Discrete Mathematics & Mathematical Reasoning

Induction

Colin Stirling

Informatics
Another proof method: Mathematical Induction

- Want to prove $\forall n \in \mathbb{N} \ (P(n))$

BASIS STEP: show $P(0)$

INDUCTIVE STEP: show $P(k) \rightarrow P(k+1)$ for all $k \in \mathbb{N}$

Assume $k$ is arbitrary and $P(k)$ is true. Show $P(k+1)$.
Another proof method: Mathematical Induction

- Want to prove $\forall n \in \mathbb{N} \ (P(n))$

- BASIS STEP show $P(0)$
Another proof method: Mathematical Induction

- Want to prove $\forall n \in \mathbb{N} \ (P(n))$
- **BASIS STEP** show $P(0)$
- **INDUCTIVE STEP** show $P(k) \rightarrow P(k + 1)$ for all $k \in \mathbb{N}$
Want to prove $\forall n \in \mathbb{N} \ (P(n))$

BASIS STEP show $P(0)$

INDUCTIVE STEP show $P(k) \rightarrow P(k + 1)$ for all $k \in \mathbb{N}$

Assume $k$ is arbitrary and $P(k)$ is true. Show $P(k + 1)$
Another proof method: Mathematical Induction

- Want to prove \( \forall n \in \mathbb{Z}^+ (P(n)) \)

- BASIS STEP show \( P(1) \)

- INDUCTIVE STEP show \( P(k) \rightarrow P(k + 1) \) for all \( k \in \mathbb{Z}^+ \)

- Assume \( k \) is arbitrary and \( P(k) \) is true. Show \( P(k + 1) \)
Another proof method: Mathematical Induction

Want to prove \( \forall n \geq m \in \mathbb{N} \ (P(n)) \)

**BASIS STEP** show \( P(m) \)

**INDUCTIVE STEP** show \( P(k) \rightarrow P(k + 1) \) for all \( k \geq m \in \mathbb{N} \)

Assume \( k \geq m \) is arbitrary and \( P(k) \) is true. Show \( P(k + 1) \)
Another proof method: Mathematical Induction

- Want to prove $\forall n \in \mathbb{Q}^+ (P(n))$
Another proof method: Mathematical Induction

- Want to prove $\forall n \in \mathbb{Q}^+ \ (P(n))$

- Can we use induction?
Another proof method: Mathematical Induction

- Want to prove $\forall n \in \mathbb{Q}^+ \ (P(n))$

- Can we use induction?

- Want to prove $\forall x \in \mathbb{R}^+ \ (P(x))$
Another proof method: Mathematical Induction

- Want to prove $\forall n \in \mathbb{Q}^+ (P(n))$

- Can we use induction?

- Want to prove $\forall x \in \mathbb{R}^+ (P(x))$

- Can we use induction?
Another proof method: Mathematical Induction

- Want to prove $\forall n \in \mathbb{Q}^+ (P(n))$

- Can we use induction?

- Want to prove $\forall x \in \mathbb{R}^+ (P(x))$

- Can we use induction?

- What justifies mathematical induction?
Another proof method: Mathematical Induction

- Want to prove  \( \forall n \in \mathbb{Q}^+ \ (P(n)) \)

- Can we use induction?

- Want to prove  \( \forall x \in \mathbb{R}^+ \ (P(x)) \)

- Can we use induction?

- What justifies mathematical induction?

- Well ordering principle: every nonempty set \( S \subseteq \mathbb{N} \) has a least element
Examples

\[ \sum_{j=1}^{n} j = \frac{n(n+1)}{2} \]
Examples

\[ \sum_{j=1}^{n} j = \frac{n(n + 1)}{2} \]

\[ \sum_{j=0}^{n} ar^j = \frac{ar^{n+1} - a}{r - 1} \text{ when } r \neq 1 \]
Examples

\[ \sum_{j=1}^{n} j = \frac{n(n+1)}{2} \]

\[ \sum_{j=0}^{n} ar^j = \frac{ar^{n+1} - a}{r - 1} \text{ when } r \neq 1 \]

for all \( n \in \mathbb{Z}^+(n < 2^n) \)
Examples

\[ \sum_{j=1}^{n} j = \frac{n(n+1)}{2} \]

\[ \sum_{j=0}^{n} a r^j = \frac{a r^{n+1} - a}{r - 1} \] when \( r \neq 1 \)

for all \( n \in \mathbb{Z}^+ \) \((n < 2^n)\)

for all integers \( n \geq 4, 2^n < n! \)
Examples

\[ \sum_{j=1}^{n} j = \frac{n(n+1)}{2} \]

\[ \sum_{j=0}^{n} a r^j = \frac{a r^{n+1} - a}{r - 1} \text{ when } r \neq 1 \]

- for all $n \in \mathbb{Z}^+(n < 2^n)$
- for all integers $n \geq 4$, $2^n < n!$
- for all $n \in \mathbb{Z}^+((n^3 - n) \text{ is divisible by } 3)$
Examples

\[ \sum_{j=1}^{n} j = \frac{n(n+1)}{2} \]

\[ \sum_{j=0}^{n} ar^j = \frac{ar^{n+1} - a}{r-1} \text{ when } r \neq 1 \]

- for all \( n \in \mathbb{Z}^+ (n < 2^n) \)
- for all integers \( n \geq 4, 2^n < n! \)
- for all \( n \in \mathbb{Z}^+ ((n^3 - n) \text{ is divisible by } 3) \)
- If \( S \) is a finite set with \( n \) elements then \( \mathcal{P}(S) \) contains \( 2^n \) elements
Odd Pie Fights An odd number of people stand in a room at mutually distinct distances. At the same time each person throws a pie at their nearest neighbour and hits them. Prove that at least one person is not hit by a pie.
More examples

- **Odd Pie Fights** An odd number of people stand in a room at mutually distinct distances. At the same time each person throws a pie at their nearest neighbour and hits them. Prove that at least one person is not hit by a pie

- **All cats have the same colour**
Two cats with different colours
Strong Induction

- Want to prove $\forall n \in \mathbb{N} (P(n))$
Strong Induction

- Want to prove $\forall n \in \mathbb{N} \ (P(n))$
- BASIS STEP show $P(0)$
Strong Induction

- **Want to prove** $\forall n \in \mathbb{N} \ (P(n))$

- **BASIS STEP** show $P(0)$

- **INDUCTIVE STEP** show $(P(0) \land \ldots \land P(k)) \rightarrow P(k + 1)$ for all $k \in \mathbb{N}$
Strong Induction

- Want to prove \( \forall n \in \mathbb{N} \ (P(n)) \)

- **BASIS STEP** show \( P(0) \)

- **INDUCTIVE STEP** show \( (P(0) \land \ldots \land P(k)) \rightarrow P(k + 1) \) for all \( k \in \mathbb{N} \)

- Assume \( k \) is arbitrary and \( P(0), \ldots, P(k) \) are true. Show \( P(k + 1) \)
Strong Induction

- Want to prove $\forall n \in \mathbb{Z}^+ \ (P(n))$

- **BASIS STEP** show $P(1)$

- **INDUCTIVE STEP** show $(P(1) \land \ldots \land P(k)) \rightarrow P(k + 1)$ for all $k \in \mathbb{Z}^+$

- Assume $k$ is arbitrary and $P(1), \ldots, P(k)$ are true. Show $P(k + 1)$
Strong Induction

- Want to prove $\forall n \geq m \in \mathbb{N} \ (P(n))$

- BASIS STEP show $P(m)$

- INDUCTIVE STEP show $(P(m) \land \ldots \land P(k)) \rightarrow P(k + 1)$ for all $k \geq m \in \mathbb{N}$

- Assume $k \geq m$ is arbitrary and $P(m), \ldots, P(k)$ are true. Show $P(k + 1)$
Examples

- If $n > 1$ is an integer, then $n$ can be written as a product of primes.
Examples

- If $n > 1$ is an integer, then $n$ can be written as a product of primes.

- Game of matches. Two players take turns removing any positive number of matches they want from one of two piles of matches. The player who removes the last match wins the game. Show that if the two piles contain the same number of matches initially then the second player can guarantee a win.
Examples

- If $n > 1$ is an integer, then $n$ can be written as a product of primes

- Game of matches. Two players take turns removing any positive number of matches they want from one of two piles of matches. The player who removes the last match wins the game. Show that if the two piles contain the same number of matches initially then the second player can guarantee a win

- If $n \geq 3$ then $f_n > \alpha^{n-2}$ (where $f_n$ is the $n$th term of the Fibonacci series and $\alpha = (1 + \sqrt{5})/2$)
Examples

- If $n > 1$ is an integer, then $n$ can be written as a product of primes.

- Game of matches. Two players take turns removing any positive number of matches they want from one of two piles of matches. The player who removes the last match wins the game. Show that if the two piles contain the same number of matches initially then the second player can guarantee a win.

- If $n \geq 3$ then $f_n > \alpha^{n-2}$ (where $f_n$ is the $n$th term of the Fibonacci series and $\alpha = (1 + \sqrt{5})/2$).

- Prove that every amount of postage of 12p or more can be formed using just 4p and 5p stamps.