Elements of Programming Languages
Tutorial 2: Substitution and alpha-equivalence
Solution notes

1. (a)
   - $(\lambda x{:}\texttt{int}.\ x)\ 1$

$$\frac{\dfrac{}{\lambda x{:}\texttt{int}.\ x \Downarrow \lambda x{:}\texttt{int}.\ x} \quad \dfrac{}{1 \Downarrow 1} \quad \dfrac{}{1 \Downarrow 1}}{(\lambda x{:}\texttt{int}.\ x)\ 1 \Downarrow 1}$$

   - $(\lambda x{:}\texttt{int}.\ x + 1)\ 42$

$$\frac{\dfrac{}{\lambda x{:}\texttt{int}.\ x+1 \Downarrow \lambda x{:}\texttt{int}.\ x+1} \quad \dfrac{}{42 \Downarrow 42} \quad \dfrac{\dfrac{}{42 \Downarrow 42} \quad \dfrac{}{1 \Downarrow 1}}{42+1 \Downarrow 43}}{(\lambda x{:}\texttt{int}.\ x+1)\ 42 \Downarrow 43}$$

   - $(\lambda x{:}\texttt{int} \to \texttt{int}.\ x)\ (\lambda x{:}\texttt{int}.\ x)\ 1$ Type annotations elided.

$$\frac{\dfrac{\dfrac{}{\lambda x.\ x \Downarrow \lambda x.\ x} \quad \dfrac{}{\lambda x.\ x \Downarrow \lambda x.\ x} \quad \dfrac{}{\lambda x.\ x \Downarrow \lambda x.\ x}}{(\lambda x.\ x)\ (\lambda x.\ x) \Downarrow \lambda x.\ x} \quad \dfrac{}{1 \Downarrow 1}}{((\lambda x.\ x)\ (\lambda x.\ x))\ 1 \Downarrow 1}$$

   - $(\star)$ $((\lambda f{:}\texttt{int} \to \texttt{int}.\ \lambda x{:}\texttt{int}.f\ (f\ x))\ (\lambda x{:}\texttt{int}.\ x+1))\ 42$ Type annotations elided.

$$\frac{\dfrac{\dfrac{}{(\lambda f.\ \lambda x.f\ (f\ x)) \Downarrow (\lambda f.\ \lambda x.f\ (f\ x))} \quad \dfrac{}{\lambda x.\ x+1 \Downarrow \lambda x.\ x+1}}{(\lambda f.\ \lambda x.f\ (f\ x))\ (\lambda x.\ x+1) \Downarrow \lambda x.(\lambda x.\ x+1)((\lambda x.\ x+1)x)} \quad \dfrac{}{42 \Downarrow 42} \quad \dfrac{\vdots}{(\lambda x.\ x+1)((\lambda x.\ x+1)42) \Downarrow 44}}{((\lambda f.\ \lambda x.f\ (f\ x))\ (\lambda x.\ x+1))\ 42 \Downarrow 44}$$

   where

$$\frac{\dfrac{}{\lambda x.\ x+1 \Downarrow \lambda x.\ x+1} \quad \dfrac{\dfrac{}{\lambda x.\ x+1 \Downarrow \lambda x.x+1} \quad \dfrac{}{42 \Downarrow 42} \quad \dfrac{}{42+1 \Downarrow 43}}{(\lambda x.\ x+1)42 \Downarrow 43} \quad \dfrac{}{43+1 \Downarrow 44}}{(\lambda x.\ x+1)((\lambda x.\ x+1)42) \Downarrow 44}$$

   (b) If $e_1 : \tau$ then we can define $\texttt{let}\ x = e_1\ \texttt{in}\ e_2$ as $(\lambda x{:}\tau.\ e_2)\ e_1$. The evaluation rule for `let` can be emulated as follows:

$$\frac{e_1 \Downarrow v_1 \quad e_2[v_1/x] \Downarrow v}{\texttt{let}\ x = e_1\ \texttt{in}\ e_2 \Downarrow v} \Longrightarrow \frac{\dfrac{}{\lambda x{:}\tau.e_2 \Downarrow \lambda x{:}\tau.\ e_2} \quad e_1 \Downarrow v_1 \quad e_2[v_1/x] \Downarrow v}{(\lambda x{:}\tau.\ e_2)\ e_1 \Downarrow v}$$

2. (a)
   - `Int => Int`

```
{x: Int => x}
```

   - `Int => Boolean => Int`

```
{x: Int => {y: Boolean => x}}
```

   - `(Int => Boolean => String) => (Int => Boolean) => (Int => String)`

```
{x: (Int => Boolean => String) =>
  {y: (Int => Boolean) =>
    {z: Int => x(z)(y(z))}}}
```

(b)

- $(\lambda x{:}\texttt{int}.\ x)\ 1$

$$\dfrac{\dfrac{}{x : \texttt{int} \vdash x : \texttt{int}}}{\vdash \lambda x{:}\texttt{int}.\ x : \texttt{int} \to \texttt{int}} \quad \dfrac{}{\vdash 1 : \texttt{int}}$$
$$\overline{\vdash (\lambda x{:}\texttt{int}.\ x)\ 1 : \texttt{int}}$$

- $(\lambda x{:}\texttt{int}.\ x + 1)\ 42$

$$\dfrac{\dfrac{\dfrac{\dfrac{}{x{:}\texttt{int} \vdash x : \texttt{int}} \quad \dfrac{}{x{:}\texttt{int} \vdash 1 : \texttt{int}}}{x : \texttt{int} \vdash x + 1 : \texttt{int}}}{\vdash \lambda x{:}\texttt{int}.\ x + 1 : \texttt{int} \to \texttt{int}} \qquad \dfrac{}{\vdash 42 : \texttt{int}}}{\vdash (\lambda x{:}\texttt{int}.\ x + 1)\ 42 : \texttt{int}}$$

- $(\lambda x{:}\texttt{int} \to \texttt{int}.\ x)\ (\lambda x{:}\texttt{int}.\ x)$

$$\dfrac{\dfrac{\dfrac{}{x{:}\texttt{int} \to \texttt{int} \vdash x : \texttt{int} \to \texttt{int}}}{\vdash (\lambda x{:}\texttt{int} \to \texttt{int}.\ x) : (\texttt{int} \to \texttt{int}) \to (\texttt{int} \to \texttt{int})} \quad \dfrac{\vdots}{\vdash \lambda x{:}\texttt{int}\ x : \texttt{int} \to \texttt{int}}}{\vdash (\lambda x{:}\texttt{int} \to \texttt{int}.\ x)\ (\lambda x{:}\texttt{int}.\ x) : \texttt{int} \to \texttt{int}}$$

- $(\lambda x{:}\tau.\ x\ x)$ **This expression cannot be typed. There is no way to choose τ so that the following derivation can be completed:**

$$\dfrac{\dfrac{\dfrac{??}{x{:}\tau \vdash x : \tau_1 \to \tau_2} \quad \dfrac{??}{x{:}\tau \vdash x{:}\tau_1}}{x{:}\tau \vdash x\ x : \tau_2}}{\vdash \lambda x{:}\tau.\ x\ x : \tau_2}$$

For if $\tau = \tau_1$ then we would also have to have $\tau = \tau_1 \to \tau_2$, i.e. $\tau_1 = \tau_1 \to \tau_2$ which is not possible if equality is structural.

3. (a) The missing rules are:

$\boxed{e_1 \equiv_\alpha e_2}$

$$\dfrac{e \equiv_\alpha e' \quad e_1 \equiv_\alpha e_1' \quad e_1 \equiv_\alpha e_1'}{\texttt{if}\ e\ \texttt{then}\ e_1\ \texttt{else}\ e_2 \equiv_\alpha \texttt{if}\ e'\ \texttt{then}\ e_1'\ \texttt{else}\ e_2'}$$

$$\dfrac{e_1(x{\leftrightarrow}z) \equiv_\alpha e_2(y{\leftrightarrow}z) \quad z \# e_1, e_2}{\lambda x.e_1 \equiv_\alpha \lambda y.e_2} \qquad \dfrac{e_1 \equiv_\alpha e_1' \quad e_1 \equiv_\alpha e_1'}{e_1\ e_2 \equiv_\alpha e_1'\ e_2'}$$

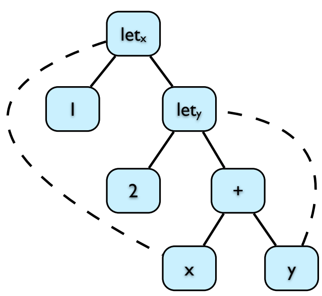
Point this out: To be precise, we should also extend $\#$ as follows:

$$\dfrac{}{x \# \lambda x{:}\tau.\ e} \qquad \dfrac{x \neq y \quad x \# e}{x \# \lambda y{:}\tau.\ e} \qquad \dfrac{x \# e_1 \quad x \# e_2}{x \# e_1\ e_2}$$

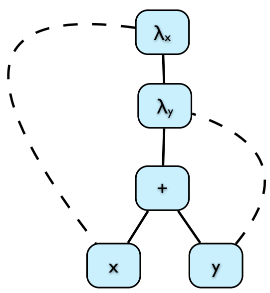
(b) Which of the following alpha-equivalence relationships hold?

$\texttt{if true then}\ y\ \texttt{else}\ z$	$\equiv_\alpha$	y	FALSE
$\texttt{let}\ x = y\ \texttt{in}\ (\texttt{if}\ x\ \texttt{then}\ y\ \texttt{else}\ z)$	$\equiv_\alpha$	$\texttt{let}\ z = y\ \texttt{in}\ (\texttt{if}\ x\ \texttt{then}\ y\ \texttt{else}\ z)$	FALSE
$\lambda x.\ (\texttt{let}\ y = x\ \texttt{in}\ y + y)$	$\equiv_\alpha$	$\lambda x.\ (\texttt{let}\ x = x\ \texttt{in}\ x + x)$	TRUE

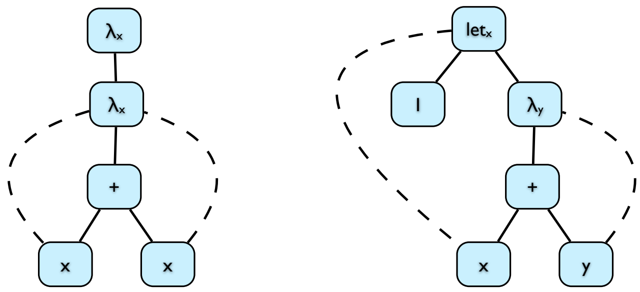
(c) The pictures should be as follows:



let x = 1 in let y = 2 in x + y



λ x. λ y. x + y



λ x. λ x. x + x   let x = 1 in λ y. x + y

4. (a)

$$\begin{aligned}
(\lambda y.\ \lambda z.\ ((x+y)+z))[y \times z/x] &= \lambda y.\ \lambda z.\ (((y \times z)+y)+z) \\
(\texttt{if}\ x == y\ \texttt{then}\ \lambda z.x\ \texttt{else}\ \lambda x.x)[z/x] &= \texttt{if}\ z == y\ \texttt{then}\ \lambda z.z\ \texttt{else}\ \lambda x.z
\end{aligned}$$

(b)

$$\begin{aligned}
\lambda y.\ \lambda z.\ ((x+y)+z) &\equiv_\alpha \lambda a.\ \lambda b.\ ((x+a)+b) \\
\texttt{if}\ x == y\ \texttt{then}\ \lambda z.x\ \texttt{else}\ \lambda x.x &\equiv_\alpha \texttt{if}\ x == y\ \texttt{then}\ \lambda c.x\ \texttt{else}\ \lambda d.d
\end{aligned}$$

(c)

$$\begin{aligned}
(\lambda a.\ \lambda b.\ ((x+a)+b))[y \times z/x] &= \lambda a.\ \lambda b.\ (((y \times z)+a)+b) \\
(\texttt{if}\ x == y\ \texttt{then}\ \lambda c.x\ \texttt{else}\ \lambda d.d)[z/x] &= \texttt{if}\ z == y\ \texttt{then}\ \lambda c.z\ \texttt{else}\ \lambda d.d
\end{aligned}$$

Illustrate that the substitutions performed without α-conversion lead to variable capture, and different binding structure from those performed after α-converting to fresh names.