Elements of Programming Languages
Lecture Notes: $\mathsf{L}_{\mathsf{Poly}}$

1 Abstract Syntax

$$
\begin{array}{rcll}
\mathit{Expr} \ni e & ::= & n \in \mathbb{N} \mid e_1 + e_2 \mid e_1 \times e_2 & \mathsf{L}_{\mathsf{Arith}} \\
& \mid & b \in \mathbb{B} \mid e_1 == e_2 \mid \mathtt{if}\ e\ \mathtt{then}\ e_1\ \mathtt{else}\ e_2 & \mathsf{L}_{\mathsf{If}} \\
& \mid & x \mid \mathtt{let}\ x = e_1\ \mathtt{in}\ e_2 & \mathsf{L}_{\mathsf{Let}} \\
& \mid & e_1\ e_2 \mid \lambda x{:}\tau.\ e & \mathsf{L}_{\mathsf{Lam}} \\
& \mid & \mathtt{rec}\ f(x:\tau_1):\tau_2.\ e & \mathsf{L}_{\mathsf{Rec}} \\
& \mid & (e_1, e_2) \mid \mathtt{fst}\ e \mid \mathtt{snd}\ e \mid \mathtt{let}\ \mathtt{pair}\ (x,y) = e_1\ \mathtt{in}\ e_2 \mid () & \\
& \mid & \mathtt{left}(e) \mid \mathtt{right}(e) \mid \mathtt{case}\ e\ \mathtt{of}\ \{\mathtt{left}(x) \Rightarrow e_1\ ;\ \mathtt{right}(y) \Rightarrow e_2\} & \mathsf{L}_{\mathsf{Data}} \\
& \mid & \Lambda A.\ e \mid e[\tau] & \mathsf{L}_{\mathsf{Poly}} \\
\\
\mathit{Type} \ni \tau & ::= & \mathtt{int} & \mathsf{L}_{\mathsf{Arith}} \\
& \mid & \mathtt{bool} & \mathsf{L}_{\mathsf{If}} \\
& \mid & \tau_1 \to \tau_2 & \mathsf{L}_{\mathsf{Lam}} \\
& \mid & \tau_1 \times \tau_2 \mid \mathtt{unit} \mid \tau_1 + \tau_2 \mid \mathtt{empty} & \mathsf{L}_{\mathsf{Data}} \\
& \mid & A \mid \forall A.\ \tau & \mathsf{L}_{\mathsf{Poly}} \\
\\
\Gamma & ::= & x_1 : \tau_1, \ldots, x_n : \tau_n & \\
\\
\mathit{Value} \ni v & ::= & n \in \mathbb{N} & \mathsf{L}_{\mathsf{Arith}} \\
& \mid & b \in \mathbb{B} & \mathsf{L}_{\mathsf{If}} \\
& \mid & \lambda x.\ e & \mathsf{L}_{\mathsf{Lam}} \\
& \mid & \mathtt{rec}\ f(x).\ e & \mathsf{L}_{\mathsf{Rec}} \\
& \mid & (v_1, v_2) \mid () \mid \mathtt{left}(v) \mid \mathtt{right}(v) & \mathsf{L}_{\mathsf{Data}} \\
& \mid & \Lambda A. e & \mathsf{L}_{\mathsf{Poly}}
\end{array}
$$

1.1 Free variables

In the following, $\oplus$ stands for any binary operator.

$$
\begin{aligned}
FV(n) &= \emptyset \\
FV(e_1 \oplus e_2) &= FV(e_1) \cup FV(e_2) \\
FV(b) &= \emptyset \\
FV(\texttt{if}\ e\ \texttt{then}\ e_1\ \texttt{else}\ e_2) &= FV(e) \cup FV(e_1) \cup FV(e_2) \\
FV(x) &= \{x\} \\
FV(\texttt{let}\ x = e_1\ \texttt{in}\ e_2) &= FV(e_1) \cup (FV(e_2) - \{x\}) \\
FV(e_1\ e_2) &= FV(e_1) \cup FV(e_2) \\
FV(\lambda x{:}\tau.e) &= FV(e) - \{x\} \\
FV(\texttt{rec}\ f(x{:}\tau) : \tau'.\ e) &= FV(e) - \{f, x\} \\
FV((e_1, e_2)) &= FV(e_1) \cup FV(e_2) \\
FV(\texttt{fst}\ e) &= FV(e) \\
FV(\texttt{snd}\ e) &= FV(e) \\
FV(\texttt{let}\ (x, y) = e_1\ \texttt{in}\ e_2) &= FV(e_1) \cup (FV(e_2) - \{x, y\}) \\
FV(\texttt{left}(e)) &= FV(e) \\
FV(\texttt{right}(e)) &= FV(e) \\
FV(\texttt{case}\ e\ \texttt{of}\ \{\texttt{left}(x) \Rightarrow e_1\ ;\ \texttt{right}(y) \Rightarrow e_2\}) &= FV(e) \cup (FV(e_1) - \{x\}) \cup (FV(e_2) - \{y\}) \\
FV(\Lambda A.e) &= FV(e) \\
FV(e[\tau]) &= FV(e)
\end{aligned}
$$

1.2 Free type variables

$$
\begin{aligned}
FTV(\texttt{int}) &= \emptyset \\
FTV(\texttt{bool}) &= \emptyset \\
FTV(\tau_1 \times \tau_2) &= FTV(\tau_1) \cup FTV(\tau_2) \\
FTV(\texttt{unit}) &= \emptyset \\
FTV(\tau_1 + \tau_2) &= FTV(\tau_1) \cup FTV(\tau_2) \\
FTV(\texttt{empty}) &= \emptyset \\
FTV(A) &= \{A\} \\
FTV(\forall A.\tau) &= FTV(\tau) - \{A\}
\end{aligned}
$$

1.3 Substitution

$$
\begin{array}{rcl}
n[e/x] & = & n \\
(e_1 \oplus e_2)[e/x] & = & e_1[e/x] \oplus e_2[e/x] \\
b[e/x] & = & b \\
(\texttt{if}\ e_0\ \texttt{then}\ e_1\ \texttt{else}\ e_2)[e/x] & = & \texttt{if}\ (e_0[e/x])\ \texttt{then}\ (e_1[e/x])\ \texttt{else}\ (e_2[e/x]) \\
x[e/x] & = & e \\
y[e/x] & = & y \quad (x \neq y) \\
(\texttt{let}\ y = e_1\ \texttt{in}\ e_2)[e/x] & = & \texttt{let}\ y = e_1[e/x]\ \texttt{in}\ e_2[e/x] \\
 & & \text{(where } y \mathrel{\#} e\text{)} \\
(\lambda y{:}\tau.\ e_0)[e/x] & = & \lambda y{:}\tau.e_0[e/x] \\
 & & \text{(where } y \mathrel{\#} e\text{)} \\
(e_1\ e_2)[e/x] & = & (e_1[e/x])\ (e_2[e/x]) \\
(\texttt{rec}\ f(y{:}\tau){:}\tau' = e_0)[e/x] & = & \texttt{rec}\ f(y{:}\tau){:}\tau' = e_0[e/x] \\
 & & \text{(where } f, y \mathrel{\#} e\text{)} \\
(e_1, e_2)[e/x] & = & (e_1[e/x], e_2[e/x]) \\
(\texttt{fst}\ e_0)[e/x] & = & \texttt{fst}\ (e_0[e/x]) \\
(\texttt{snd}\ e_0)[e/x] & = & \texttt{snd}\ (e_0[e/x]) \\
(\texttt{let}\ (y, z) = e_1\ \texttt{in}\ e_2)[e/x] & = & \texttt{let}\ (y, z) = e_1[e/x]\ \texttt{in}\ e_2[e/x] \\
 & & \text{(where } y, z \mathrel{\#} e\text{)} \\
(\texttt{left}(e_0))[e/x] & = & \texttt{left}(e_0[e/x]) \\
(\texttt{right}(e_0))[e/x] & = & \texttt{right}(e_0[e/x]) \\
\texttt{case}\ e_0\ \texttt{of}\ \{\texttt{left}(y) \Rightarrow e_1\ ;\ \texttt{right}(z) \Rightarrow e_2\})[e/x] & = & \texttt{case}\ (e_0[e/x])\ \texttt{of}\ \{\texttt{left}(y) \Rightarrow e_1[e/x]\ ;\ \texttt{right}(z) \Rightarrow e_2[e/x]\} \\
 & & \text{(where } y, z \mathrel{\#} e\text{)} \\
(\Lambda A.e_0)[e/x] & = & \Lambda A.(e_0[e/x]) \\
(e_0[\tau])[e/x] & = & (e_0[e/x])[\tau]
\end{array}
$$

2 Evaluation

$\boxed{e \Downarrow v}$ for $\mathsf{L_{Arith}}$

$$\frac{}{v \Downarrow v} \qquad \frac{e_1 \Downarrow v_1 \quad e_2 \Downarrow v_2}{e_1 + e_2 \Downarrow v_1 +_{\mathbb{N}} v_2} \qquad \frac{e_1 \Downarrow v_1 \quad e_2 \Downarrow v_2}{e_1 \times e_2 \Downarrow v_1 \times_{\mathbb{N}} v_2}$$

$\boxed{e \Downarrow v}$ for $\mathsf{L_{If}}$

$$\frac{e_1 \Downarrow v \quad e_2 \Downarrow v}{e_1 == e_2 \Downarrow \texttt{true}} \qquad \frac{e_1 \Downarrow v_1 \quad e_2 \Downarrow v_2 \quad v_1 \neq v_2}{e_1 == e_2 \Downarrow \texttt{false}}$$

$$\frac{e \Downarrow \texttt{true} \quad e_1 \Downarrow v_1}{\texttt{if}\ e\ \texttt{then}\ e_1\ \texttt{else}\ e_2 \Downarrow v_1} \qquad \frac{e \Downarrow \texttt{false} \quad e_2 \Downarrow v_2}{\texttt{if}\ e\ \texttt{then}\ e_1\ \texttt{else}\ e_2 \Downarrow v_2}$$

$\boxed{e \Downarrow v}$ for $\mathsf{L_{Let}}$

$$\frac{e_1 \Downarrow v_1 \quad e_2[v_1/x] \Downarrow v_2}{\texttt{let}\ x = e_1\ \texttt{in}\ e_2 \Downarrow v_2}$$

$\boxed{e \Downarrow v}$ for $\mathsf{L_{Lam}}$

$$\frac{}{\lambda x.\, e \Downarrow \lambda x.\, e} \qquad \frac{e_1 \Downarrow \lambda x.e \quad e_2 \Downarrow v_2 \quad e[v_2/x] \Downarrow v}{e_1\ e_2 \Downarrow v}$$

$\boxed{e \Downarrow v}$ for $\mathsf{L_{Rec}}$

$$\frac{}{\texttt{rec}\ f(x).\, e \Downarrow \texttt{rec}\ f(x).\, e} \qquad \frac{e_1 \Downarrow \texttt{rec}\ f(x).\, e \quad e_2 \Downarrow v_2 \quad e[\texttt{rec}\ f(x).\, e/f, v_2/x] \Downarrow v}{e_1\ e_2 \Downarrow v}$$

$\boxed{e \Downarrow v}$ for $\mathsf{L_{Data}}$

$$\frac{e_1 \Downarrow v_1 \quad e_2 \Downarrow v_2}{(e_1, e_2) \Downarrow (v_1, v_2)} \qquad \frac{e \Downarrow (v_1, v_2)}{\texttt{fst}\ e \Downarrow v_1} \qquad \frac{e \Downarrow (v_1, v_2)}{\texttt{snd}\ e \Downarrow v_2} \qquad \frac{e_1 \Downarrow (v_1, v_2) \quad e_2[v_1/x, v_2/y] \Downarrow v}{\texttt{let pair}\ (x, y) = e_1\ \texttt{in}\ e_2 \Downarrow v}$$

$$\frac{e \Downarrow v}{\texttt{left}(e) \Downarrow \texttt{left}(v)} \qquad \frac{e \Downarrow v}{\texttt{right}(e) \Downarrow \texttt{right}(v)}$$

$$\frac{e \Downarrow \texttt{left}(v_1) \quad e_1[v_1/x] \Downarrow v}{\texttt{case}\ e\ \texttt{of}\ \{\texttt{left}(x) \Rightarrow e_1\ ;\ \texttt{right}(y) \Rightarrow e_2\} \Downarrow v} \qquad \frac{e \Downarrow \texttt{right}(v_2) \quad e_2[v_2/y] \Downarrow v}{\texttt{case}\ e\ \texttt{of}\ \{\texttt{left}(x) \Rightarrow e_1\ ;\ \texttt{right}(y) \Rightarrow e_2\} \Downarrow v}$$

$\boxed{e \Downarrow v}$ for $\mathsf{L}_{\mathsf{Poly}}$

$$\frac{}{\Lambda A.\, e \Downarrow \Lambda A.\, e} \qquad \frac{e \Downarrow \Lambda A.\, e_0 \quad e_0[\tau/A] \Downarrow v}{e[\tau] \Downarrow v}$$

3 Types

$\boxed{\Gamma \vdash e : \tau}$ for $\mathsf{L_{Arith}}$

$$\frac{}{\Gamma \vdash n : \texttt{int}} \qquad \frac{\Gamma \vdash e_1 : \texttt{int} \quad \Gamma \vdash e_2 : \texttt{int}}{\Gamma \vdash e_1 + e_2 : \texttt{int}} \qquad \frac{\Gamma \vdash e_1 : \texttt{int} \quad \Gamma \vdash e_2 : \texttt{int}}{\Gamma \vdash e_1 \times e_2 : \texttt{int}}$$

$\boxed{\Gamma \vdash e : \tau}$ for $\mathsf{L_{If}}$

$$\frac{}{\Gamma \vdash b : \texttt{bool}} \qquad \frac{\Gamma \vdash e_1 : \tau \quad \Gamma \vdash e_2 : \tau}{\Gamma \vdash e_1 == e_2 : \texttt{bool}} \qquad \frac{\Gamma \vdash e : \texttt{bool} \quad \Gamma \vdash e_1 : \tau \quad \Gamma \vdash e_2 : \tau}{\Gamma \vdash \texttt{if}\ e\ \texttt{then}\ e_1\ \texttt{else}\ e_2 : \tau}$$

$\boxed{\Gamma \vdash e : \tau}$ for $\mathsf{L_{Let}}$

$$\frac{\Gamma(x) = \tau}{\Gamma \vdash x : \tau} \qquad \frac{\Gamma \vdash e_1 : \tau_1 \quad \Gamma, x : \tau_1 \vdash e_2 : \tau_2}{\Gamma \vdash \texttt{let}\ x = e_1\ \texttt{in}\ e_2 : \tau_2}$$

$\boxed{\Gamma \vdash e : \tau}$ for $\mathsf{L_{Lam}}$

$$\frac{\Gamma, x{:}\tau_1 \vdash e : \tau_2}{\Gamma \vdash \lambda x{:}\tau_1.\, e : \tau_1 \to \tau_2} \qquad \frac{\Gamma \vdash e_1 : \tau_1 \to \tau_2 \quad \Gamma \vdash e_2 : \tau_1}{\Gamma \vdash e_1\ e_2 : \tau_2}$$

$\boxed{\Gamma \vdash e : \tau}$ for $\mathsf{L_{Rec}}$

$$\frac{\Gamma, f : \tau_1 \to \tau_2, x : \tau_1 \vdash e : \tau_2}{\Gamma \vdash \texttt{rec}\ f(x{:}\tau_1) : \tau_2.\, e : \tau_1 \to \tau_2}$$

$\boxed{\Gamma \vdash e : \tau}$ for $\mathsf{L_{Data}}$

$$\frac{\Gamma \vdash e_1 : \tau_1 \quad \Gamma \vdash e_2 : \tau_2}{\Gamma \vdash (e_1, e_2) : \tau_1 \times \tau_2} \qquad \frac{\Gamma \vdash e : \tau_1 \times \tau_2}{\Gamma \vdash \texttt{fst}\ e : \tau_1} \qquad \frac{\Gamma \vdash e : \tau_1 \times \tau_2}{\Gamma \vdash \texttt{snd}\ e : \tau_2} \qquad \frac{\Gamma \vdash e_1 : \tau_1 \times \tau_2 \quad \Gamma, x : \tau_1, y : \tau_2 \vdash e_2 : \tau}{\Gamma \vdash \texttt{let pair}\ (x, y) = e_1\ \texttt{in}\ e_2 : \tau}$$

$$\frac{\Gamma \vdash e : \tau_1}{\Gamma \vdash \texttt{left}(e) : \tau_1 + \tau_2} \qquad \frac{\Gamma \vdash e : \tau_2}{\Gamma \vdash \texttt{right}(e) : \tau_1 + \tau_2} \qquad \frac{\Gamma \vdash e : \tau_1 + \tau_2 \quad \Gamma, x : \tau_1 \vdash e_1 : \tau \quad \Gamma, y : \tau_2 \vdash e_2 : \tau}{\Gamma \vdash \texttt{case}\ e\ \texttt{of}\ \{\texttt{left}(x) \Rightarrow e_1\ ;\ \texttt{right}(y) \Rightarrow e_2\} : \tau}$$

$\boxed{\Gamma \vdash e : \tau}$ for $\mathsf{L_{Poly}}$

$$\frac{\Gamma \vdash e : \tau \quad A \# \Gamma}{\Gamma \vdash \Lambda A.\, e : \forall A.\, \tau} \qquad \frac{\Gamma \vdash e : \forall A.\, \tau}{\Gamma \vdash e[\tau_0] : \tau[\tau_0/A]}$$