Module Title: dmmr
Exam Diet (Dec/April/Aug): April 2017
Brief notes on answers:

1. (a) Any correct bijection here such as $f(x) = -x$ which is different from the identity function.

   (i). $f : A \to \mathbb{Z}^+$ given as $f(x) = \sqrt{x}$ is a bijection. 3 marks for the details.

   (ii). $f : A \to \mathbb{Z}^+$ given as $f(x) = 2x$, $x > 0$ and $f(x) = -(2x-1)$ for $x \le 0$ is a bijection; 4 marks for the details.

2. (a) For the base case $n = 1$. LHS is a as is RHS. 1 mark for this. For the inductive step assume it holds for $n = j$. Show it for $n = j + 1$.

$$\sum_{k=1}^{j+1}(a + (k-1)r) = \sum_{k=1}^{j}(a + (k-1)r) + (a + jr)$$

   Using the IH this is

$$\frac{j}{2}(2a + (j-1)r) + (a + jr)$$

   Now the result follows as this is equal to $\frac{j+1}{2}(2a + jr)$. 6 marks here; 3 for using induction hypothesis and 3 for getting it all correct.

   (b) Using Euclid's algorithm: $= \gcd(89, 55) = \gcd(55, 34) = \gcd(34, 21) = \gcd(21, 13) = \gcd(13, 8) = \gcd(8, 5) = \gcd(5, 3) = \gcd(3, 2) = \gcd(2, 1) = \gcd(1, 0) = 1$. 3 marks for full answer.

3. (a) Note that since we know that for $i \in \{1, 2, 3\}$, f(7-i) = 5- f(i), we know that the values $f(i)$ of the function, for all $i \in \{1, 2, 3\}$, determine the values of the function on the entire domain $\{1, 2, 3, 4, 5, 6\}$. (Namely, $f(1)$ determines $f(6)$, and $f(2)$ determines $f(5)$, and $f(3)$ determines $f(4)$.)

   Moreover, for each $i \in \{1, 2, 3\}$, we are free to let $f(i)$ be any value in $\{1, 2, 3, 4\}$. Thus, the number of such functions is the number of distinct functions $g : \{1, 2, 3\} \to \{1, 2, 3, 4\}$. There are thus $4^3 = 64$ distinct such functions.

   (b) Each function $f : \{1, \dots, 7\} \to \{1, \dots, 7\}$ can be described by a sequence $(f(1), f(2), \dots, f(7))$ of numbers, each in $\{1, \dots, 7\}$ such that the i'th number $f(i)$ is not i. Thus, there are $7 - 1 = 6$ possible choices for the i'th number, $f(i)$, for all $i \in \{1, \dots, 7\}$. By the product rule, there are thus 6^7 such functions.

4. By the binomial theorem, we know that
   $(n+1)^d = \sum_{i=0}^{d} \binom{d}{i} n^i 1^{d-i} = \sum_{i=0}^{d} \binom{d}{i} n^i \ge \sum_{i=0}^{d} n^i \ge \sum_{i=0}^{d} \binom{n}{i}$.
   The last inequality holds because $n^i \ge \binom{n}{i}$. To see why this is true, note that: $\binom{n}{i} = \frac{n(n-1)\dots(n-i+1)}{i!} \le n(n-1)\dots(n-i+1) \le n^i$.

5. This is a straightforward application of Bayes' Theorem. Let A be the event that a chimp has disease A, and let G be the event that a chimp has that specific gene. We are told that $P(G|A) = 3/5$ and $P(G|\bar{A}) = 1/10$, and that $P(A) = 1/6$ and (thus) $P(\bar{A}) = 5/6$. We are interested in knowing $P(A|G)$. By Bayes' Theorem, this is:

$$\begin{aligned} P(A|G) &= \frac{P(G|A)P(A)}{P(G|A)P(A) + P(G|\bar{A})P(\bar{A})} = \frac{(3/5)(1/6)}{(3/5)(1/6) + (1/10)(5/6)} \\ &= \frac{(1/10)}{(1/10) + (1/12)} = \frac{(12/120)}{(22/120)} = \frac{6}{11}. \end{aligned}$$

PART B

6. (a) True. If $C \in \mathcal{P}(A)$ or $\mathcal{P}(B)$ then $C \subseteq A$ or $C \subseteq B$ so therefore $C \in \mathcal{P}(A \cup B)$. 3 marks for convincing explanation; only 1 mark without explanation.
   (b) True. If $C \in \mathcal{P}(A)$ and $C \in \mathcal{P}(B)$ then $C \subseteq A$ and $C \subseteq B$ so $C \subseteq A \cap B$ and so $C \in \mathcal{P}(A \cap B)$. 3 marks for convincing explanation, 1 mark without explanation.
   (c) False. If $A = \{0, 1\}$ and $B = \{1\}$ then $\{0, 1\} \in \mathcal{P}(A)$ but not in $\mathcal{P}(B)$; however $\mathcal{P}(A - B) = \mathcal{P}(\{0\})$. As before 3 marks for explanation; 1 mark without explanation.
   (d) False. $\mathcal{P}(A) \times \mathcal{P}(B)$ is a set of set pairs whereas $\mathcal{P}(A \times B)$ is a set of sets of pairs: for instance if $A = \{0, 1\} = B$ then (A, B) is in the first but not in the second. 3 marks for explanation; 1 mark without explanation.
   (e) False. If $A = \{0\}$ and $B = \{1\}$ then $\{0, 1\} \in \mathcal{P}(A \cup B)$ but not in $\mathcal{P}(A) \cup \mathcal{P}(B)$. 3 marks for explanation; 1 mark without explanation.
   (f) True. If $C \in \mathcal{P}(A \cap B)$ then $C \subseteq A$ and $C \subseteq B$ so $C \in \mathcal{P}(A) \cap \mathcal{P}(B)$ 4 marks for explanation; 1 mark without explanation.
   (g) True. If $C \in \mathcal{P}(A - B)$ then $C \subseteq A$ and $C \cap B = \emptyset$ so $C \in \mathcal{P}(A) \cap \mathcal{P}(B)$ 3 marks for explanation; 1 mark without explanation.
   (h) False. $\mathcal{P}(A \times B)$ is a set of sets of pairs whereas $\mathcal{P}(A) \times \mathcal{P}(B)$ is a set of set pairs: if $A = \{0\}$ and $B = \{1\}$ then $\{(0, 1)\}$ is an element of the first but not the second. 3 marks for explanation; 1 mark without explanation.

7. (a) We did this proof in lectures. Consider $(a_1 M_1 y_1 + \ldots + a_n M_n y_n) \bmod m_i$. By assumption m_i divides M_j if $j \neq i$. So it is equal to $a_i M_i y_i \bmod m_i$ because $cm + d \bmod m = d \bmod m$. Now we assume y_i is inverse of M_i mod m_i; so $M_i y_i \equiv 1 \pmod{m_i}$. Therefore, $x \equiv a_i \pmod{m}$. 10 marks for adding in all the details of this answer such as explaining inverses.
   (b) Here $M_1 = 105$ and $y_1 = 1$; $M_2 = 70$ and $y_2 = 1$; $M_3 = 42$ and $y_3 = 3$; $M_4 = 30$ and $y_4 = 4$. So the answer is $105 + 140 + 504 + 720 \bmod 210 = 209$. 10 marks for getting all the details right.
   (c) Assume z is another solution; then $z \equiv x \pmod{m_1}$, …, $z \equiv x \pmod{m_n}$. But then since $\gcd(m_i, m_j) = 1$ for $i \neq j$ it follows that $z \equiv x \pmod{m}$ as required. 5 marks for full argument.

8. (a) We prove this by contradiction. Suppose not. Then $d(u) > (n/2)$ for all $u \in V$. But then note that the edge set E is non-empty and for any pair $u, v \in V$ such that $\{u, v\} \in E$ we have $d(u) + d(v) > n$. But then since there are only $n = |V|$ vertices, by the pigeon hole principle there must be some vertex w such that there is an edge between w and both u and v. In other words, there must be a triangle. This is a contradiction. So, there can not exist any pair of vertices $u, v \in E$ such that $\{u, v\} \in E$ and $d(u) + d(v) > n$, and hence it can not be the case that all vertices u have degree $d(u) > n/2$.
   (b) We prove by induction on $n \geq 3$ that if G is triangle-free then $m \leq \frac{n^2}{4}$. First, we establish the base cases $n = 3$ and $n = 4$. When $n = 3$, if G is triangle-free it has at most $m = 2$ edges, and $m \leq 2 \leq \frac{n^2}{4} = \frac{9}{4}$. Next, $n = 4$. In this case we can

see that if G is triangle-free, the most number of edges G could have is $m = 4$, given by a "rectangle graph". And again, we see that $m \leq 4 \leq \frac{n^2}{4} = \frac{16}{4} = 4$.

Now, for the inductive step, suppose that for some $n \geq 5$ the claim holds when the number of vertices is $|V| \leq n - 2$. We show that it holds when the number of vertices is $|V| = n$.

Consider any pair of vertices $u, v \in V$, such that $\{u, v\} \in E$. (If there is no such pair, then we are done, because $m = 0 \leq (n^2/4)$.)

We have already argued in part (a) that since G is triangle-free it must be the case that $d(u) + d(v) \leq n$. Now consider the graph G' obtained by G by completely removing the vertices u and v and all edges incident to them.

The subgraph G' is clearly also triangle-free, since G is traingle-free.

Moreover, G' has $n - 2$ vertices. Thus, by the induction hypothesis, G' has at most $(n-2)^2/4$ edges. But since $d(u) + d(v) \leq n$, and since $\{u, v\} \in E$, we see that removing u and v can lead to the removal of at most $n - 1$ edges from G (because the edge $\{u, v\}$ is double-counted in $d(u) + d(v)$). Thus m, the number of edges of G, satisfies:

$$m \leq \frac{(n-2)^2}{4} + (n-1) = \frac{(n-2)^2 + 4(n-1)}{4} = \frac{n^2 - 4n + 4 + 4n - 4}{4} = \frac{n^2}{4}.$$

(Note: for the base case, we do need to consider both $n = 3$ and $n = 4$, as we have, because the inductive step for n uses the inductive hypothesis applied to $n - 2$, and consequently, to establish the claim for all $n \geq 3$ our base case has to include both $n = 3$ and $n = 4$. Otherwise, for example, if we only had $n = 3$ as a base case, then the inductive argument as given wouldn't work for $n = 4$, unless we also added the $n = 2$ case as a base case.)

(c) This is an immediate consequence of the special case of Ramsey's theorem, proved in class, that among any group of 6 people (vertices), every pair of which are either friends or enemies (edges or non-edges), there are either 3 who are mutual friends (a triangle), or 3 who are mutual enemies (a group of 3 vertices without any edges between them.

Since we are told that there are at least 6 vertices, and that there are no triangles, there must be (at least) 3 nodes such that there are no edges between any of them.