Sets[1]

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September 24, 2014

[1]Slides mainly borrowed from Richard Mayr

Sets

- A set is a well-defined finite or infinite collection of objects
  - The proper mathematical definition of set is much more complicated
  - We will not formally study Set Theory here so we do not need to know what well-defined means
  - We will be naively looking at ubiquitous structures that are available within it

- The objects in the set are called the elements or members of the set

- If s is a member of the set $\mathcal{S}$, then we write $s \in \mathcal{S}$

- If s is not a member of the set $\mathcal{S}$, then we write $s \notin \mathcal{S}$

Describing a set: Roster method

- Roster method: list all the elements of the set between braces
  Example The set of vowels in the English alphabet can be described by $\mathcal{V} = \{a, e, i, o, u, y\}$

[2] Do not abuse of this. Patterns are not always as clear as the writer thinks

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  Example The set of vowels in the English alphabet can be described by $\mathcal{V} = \{a, e, i, o, u, y\}$

- Dots "..." may be used to describe a set without listing all of the members when the pattern is clear[2]
  Example The set of letters in the English alphabet can be described by $\mathcal{L} = \{a, b, c, \ldots, z\}$
  Example The set of natural numbers can be described by $\mathbb{N} = \{0, 1, 2, 3, \ldots\}$

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Some important sets

$\mathbb{B} = \{\text{true}, \text{false}\}$: Boolean values
$\mathbb{N} = \{0, 1, 2, 3, \ldots\}$: Natural numbers
$\mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$: Integers
$\mathbb{Z}^+ = \{1, 2, 3, \ldots\}$: Positive integers
$\mathbb{R}$: Real numbers
$\mathbb{R}^+$: Positive real numbers
$\mathbb{Q}$: Rational numbers
$\mathbb{C}$: Complex numbers

Describing a set: set builder notation

- Characterize the elements of the set by the property (or properties) they must satisfy to be members
- A predicate can be used

Example $\mathcal{S} = \{x \mid x \text{ is a positive integer less than } 100\}$
$\mathcal{S} = \{x \mid x \in \mathbb{Z}^+ \wedge x < 100\}$
$\mathcal{S} = \{x \in \mathbb{Z}^+ \mid x < 100\}$

Example $\mathcal{P} = \{x \mid P(x)\}$ where $P(x) = \text{true}$ iff x is a prime number

Example $\mathbb{Q}^+ = \{q \mid \exists n, m \in \mathbb{Z}^+.\ q = n/m\}$

Describing a set: interval notation

Used to describe subsets of sets upon which an order is defined, *e.g.* numbers

$$\begin{aligned}
[a, b] &= \{x \mid a \leq x \leq b\} \\
[a, b) &= \{x \mid a \leq x < b\} \\
(a, b] &= \{x \mid a < x \leq b\} \\
(a, b) &= \{x \mid a < x < b\}
\end{aligned}$$

- closed interval: $[a, b]$
- open interval: (a, b)
- half-open intervals: $[a, b)$ and $(a, b]$

Universal set and Empty set

- The universal set $\mathcal{U}$ is the set containing everything currently under consideration
  - Content depends on the context
  - Sometimes explicitly stated, sometimes implicit
- The empty set is the set with no elements. Symbolized by $\emptyset$ or $\{\}$ and defined by

$$\forall x \in \mathcal{U}.\ x \notin \emptyset$$

Truth Sets and Characteristic Predicates

We fix a domain $\mathcal{U}$

- Let $P(x)$ be a predicate on $\mathcal{U}$. The truth set of P is the subset of $\mathcal{U}$ where P is true, *i.e.* the set

$$\{x \in \mathcal{U} \mid P(x)\}$$

- Let $\mathcal{S} \subseteq \mathcal{U}$ be a subset of $\mathcal{U}$. The characteristic predicate of $\mathcal{S}$ is the predicate P that is true exactly on $\mathcal{S}$, *i.e.*

$$P(x) \leftrightarrow x \in \mathcal{S}$$

Some remarks

- Sets can be elements of other sets,
  Example $\mathcal{S} = \{\{1, 2, 3\}, a, \{u\}, \{b, c\}\}$

- The empty set is different from the set containing the

$$\emptyset \neq \{\emptyset\}$$

Russell's Paradox

(After Bertrand Russell (18721970); Logician, mathematician and philosopher. Nobel Prize in Literature 1950)

- Naive set theory contains contradictions
  Let $\mathcal{S}$ be the set of all sets which are not members of themselves
  $$\mathcal{S} = \{\mathcal{S}' \mid \mathcal{S}' \notin \mathcal{S}'\}$$
  "Is $\mathcal{S}$ a member of itself?", *i.e.* $\mathcal{S} \in \mathcal{S}$?

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- Modern formulations (such as Zermelo-Fraenkel) avoid such obvious problems by stricter axioms about set construction[3]. However, it is impossible to prove in ZF that ZF is consistent (unless ZF is inconsistent)

[3] Well-definedness condition in definition of a set

Set equality

Definition

Two sets A and B are equal, denoted $A = B$, iff they have the same elements

$$\forall A, B.\ (A = B \ \leftrightarrow\ \forall x.\ (x \in A \leftrightarrow x \in B))$$

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Example The order is not important

$$\{a, e, i, o, u, y\} = \{y, u, o, i, e, a\}$$

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Example The order is not important

$$\{a, e, i, o, u, y\} = \{y, u, o, i, e, a\}$$

Example Repetitions are not important

$$\{a, e, i, o, u, y\} = \{a, a, e, e, i, i, o, o, u, u, y, y\}$$

Subsets and supersets

Definition

A set A is a subset of a set B (and B is a superset of A), denoted $A \subseteq B$, iff all elements of A are elements of B

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Example $\forall \mathcal{S}.\ \emptyset \subseteq \mathcal{S}$

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Example $\forall \mathcal{S}.\ \emptyset \subseteq \mathcal{S}$

Example $\forall \mathcal{S}.\ \mathcal{S} \subseteq \mathcal{S}$

Definition

A is a proper subset of B iff $A \subseteq B$ and $A \neq B$. This is denoted by $A \subset B$. Equivalently,
$\forall A, B.\ (A \subset B \ \leftrightarrow\ \forall x.\ (x \in A \to x \in B) \wedge \exists x.\ (x \in B \wedge x \notin A))$

Set cardinality

Definition

If there are exactly n distinct elements in a set $\mathcal{S}$, where n is a non-negative integer, we say that $\mathcal{S}$ is finite. Otherwise it is infinite

Definition

The cardinality of a finite set $\mathcal{S}$, denoted by $|\mathcal{S}|$, is the number of (distinct) elements of $\mathcal{S}$

Examples

$|\emptyset| = 0$
$|\{1, 2, 3\}| = 3$
$|\{\emptyset\}| = 1$
$\mathbb{Z}$ is infinite

Powerset

Definition

The set of all subsets of a set $\mathcal{S}$ is called the power set of $\mathcal{S}$. It is denoted by $\mathcal{P}(\mathcal{S})$ or $2^{\mathcal{S}}$. Formally

$$\mathcal{P}(\mathcal{S}) = \{\mathcal{S}' \mid \mathcal{S}' \subseteq \mathcal{S}\}$$

In particular,

$$\left.\begin{array}{l} \bullet\ \mathcal{S} \in \mathcal{P}(\mathcal{S}) \\ \bullet\ \emptyset \in \mathcal{P}(\mathcal{S}) \end{array}\right\} \Rightarrow \forall \mathcal{S}.\ \mathcal{P}(\mathcal{S}) \neq \emptyset$$

Example $\mathcal{P}(\{a, b\}) = \{\emptyset, \{a\}, \{b\}, \{a, b\}\}$

Example

$$\mathcal{P}(\emptyset) = \{\emptyset\}$$
$$\mathcal{P}(\mathcal{P}(\emptyset)) = \{\emptyset, \{\emptyset\}\}$$
$$\mathcal{P}(\mathcal{P}(\mathcal{P}(\emptyset))) = \{\emptyset, \{\emptyset\}, \{\{\emptyset\}\}, \{\emptyset, \{\emptyset\}\}\}$$

If $|\mathcal{S}| = n$ then $|\mathcal{P}(\mathcal{S})| = 2^n$. Proof by induction on n; see later

Tuples

- The ordered n-tuple $(a_1, a_2, \ldots, a_n)$ is the ordered collection of n elements, where a_1 is the first, a_2 the second, *etc.*, and a_n the n^{th} (*i.e.* the last)

- Two n-tuples are equal iff their corresponding elements are equal:
$$(a_1, a_2, ..., a_n) = (b_1, b_2, ..., b_n) \leftrightarrow a_1 = b_1 \wedge a_2 = b_2 \wedge \cdots \wedge a_n = b_n$$

- 2-tuples are called ordered pairs

Cartesian Product

Definition

The Cartesian product of two sets A and B, denoted by $A \times B$, is the set of all ordered pairs (a, b) where $a \in A$ and $b \in B$

$$A \times B = \{(a, b) \mid a \in A \wedge b \in B\}$$

Definition

The Cartesian product of n sets $A_1, A_2, \ldots, A_n$, denoted by $A_1 \times A_2 \times \ldots \times A_n$, is the set of all tuples $(a_1, a_2, \ldots, a_n)$ where $a_i \in A_i$ for $1 \leq i \leq n$

$$A_1 \times A_2 \times \ldots \times A_n = \{(a_1, a_2, \ldots, a_n) \mid a_i \in A_i \text{ for } 1 \leq i \leq n\}$$

Example For $A = \{0, 1\}$, $B = \{1, 2\}$ and $C = \{0, 1, 2\}$

$$A \times B \times C = \left\{ \begin{array}{l} (0,1,0),(0,1,1),(0,1,2),(0,2,0),(0,2,1),(0,2,2), \\ (1,1,0),(1,1,1),(1,1,2),(1,2,0),(1,2,1),(1,1,2) \end{array} \right\}$$

The powerset Boolean algebra

$$(\mathcal{P}(\mathcal{U}), \emptyset, \mathcal{U}, \cup, \cap, \bar{\cdot})$$

$$A \cup B = \{x \in \mathcal{U} \mid x \in A \vee x \in B\}$$
$$A \cap B = \{x \in \mathcal{U} \mid x \in A \wedge x \in B\}$$
$$\overline{A} = \{x \in \mathcal{U} \mid \neg(x \in A)\}$$

- $|A \cup B| = |A| + |B| - |A \cap B|$
  In particular, $|A \cup B| \leq |A| + |B|$
- $|A \cap B| \leq |A|$
  $|A \cap B| \leq |B|$

Set difference

Definition

The difference between sets A and B, denoted $A - B$ is the set containing the elements of A that are not in B:

$$A - B = \{x \mid x \in A \wedge x \notin B\}$$

Example $\{1, 2, 3\} - \{2, 4, 6\} = \{1, 3\}$

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- $|A - B| = |A| - |A \cap B| \leq |A|$
- $A - B = A \cap \overline{B}$

Set Identities

(proofs on the board)

Identity laws: $A \cup \emptyset = A,\ A \cap \mathcal{U} = A$

Domination laws: $A \cup \mathcal{U} = \mathcal{U},\ A \cap \emptyset = \emptyset$

Idempotent laws: $A \cup A = A,\ A \cap A = A$

Complementation law: $\overline{(\overline{A})} = A$

Complement laws: $A \cup \overline{A} = \mathcal{U},\ A \cap \overline{A} = \emptyset$

Commutative laws: $A \cup B = B \cup A,\ A \cap B = B \cap A$

Associative laws: $A \cup (B \cup C) = (A \cup B) \cup C,$
$A \cap (B \cap C) = (A \cap B) \cap C$

Distributive laws: $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

Absorption laws: $A \cup (A \cap B) = A,\ A \cap (A \cup B) = A$

De Morgan's laws: $\overline{A \cup B} = \overline{A} \cap \overline{B},\ \overline{A \cap B} = \overline{A} \cup \overline{B}$

A theorem

Theorem

Let

$$\mathcal{F} \stackrel{\text{def}}{=} \{\mathcal{S} \subseteq \mathbb{R} \mid (0 \in \mathcal{S}) \wedge \forall x \in \mathbb{R}.\ (x \in \mathcal{S} \rightarrow (x+1) \in \mathcal{S})\}$$

Then

1. $\mathbb{N} \in \mathcal{F}$
2. $\mathbb{N} \subseteq \bigcap \mathcal{F}$[a]

---

[a] $\bigcap \mathcal{F}$ denotes the intersection of all the sets in $\mathcal{F}$, *i.e.* $\bigcap \mathcal{F} \stackrel{\text{def}}{=} \bigcap_{\mathcal{S} \in \mathcal{F}} \mathcal{S}$

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Then

1. $\mathbb{N} \in \mathcal{F}$
2. $\mathbb{N} \subseteq \bigcap \mathcal{F}$[a]

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[a] $\bigcap \mathcal{F}$ denotes the intersection of all the sets in $\mathcal{F}$, *i.e.* $\bigcap \mathcal{F} \stackrel{\text{def}}{=} \bigcap_{\mathcal{S} \in \mathcal{F}} \mathcal{S}$

Corollary

$$\bigcap \mathcal{F} = \mathbb{N}$$

(proofs on the board)