Formal Verification

Lecture 8: Operations on Binary Decision Diagrams (BDDs)

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Diagrams from Huth & Ryan, LiCS, 2nd Ed.

Recap

- Previously:
  - (Reduced, Ordered) Binary Decision Diagrams ((RO)BDDs)
- This time:
  - Operations on ROBDDs
    `reduce`, `apply`, `restrict`, `exists`
  - Symbolic Model Checking with BDDs

Binary Decision Diagrams

Binary Decision Diagrams: DAGs, such that

- Unique root node
- Variables on non-terminal nodes
- Truth-values on terminal nodes
- Exactly two edges from each non-terminal node, labelled 0, 1

Some notation, for a given BDD node n:

- If n is a non-terminal node:
  $\mathrm{var}(n)$ — the variable label on node n;
  $\mathrm{lo}(n)$ — the node reached by following the 0 edge from n;
  $\mathrm{hi}(n)$ — the node reached by following the 1 edge from n;
- If n is a terminal node:
  $\mathrm{val}(n)$ — the truth value labelling n

For a BDD B, the root node is called $\mathrm{root}(B)$.

`reduce`

`reduce` constructs a ROBDD from an OBDD.

1. Label each OBDD node n with an integer $\mathrm{id}(n)$,
2. in a single bottom-up pass, such that:
3. two OBDD nodes m and n have the same label ($\mathrm{id}(m) = \mathrm{id}(n)$) if and only if m and n represent the same boolean function.

The ROBDD is then created by using one node from each class of nodes with the same label.

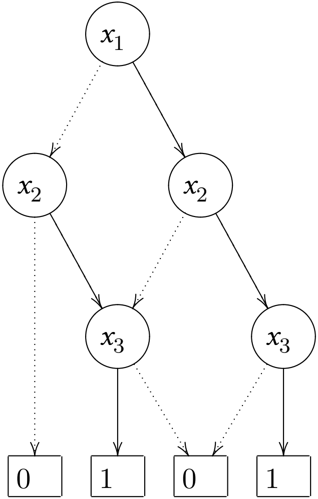
`reduce`

Assignment of labels follows the rules for performing reductions.

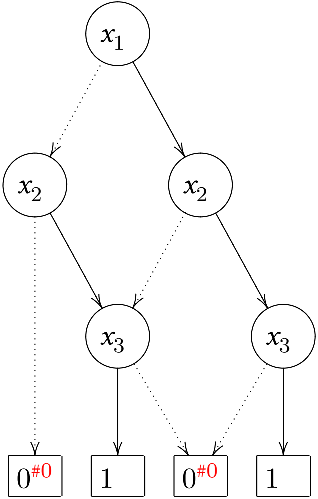
To label a node n:

- Remove duplicate terminals:
  if n is a terminal node (*i.e.*, $\boxed{0}$ or $\boxed{1}$), then set $\mathrm{id}(n)$ to be $\mathrm{val}(n)$.
- Remove redundant tests:
  if $\mathrm{id}(\mathrm{lo}(n)) = \mathrm{id}(\mathrm{hi}(n))$ then set $\mathrm{id}(n)$ to be $\mathrm{id}(\mathrm{lo}(n))$.
- Remove duplicate nodes:
  if there exists a node m that has already been labelled such that
  $$\left\{\begin{array}{c} \mathrm{var}(m) = \mathrm{var}(n) \\ \mathrm{lo}(m) = \mathrm{lo}(n) \\ \mathrm{hi}(m) = \mathrm{hi}(n) \end{array}\right\}, \text{ set } \mathrm{id}(n) \text{ to } \mathrm{id}(m).$$
  Use a hashtable with $\langle \mathrm{var}(n), \mathrm{lo}(n), \mathrm{hi}(n) \rangle$ keys for $O(1)$ lookup time.
- Otherwise, set $\mathrm{id}(n)$ to an unused number.

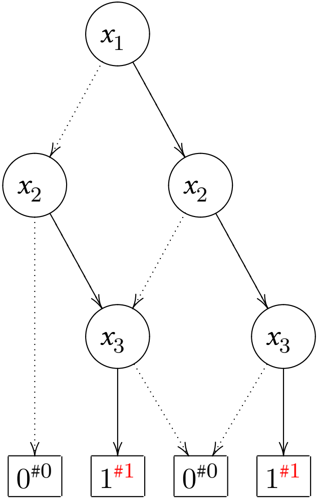
`reduce` Example



`reduce` Example



reduce Example



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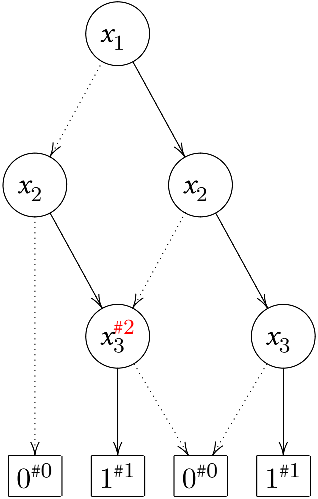

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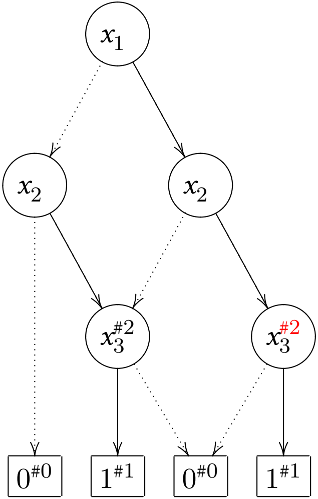

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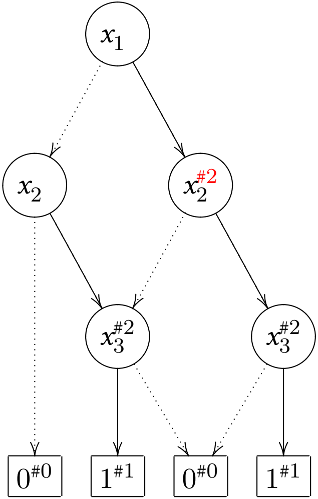

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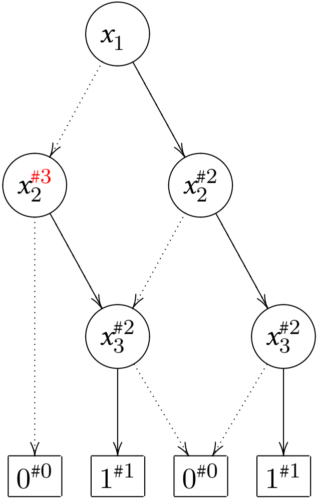

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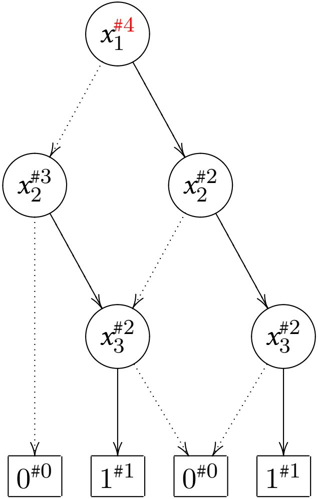
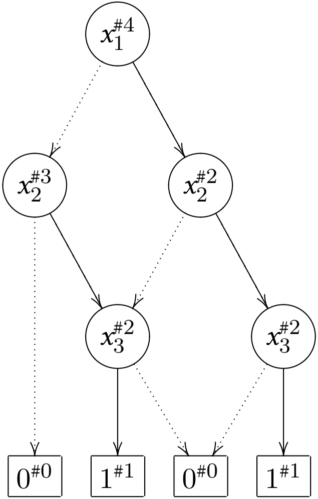
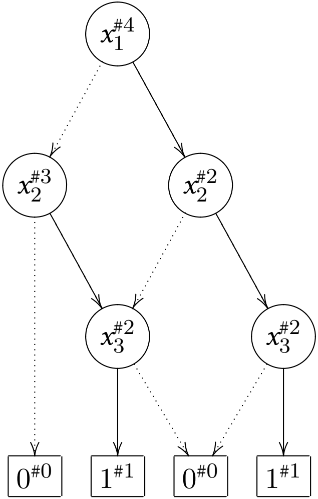


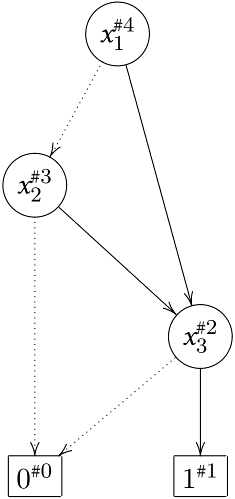
`reduce` Example



`reduce` Example



Reduces to



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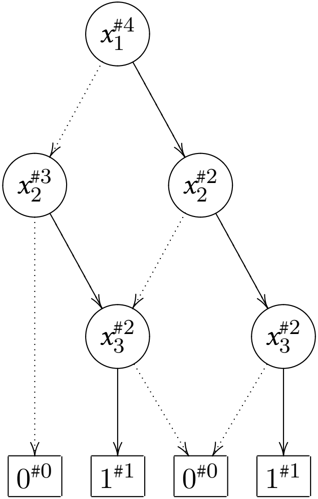

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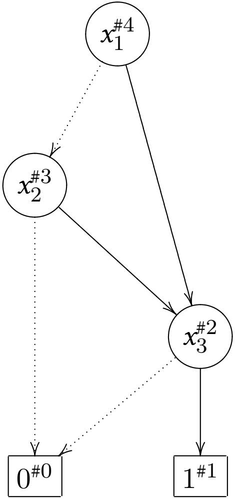

In practice, labelling and construction are interleaved.

`apply`

Given compatible OBDDs B_f and B_g that represent formulas f and g, `apply`$(\square, B_f, B_g)$ computes an OBDD representing $f \square g$.

- where $\square$ represents some binary operation on boolean formulas

  for example, $\wedge$, $\vee$, $\oplus$

- Unary operations can be handled too.

  for example, negation: $\neg x = x \oplus 1$

`apply`: Shannon expansions

For any boolean formula f and variable x, it can be written as:

$$f \equiv (\neg x \wedge f[0/x]) \vee (x \wedge f[1/x])$$

This is the **Shannon expansion** of f (originally due to G. Boole).

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In particular: $f \square g$ can be expanded like so:

$$f \square g \equiv (\neg x \wedge (f[0/x] \square g[0/x])) \vee (x \wedge (f[1/x] \square g[1/x]))$$

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In particular: $f \Box g$ can be expanded like so:

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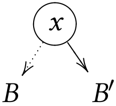
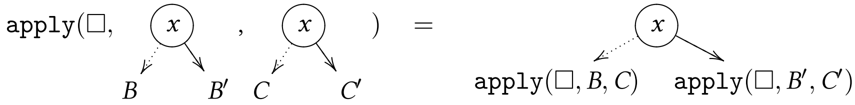
If a BDD 

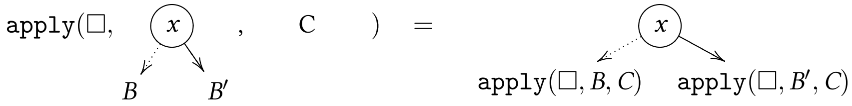
represents a boolean function f, then:

1. B represents $f[0/x]$ and B' represents $f[1/x]$; and
2. The BDD is effectively a compressed representation of f in Shannon normal form.

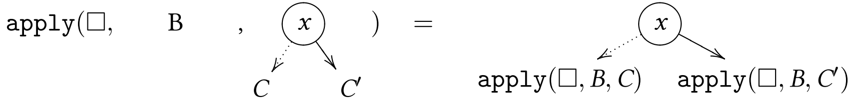
So: implement `apply` recursively on the structure of the BDDs.

`apply`: cases





when C is terminal node, or non-terminal with $\mathrm{var}(\mathrm{root}(C)) > x$

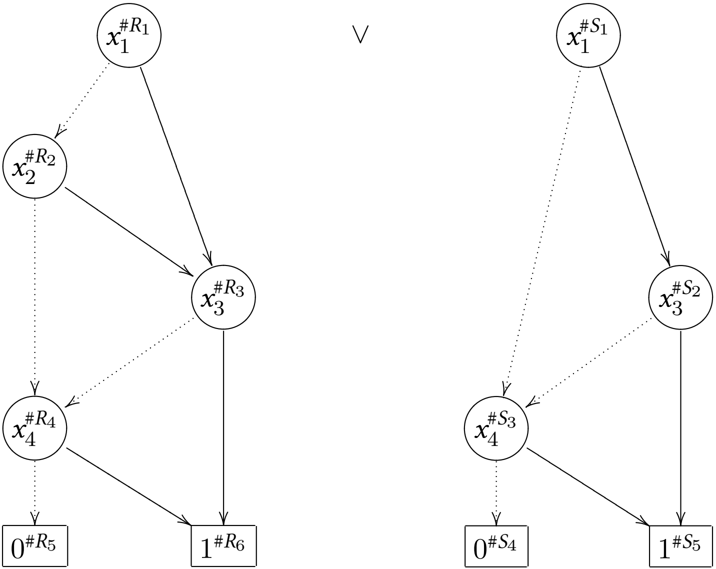


when B is terminal node, or non-terminal with $\mathrm{var}(\mathrm{root}(B)) > x$

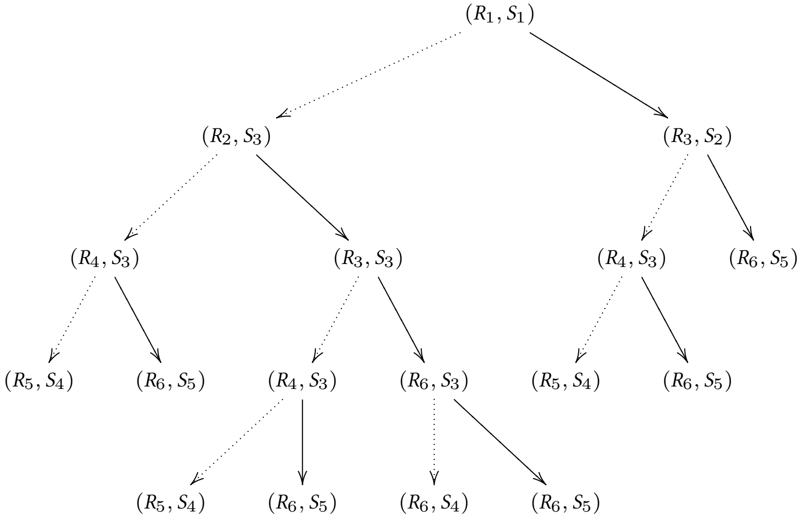
$$\texttt{apply}(\square, \boxed{u}, \boxed{v}) = \boxed{u \,\square\, v}$$

`apply`: example

Compute $\texttt{apply}(\vee, B_f, B_g)$, where B_f and B_g are:



`apply`: recursive calls



`apply`: memoisation

The recursive `apply` implementation will generate an OBBD.

- Apply `reduce` to convert it back to a ROBDD.

However, as can be seen from the tree of recursive calls, there are many calls to `apply` with the same arguments.

- Each invocation of `apply` where at least one of the arguments is non-terminal generates two further calls to `apply`: the number of calls is worst-case exponential in the sizes of the original diagrams.

We are not taking into account the **sharing** in BDDs.

We can greatly improve the run-time by using **memoisation**: remembering the results of previous calls.

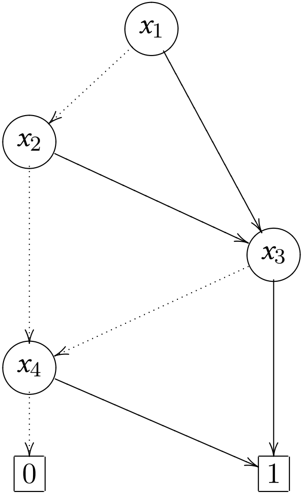
`apply`: memoised recursive calls

Memoisation results in at most $|B_f| \cdot |B_g|$ calls to `apply`.

$$
\begin{tikzcd}
 & (R_1, S_1)(x_1) \arrow[dl, dotted] \arrow[dr] & \\
(R_2, S_3)(x_2) \arrow[dd, dotted] \arrow[dr] & & (R_3, S_2)(x_3) \arrow[ddll, dotted, bend right] \arrow[dddl] \\
 & (R_3, S_3)(x_3) \arrow[dl, dotted] \arrow[dr] & \\
(R_4, S_3)(x_4) \arrow[d, dotted] \arrow[dr] & & (R_6, S_3)(x_4) \arrow[dl] \arrow[d, dotted] \\
(R_5, S_4)(0) & (R_6, S_5)(1) & (R_6, S_4)(1)
\end{tikzcd}
$$

`apply`: Result

If we are careful to never create the same BDD node twice (using the same lookup table technique as `reduce`), then with memoisation, we automatically get a reduced BDD:



Other Operations

`restrict`$(0, x, B_f)$ computes ROBDD for $f[0/x]$

1. For each node n labelled with x, incoming edges are redirected to $\mathrm{lo}(n)$, and the node n is removed.
2. Resulting BDD then reduced with `reduce`.
3. (again, `reduce` can be interleaved with the removal.)

`exists`(x, B_f) computes ROBDD for $\exists x.\ f$.

1. Uses the identity

$$(\exists x.\ f) \equiv f[0/x] \vee f[1/x]$$

2. Realised using the `restrict` and `apply` functions:

$$\texttt{apply}(\vee, \texttt{restrict}(0, x, B_f), \texttt{restrict}(1, x, B_f))$$

Time Complexities

Algorithm	Input OBDDs	Output OBDD	Time complexity
`reduce`	B	reduced B	$O(\lvert B\rvert \cdot \log \lvert B\rvert)$
`apply`	B_f, B_g (reduced)	$B_{f \square g}$ (reduced)	$O(\lvert B_f\rvert \cdot \lvert B_g\rvert)$
`restrict`	B_f (reduced)	$B_{f[0/x]}$ or $B_{f[1/x]}$ (red'd)	$O(\lvert B_f\rvert \cdot \log \lvert B_f\rvert)$
$\exists$	B_f (reduced)	$B_{\exists x_1 \ldots x_n . f}$ (reduced)	NP-complete

H&R, Figure 6.23

Implementing CTL Model Checking using BDDs

Recall:

1. CTL model checking computes a set of states $[\![\phi]\!]$ for every sub-formula ϕ of the original formula.
2. Sets of states will be represented using ROBDDs

States are represented by boolean vectors $\langle v_1, \ldots, v_n \rangle$ (i.e. $v_i \in \{0, 1\}$).

Sets of states are represented using ROBDDs on n variables $x_1, \ldots, x_n$ (the Atoms) that describe the **characteristic function** of the set (see H&R 6.3.1 for a detailed description).

- Set operations $(\cap, \cup, ^{-})$ implemented using the operations on BBDs

For example, the definition

$$[\![\phi \wedge \psi]\!] = [\![\phi]\!] \cap [\![\psi]\!]$$

is implemented by:

$$B_{[\![\phi \wedge \psi]\!]} = \mathtt{apply}(\wedge, B_{[\![\phi]\!]}, B_{[\![\psi]\!]})$$

Implementing CTL Model Checking using BDDs

Transition relations $(\rightarrow) \subseteq S \times S$ are represented by ROBDDs on $2n$ variables.

- If the variables $x_1, \ldots, x_n$ describe the current state, and the variables $x_1', \ldots, x_n'$ describe the next state, then a good ordering is $x_1, x_1', x_2, x_2', \ldots, x_n, x_n'$ (interleaving).

When translating from the model description, the boolean formulas describing the:

1. initial state set
2. transition relation
3. defined variables

are translated into ROBDDs by using the `apply` algorithm, following the structure of the original formula.

This avoids exponential blow-up from first constructing a decision tree and then reducing.

Implementing CTL Model Checking using BDDs

The function applications

$$\begin{array}{rcl} \mathrm{pre}_\exists(Y) & \doteq & \{s \in S \mid \exists s' \in S.\ (s \to s') \wedge s' \in Y\} \\ \mathrm{pre}_\forall(Y) & \doteq & \{s \in S \mid \forall s' \in S.\ (s \to s') \to s' \in Y\} \end{array}$$

are implemented using BDDs like so:

$$B_{\mathrm{pre}_\exists(Y)} = \mathtt{exists}(\overrightarrow{x'}, \mathtt{apply}(\wedge, B_\to, B_{Y'}))$$

where

- $B_\to$ is the ROBDD representing the transition relation $\to$;
- $B_{Y'}$ is the ROBDD representing the set Y with the variables $x_1, \ldots, x_n$ renamed to $x'_1, \ldots, x'_n$.

And:

$$\mathrm{pre}_\forall(Y) = S - \mathrm{pre}_\exists(S - Y)$$

where $S - Y$ is implemented by negation (via apply).

Implementing CTL Model Checking using BDDs

To implement the temporal connectives, we compute fix points.

$$
\begin{array}{lcl}
[\![\mathbf{EF}\ \phi]\!] & = & \mu Y.\ [\![\phi]\!] \cup \mathrm{pre}_{\exists}(Y) \\
[\![\mathbf{EG}\ \phi]\!] & = & \nu Y.\ [\![\phi]\!] \cap \mathrm{pre}_{\exists}(Y) \\
\ldots & &
\end{array}
$$

By Knaster-Tarski, we know that:

- $F^{|S|}(\emptyset)$ is the *least* fixed point of F: $\mu Y.F(Y)$
- $F^{|S|}(S)$ is the *greatest* fixed point of F: $\nu Y.F(Y)$

Compute $[\![\mathbf{EF}\ \phi]\!]$ using the sequence (of ROBDDs)

$$Y^0 = \emptyset,\ Y^1 = [\![\phi]\!] \cup \mathrm{pre}_{\exists}(\emptyset),\ Y^2 = [\![\phi]\!] \cup \mathrm{pre}_{\exists}([\![\phi]\!] \cup \mathrm{pre}_{\exists}(\emptyset)), \ldots$$

Usually, we won't need $|S|$ steps: we can stop when $Y_i = Y_{i+1}$

- This check is very cheap with ROBDDs.

Summary

- Operations on BDDs (H&R 6.2)
  - `reduce`
  - `apply`
  - `restrict`, `exists`
- Symbolic Model Checking (H&R 6.3)
  - Representing states and transitions as BDDs
  - Implementing the CTL MC algorithm with BDDs