Discrete Mathematics & Mathematical Reasoning
Induction

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Informatics

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- What justifies mathematical induction?
- Well ordering principle: every nonempty set $S \subseteq \mathbb{N}$ has a least element

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- If S is a finite set with n elements then $\mathcal{P}(S)$ contains 2^n elements

More examples

- Odd Pie Fights An odd number of people stand in a room at mutually distinct distances. At the same time each person throws a pie at their nearest neighbour and hits them. Prove that at least one person is not hit by a pie

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- All cats have the same colour

Two cats with different colours

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- Prove that every amount of postage of 12p or more can be formed using just 4p and 5p stamps