Discrete Mathematics & Mathematical Reasoning
Multiplicative Inverses and Some Cryptography

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Informatics

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- Notice $\gcd(8, 15) = 1$ whereas $\gcd(12, 15) = 3$

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By Bézout's theorem there are s and t such that

$$sm + tx = 1 = \gcd(m, x)$$

So, $sm + tx \equiv 1 \pmod{m}$. As $sm \equiv 0 \pmod{m}$, so $tx \equiv 1 (\text{mod } m)$.
For uniqueness mod m. Assume $tx \equiv 1 \pmod{m}$ and $ux \equiv 1 \pmod{m}$.
Therefore, $tx \equiv ux \pmod{m}$. Since $\gcd(m, x) = 1$ it follows that $t \equiv u \pmod{m}$. □

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Compute the multiplicative inverse using extended euclidean algorithm

Chinese remainder theorem

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Let $m_1, m_2, \ldots, m_n$ be pairwise relatively prime positive integers greater than 1 and $a_1, a_2, \ldots, a_n$ be arbitrary integers. Then the system

$$\begin{aligned} x &\equiv a_1 \ (mod\ m_1) \\ x &\equiv a_2 \ (mod\ m_2) \\ &\vdots \\ x &\equiv a_n \ (mod\ m_n) \end{aligned}$$

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In the book □

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- $x = 140 + 63 + 75 = 278 \equiv 68 \pmod{105}$

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Assume $p \not| a$ and so, therefore, $\gcd(p, a) = 1$. Then $a, 2a, \ldots, (p-1)a$ are not pairwise congruent modulo p; if $ia \equiv ja \pmod p$ because $\gcd(p, a) = 1$ then $i \equiv j \pmod p$ which is impossible. Therefore, each element $ja \bmod p$ is a distinct element in the set $\{1, \ldots, p-1\}$. This means that the product $a \cdot 2a \cdots (p-1)a \equiv 1 \cdot 2 \cdots p-1 \pmod p$. Therefore, $(p-1)!a^{p-1} \equiv (p-1)! \pmod p$. Now because $\gcd(p, q) = 1$ for $1 \leq q \leq p-1$ it follows that $a^{p-1} \equiv 1 \pmod p$. Therefore, also $a^p \equiv a \pmod p$ and when $p|a$ then clearly $a^p \equiv a \pmod p$. □

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- $2^{340} \equiv 1 \pmod{11}$ because $2^{10} \equiv 1 \pmod{11}$

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- What is WKLV LV D VHFSHW ?

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- The challenge: De can't be feasibly computed from En; and given En(M) one can't feasibly compute M

RSA Cryptosystem: Rivest, Shamir and Adleman

- Choose two distinct prime numbers p and q
- Let $n = pq$ and $k = (p-1)(q-1)$
- Choose integer e where $1 < e < k$ and $\gcd(e, k) = 1$
- (n, e) is released as the public key
- Let d be the multiplicative inverse of e modulo k, so $de \equiv 1 \pmod{k}$
- (n, d) is the private key and kept secret

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2. Given m, she can recover the original message M by reversing the padding scheme

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- $0981^{937} \bmod 2537 = 0704$ and $0461^{937} \bmod 2537 = 1115$
- So decrypted message is HELP

RSA: correctness of decryption

Given that $c = m^e$ mod n, is $m = c^d$ mod n?

$$c^d = (m^e)^d \equiv m^{ed} \pmod{n}$$

By construction, d and e are each others multiplicative inverses modulo k, i.e. $ed \equiv 1 \pmod{k}$. Also $k = (p-1)(q-1)$. Thus $ed - 1 = h(p-1)(q-1)$ for some integer h. We consider m^{ed} mod p
If $p \not| m$ then
$m^{ed} = m^{h(p-1)(q-1)}m = (m^{p-1})^{h(q-1)}m \equiv 1^{h(q-1)}m \equiv m \pmod{p}$ (by Fermat's little theorem)
Otherwise $m^{ed} \equiv 0 \equiv m \pmod{p}$
Symmetrically, $m^{ed} \equiv m \pmod{q}$
Since p, q are distinct primes, we have $m^{ed} \equiv m \pmod{pq}$. Since $n = pq$, we have $c^d = m^{ed} \equiv m \pmod{n}$