# Discrete Mathematics & Mathematical Reasoning
# Induction

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Informatics

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- What justifies mathematical induction?
- Well ordering principle: every nonempty set $S \subseteq \mathbb{N}$ has a least element

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- for all $n \in \mathbb{N}$ ($7^{n+2} + 8^{2n+1}$ is divisible by 57)

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- Odd Pie Fights An odd number of people stand in a room at mutually distinct distances. At the same time each person throws a pie at their nearest neighbour and hits them. Prove that at least one person is not hit by a pie

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- All horses have the same colour

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- Prove that every amount of postage of 12p or more can be formed using just 4p and 5p stamps
- Game of matches Two players take turns removing any positive number of matches they want from one of two piles of matches. The player who removes the last match wins the game. Show that if the two piles contain the same number of matches initially then the second player can guarantee a win