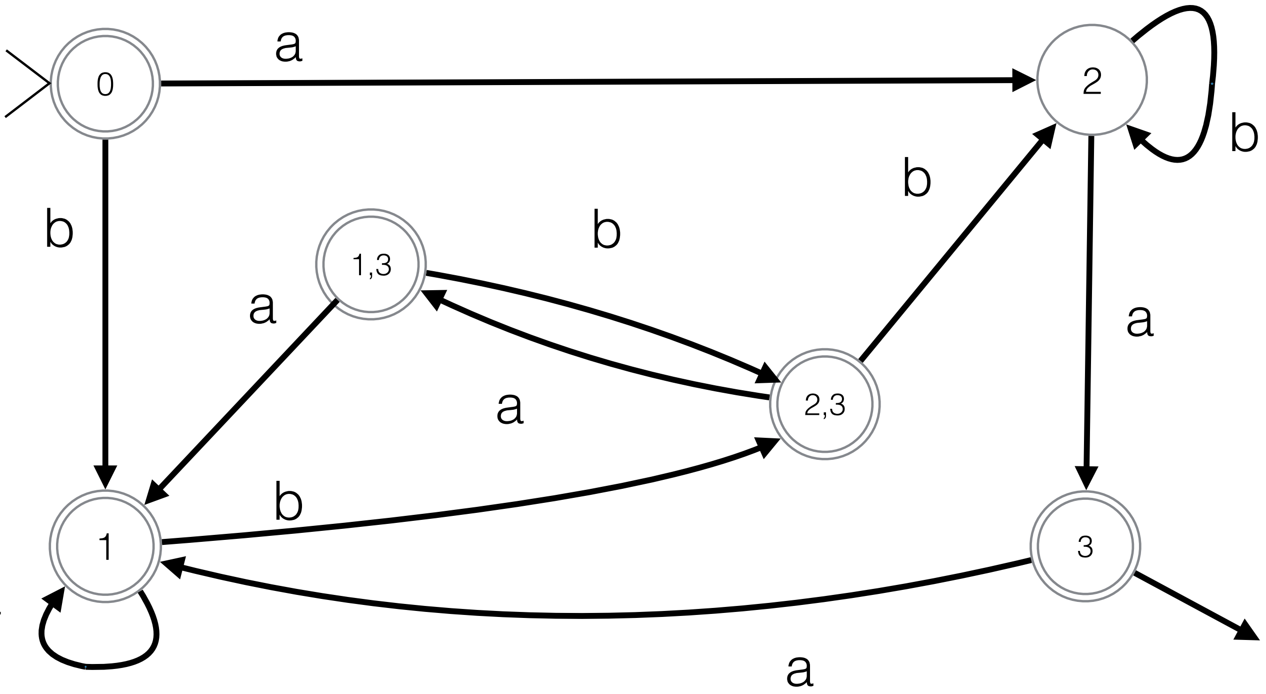


| | a | b |
|---|---|---|
| **0** | 2 | 1 |
| **1** | 1 | 2,3 |
| **2** | 3 | 2 |
| **3** | 1 | |
| | | |
| **2,3** | 1,3 | 2 |
| | | |



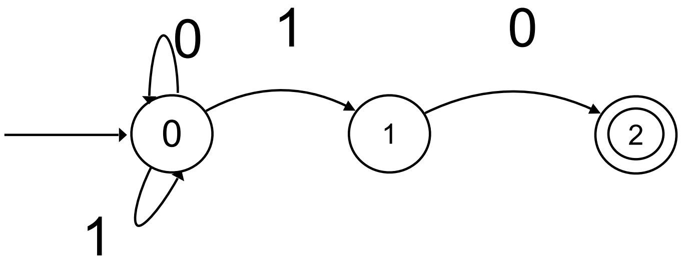
| | a | b |
|---|---|---|
| **0** | 2 | 1 |
| **1** | 1 | 2,3 |
| **2** | 3 | 2 |
| **3** | 1 | |
| | | |
| **2,3** | 1,3 | 2 |
| **1,3** | 1 | 2,3 |

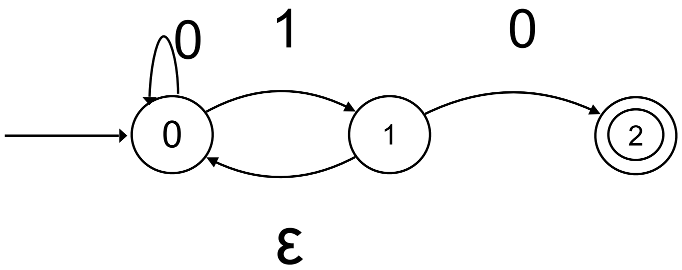


# Internal Transitions

We sometimes add an internal transition ε to a non-deterministic machine (NFA)This is a state change that consumes no input.




| | 0 | 1 | ε |
|---|---|---|---|
| **0** | 0 | 1 | |
| **1** | 2 | | 0 |
| **2** | | | |

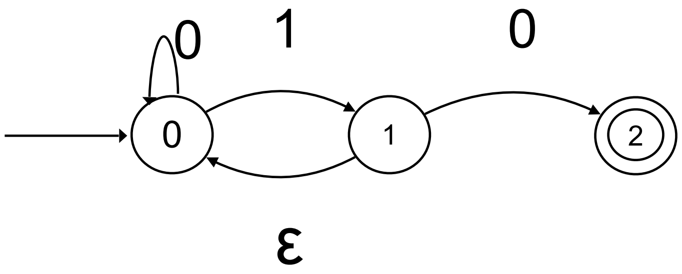
# Internal Transitions

We sometimes add **internal transitions** – labelled ε – to a non-deterministic machine (NFA).

This is a state change that consumes no input.

It introduces non-determinism in the observed behaviour of the machine.

| | 0 | 1 | ε |
|---|---|---|---|
| **0** | 0 | 1 | |
| **1** | 2 | | 0 |
| **2** | | | |




| | 0ε* | 1ε* |
|---|---|---|
| **0** | 0 | 1,0 |
| **1** | 2 | |
| **2** | | |

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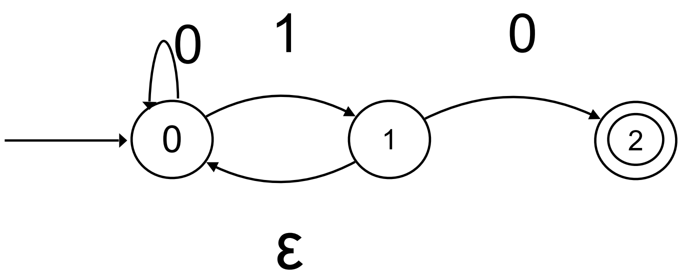


| | 0 | 1 | ε |
|---|---|---|---|
| **0** | 0 | 1 | |
| **1** | 2 | | 0 |
| **2** | | | |

| | 0ε* | 1ε* |
|---|---|---|
| **0** | 0 | 0,1 |
| **0,1** | 0,2 | 0,1 |
| **0,2** | 0 | 0,1 |

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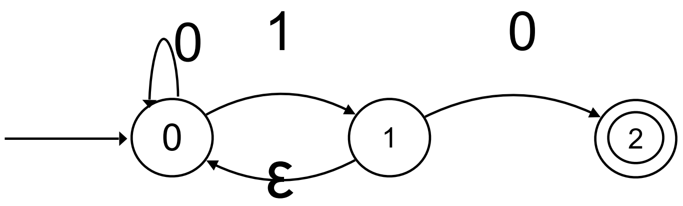
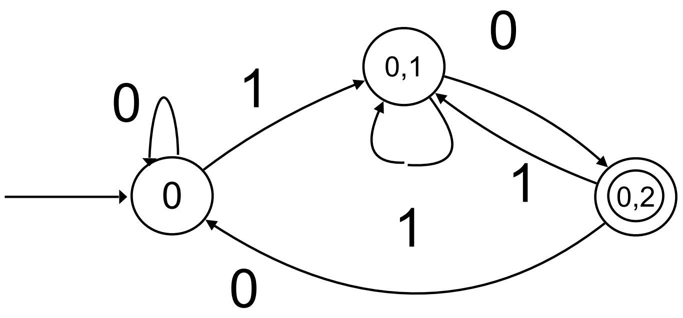
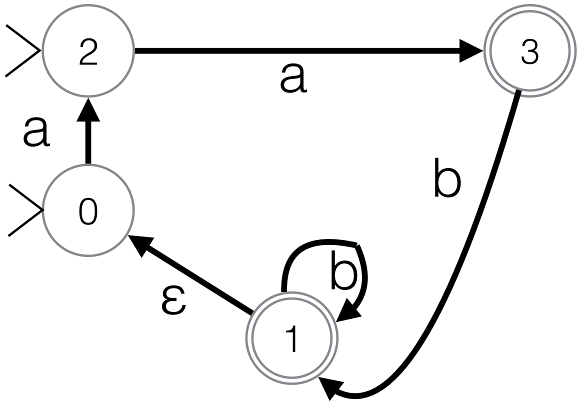



| | 0 | 1 | ε |
|---|---|---|---|
| **0** | 0 | 1 | |
| **1** | 2 | | 0 |
| **2** | | | |

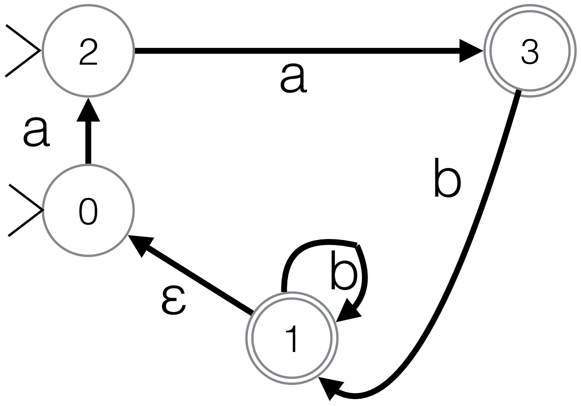
| | 0ε* | 1ε* |
|---|---|---|
| **0** | 0 | 0,1 |
| **0,1** | 0,2 | 0,1 |
| **0,2** | 0 | 0,1 |

| | a | b | ε |
|---|---|---|---|
| 0 | 2 | | |
| 1 | | 1 | 0 |
| 2 | 3 | | |
| 3 | | 1 | |

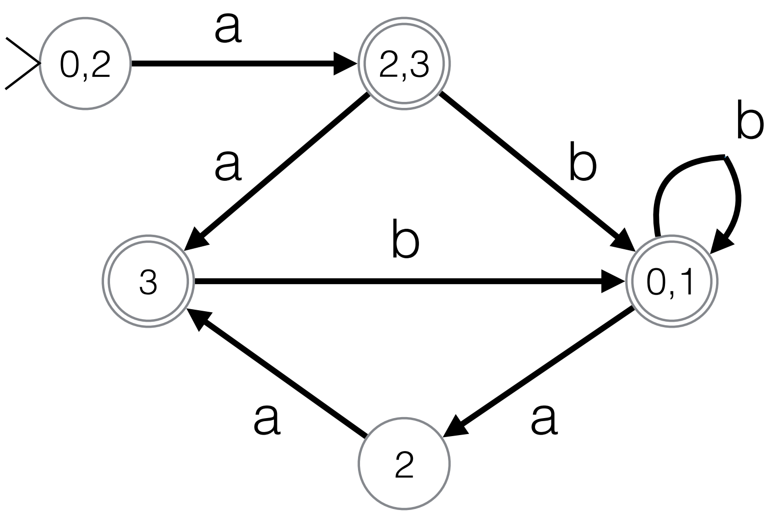



| | aε* | bε* |
|---|---|---|
| 0,2 | 2,3 | |
| 2,3 | | |
| | | |
| | | |
| | | |

| | a | b | ε |
|---|---|---|---|
| 0 | 2 | | |
| 1 | | 1 | 0 |
| 2 | 3 | | |
| 3 | | 1 | |




| | aε* | bε* |
|---|---|---|
| 0,2 | 2,3 | |
| 2,3 | 3 | 0,1 |
| 3 | | 0,1 |
| 0,1 | 2 | 0,1 |
| 2 | 3 | |



# http://xkcd.com/