Discrete Mathematics & Mathematical Reasoning
Basic Structures: Sets, Functions and Relations

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Informatics

Some slides based on ones by Myrto Arapinis

Some important sets

$\mathbb{B} = \{\text{true}, \text{false}\}$ Boolean values

$\mathbb{N} = \{0, 1, 2, 3, \ldots\}$ Natural numbers

$\mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$ Integers

$\mathbb{Z}^+ = \{1, 2, 3, \ldots\}$ Positive integers

$\mathbb{R}$ Real numbers

$\mathbb{R}^+$ Positive real numbers

$\mathbb{Q}$ Rational numbers

$\mathbb{C}$ Complex numbers

$\emptyset$ Empty set

Sets defined using comprehension

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- Example Subsets of sets upon which an order is defined

$$
\begin{array}{lcll}
[a,b] & = & \{x \mid a \leq x \leq b\} & \text{closed interval} \\
[a,b) & = & \{x \mid a \leq x < b\} & \\
(a,b] & = & \{x \mid a < x \leq b\} & \\
(a,b) & = & \{x \mid a < x < b\} & \text{open interval}
\end{array}
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- $A \times B$ cartesian product (tuple sets)

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- Modern formulations (such as Zermelo-Fraenkel set theory) restrict comprehension. (However, it is impossible to prove in ZF that ZF is consistent unless ZF is inconsistent.)

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- $f : A \rightarrow B$ if f is a function from A to B

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Definition

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Function composition

Definition

Let $f : B \to C$ and $g : A \to B$. The composition function $f \circ g : A \to C$ is $(f \circ g)(a) = f(g(a))$

Results about function composition

Theorem

The composition of two functions is a function

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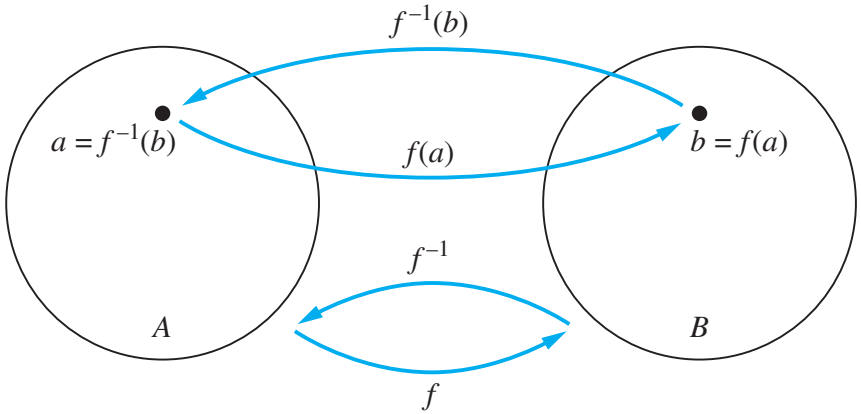
Corollary

The composition of two bijections is a bijection

Inverse function

Definition

If $f : A \to B$ is a bijection, then the inverse of f, written $f^{-1} : B \to A$ is $f^{-1}(b) = a$ iff $f(a) = b$



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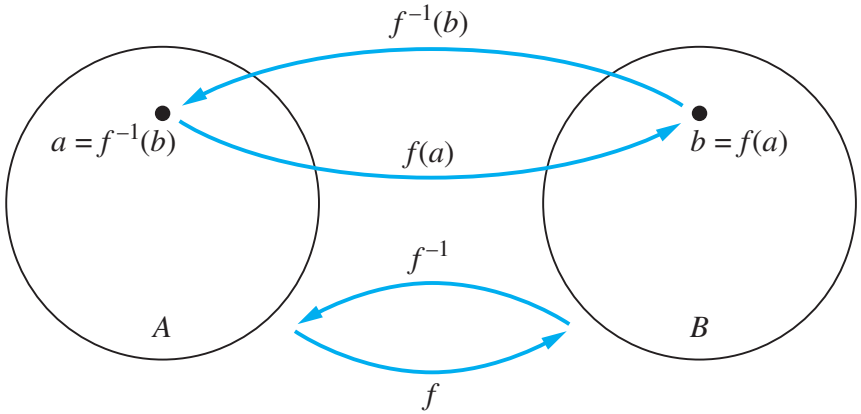

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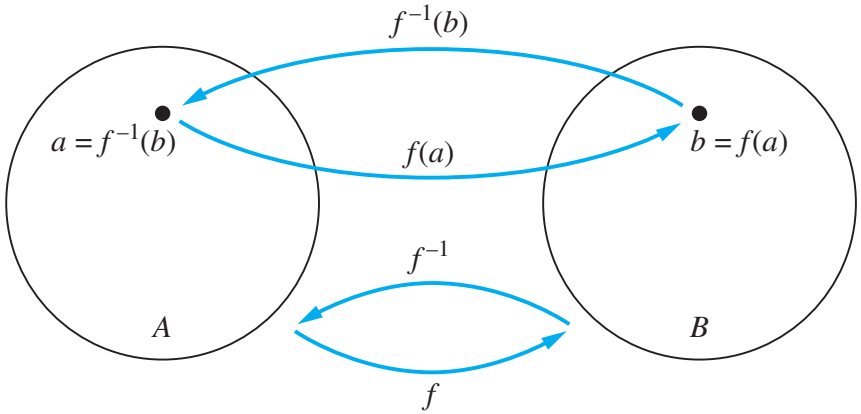

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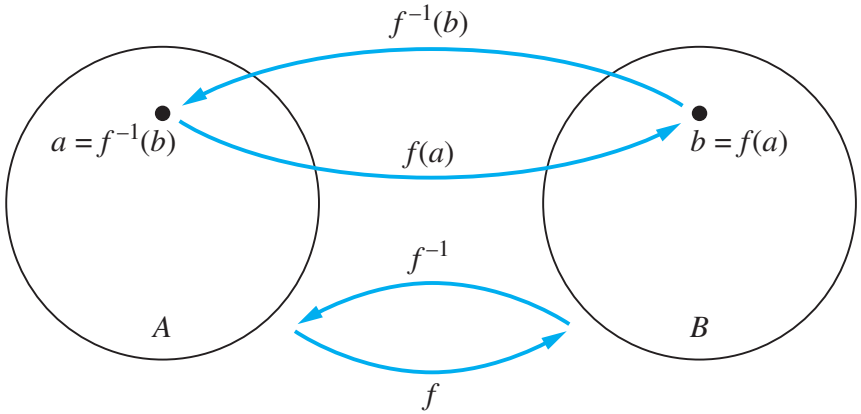

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What is $f^{-1} \circ f$? and $f \circ f^{-1}$?

The floor and ceiling functions

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$$\lfloor -6.1 \rfloor = -7 \qquad \lceil 6.1 \rceil = 7$$

Useful tips about floors and ceilings

- When showing properties of floors is to let $x = n + \epsilon$ if $\lfloor x \rfloor = n$ where $0 \leq \epsilon < 1$
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- Proof in book

Prove $\lceil x \rceil + \lceil y \rceil = \lceil x + y \rceil$

Prove $\lceil x \rceil + \lceil y \rceil = \lceil x + y \rceil$

False; counterexample $x = 1/2$ and $y = 1/2$

The factorial function

Definition

The factorial function $f : \mathbb{N} \to \mathbb{N}$, denoted as $f(n) = n!$ assigns to n the product of the first n positive integers

$$f(0) = 0! = 1$$

and

$$f(n) = n! = 1 \cdot 2 \cdot \cdots \cdot (n-1) \cdot n$$

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Definition

Given sets $A_1, \ldots, A_n$, a subset $R \subseteq A_1 \times \cdots \times A_n$ is an n-ary relation

Examples

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- Let $m > 1$ be an integer. $R = \{(a, b) \mid a \bmod m = b \bmod m\}$
- Written as $a \equiv b \pmod{m}$

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A binary relation R on A is called

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- symmetric iff $\forall x, y \in A\ ((x, y) \in R \rightarrow (y, x) \in R)$
- $=$ is symmetric, but $\leq$, $<$, and $|$ are not

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- $\leq$, $=$, and $|$ are reflexive, but $<$ is not
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- $=$ is symmetric, but $\leq$, $<$, and $|$ are not
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- transitive iff $\forall x, y, z \in A\ (((x, y) \in R \wedge (y, z) \in R) \rightarrow (x, z) \in R)$
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Equivalence relations

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- $|$ on integers is not an equivalence relation.
- For $m > 1$ be an integer the relation $\equiv \pmod m$ is an equivalence relation on integers

Equivalence classes

Definition

Let R be an equivalence relation on a set A and $a \in A$. Let

$$[a]_R = \{s \mid (a, s) \in R\}$$

be the equivalence class of a w.r.t. R

If $b \in [a]_R$ then b is called a representative of the equivalence class. Every member of the class can be a representative

Theorem

Result

Let R be an equivalence on A and $a, b \in A$. The following three statements are equivalent

1. aRb
2. $[a]_R = [b]_R$
3. $[a]_R \cap [b]_R \neq \emptyset$

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Proof in book

Partitions of a set

Definition

A partition of a set A is a collection of disjoint, nonempty subsets that have A as their union. In other words, the collection of subsets $A_i \subseteq A$ with $i \in I$ (where I is an index set) forms a partition of A iff

1. $A_i \neq \emptyset$ for all $i \in I$
2. $A_i \cap A_j = \emptyset$ for all $i \neq j \in I$
3. $\bigcup_{i \in I} A_i = A$

Result

Theorem

1. If R is an equivalence on A, then the equivalence classes of R form a partition of A
2. Conversely, given a partition $\{A_i \mid i \in I\}$ of A there exists an equivalence relation R that has exactly the sets A_i, $i \in I$, as its equivalence classes

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