DMMR Tutorial sheet 7

Graphs

1. Suppose there is a finite set A of job applicants and a finite set J of job openings, and that for some fixed positive integer $k \geq 1$, every job applicant $a \in A$ has applied to exactly k jobs in J, and every job opening $j \in J$ has received exactly k job applications from applicants in A.

   Prove that $|A| = |J|$, and that it is possible to match each job applicant $a \in A$ with a unique job $f(a) \in J$ which a has applied for, such that all applicants and all jobs are "matched", and no job (no applicant) is matched to more than one applicant (one job, respectively). In other words, prove that there is a bijective function $f : A \to J$, such that, for all $a \in A$, a has applied to $f(a)$.

   [Hint: apply the generalized pigeonhole principle to show that Hall's Theorem applies in this setting.]

   Solution:
   Consider the bipartite graph $G = (A \cup J, E)$, with bipartition (A, J). There is one vertex for each applicant $a \in A$, and one vertex for each job opening $j \in J$. There is an edge $\{a, j\} \in E$ if and only if applicant $a \in A$ has applied to job $j \in J$.

   From the fact that every applicant has applied to exactly $k \geq 1$ jobs, and that every job has received k applications, we know that G is a k-*regular* bipartite graph, meaning every vertex in G has degree exactly k.

   To see that $|A| = |J|$, note that if we count the edges of G by summing many edges are incident on A, we see that there are $k|A|$ edges, while if we count the edges by counting how many are incident on J, we see that there are $k|J|$ edges. Since $k \geq 1$, we must have $|A| = |J|$.

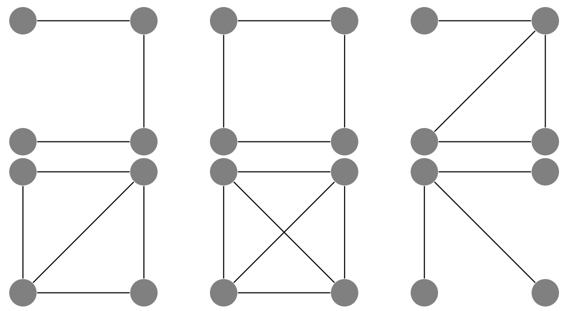
   We are asked to show that there must exist a perfect matching in G. We show this by showing that the conditions of Hall's theorem hold. Recall that Hall's theorem tells that (given that $|A| = |J|$), there is a perfect matching in G if and only if for every subset $S \subseteq A$, $|N(S)| \geq |S|$, where $N(S)$ denotes the set of neighbours of nodes in S.

   For any subset $S \subseteq A$, since every applicant $a \in S$ has exactly k jobs as neighbours, we know that there are $k|S|$ edges going out of S. Suppose, for contradiction, that $|N(S)| < |S|$. Then by the generalized pigeonhole principle, there must be some job $j \in N(S)$, such that j has strictly more than k applicants. But this contradicts the assumption that G is k-regular, i.e., every job receives exactly k applications.

   Thus, Hall's Theorem applies, and implies there exists a perfect matching between applicants in A and jobs in J. $\square$

2. How many non-isomorphic (simple, undirected) graphs are there with exactly 4 vertices? Justify your answer.

   Solution:
   Let us divide in cases, according to the number of vertices of the largest connected subgraph (LCS). Clearly, there is only one graph where the LCS has 1 vertex (all vertices have degree 0). There are 2 graphs where LCS has 2 vertices. There are 2 graphs where LCS has 3 vertices, and there are 6 graphs where the LCS has 4 vertices. These six graphs are represented below.



Thus, the total number of non-isomorphic simple undirected graphs with 4 vertices is $1+2+2+6=11$. $\square$

3. Suppose $G=(V,E)$ is a directed graph, and u and v are vertices of G. Show that either u and v are in the same strongly connected component of G, or they are in disjoint strongly connected components of G.

   Solution:
   Suppose the strongly connected components C_u of u and C_v of v are not disjoint, i.e., there is a non-empty intersection. We will prove that the subgraph with vertices $C_u \cup C_v$ is strongly connected. It suffices to prove that any point in C_u is strongly connected to any point in C_v. Let $u' \in C_u$ and $v' \in C_v$, and $x \in C_u \cap C_v$. Clearly, there are directed paths from u' to x and from x to v'. Thus, there is a path from u' to v'. Similarly, there must also be a path from v' to u'. Thus, the subgraph with vertices $C_u \cup C_v$ is strongly connected. $\square$

4. Recall that the n-dimensional **hypercube**, or n-cube, is the simple undirected graph whose nodes are bit strings of length n, and such that there is an edge between a pair of nodes if and only if their bit strings differ in exactly one bit position.

   (a) For what values of $n \geq 1$ does the n-cube have an Euler circuit?

   (b) Prove by induction that for all $n \geq 2$, the n-cube has a Hamiltonian circuit.

   Solution:

   (a) Any vertex in a n-cube has degree n and every n-cube is connected (to give a path, change the bits one by one from one vertex to the other). An Euler circuit exists in a connected graph if and only if every vertex has even degree. Thus, there is an Euler circuit if and only if n is even.

   (b) The base case ($n=2$) is trivial (the 2-cube is the 4-cycle C_4, which clearly has a Hamiltonian circuit). Let us assume that there is a Hamiltonian circuit for the n-cube and prove that there must also be one for the $(n+1)$-cube. Take the $(n+1)$-cube and consider the subgrah G_0 with the vertices of the form $(b_1,\ldots,b_n,0)$, and the subgraph G_1 with the vertices of the form $(b_1,\ldots,b_n,1)$. Clearly, G_0 and G_1 are both isomorphic to the n-cube. By the inductive hypothesis we can find a Hamiltonian circuit for each. Let us take the 'same' circuit for both (we can get one of the circuits by only changing the last coordinate of each vertex in the other circuit). Now take two vertices that are adjacent in the Hamiltonian cycle for G_0: $(x_1,\ldots,x_n,0)$, $(y_1,\ldots,y_n,0)$, and the corresponding vertices in G_1: $(x_1,\ldots,x_n,1)$, $(y_1,\ldots,y_n,1)$. Then, by dropping the edge between $(x_1,\ldots,x_n,0)$ and $(y_1,\ldots,y_n,0)$ in the Hamiltonian circuit, we get a Hamiltonian path $P_0=(x_1,\ldots,x_n,0),\ldots,(y_1,\ldots,y_n,0)$ of G_0, and likewise we get a Hamiltonian path

$P_1 = (y_1, \dots, y_n, 1), \dots, (x_1, \dots, x_n, 1)$ of G_1, which goes in the opposite direction. Now consider the Hamiltonian circuit for G obtained from P_0 and P_1, by composing the paths P_0 and P_1, using the edge between $(y_1, \dots, y_n, 0)$ and $(y_1, \dots, y_n, 1)$, and the edge between $(x_1, \dots, x_n, 1)$ and $(x_1, \dots, x_n, 0)$. Specifically, consider the Hamiltonian circuit $C = P_0, P_1, (x_1, \dots, x_n, 0)$. This is clearly a Hamiltonian circuit of the $(n+1)$-cube, since it is a circuit that traverses each vertex of the $(n+1)$-cube exactly once (and returns at the end to where it started).

(It is worth pointing out that a Hamiltonian circuit on a hypercube is also known as a **Gray code**, and has applications in coding theory and other areas.) $\square$

5. Consider a directed graph $G = (V, E)$, and let $s, t \in V$ be two distinct *and non-adjacent* vertices of G. A directed s-t-path in G is a sequence of vertices $s = v_0, v_1, \dots, v_k = t$, such that $(v_{i-1}, v_i) \in E$, for all $i \in \{1, \dots, k\}$. Two distinct directed s-t-paths are called "internally vertex-disjoint" if they share no vertex in common other than s and t themselves, i.e., the intersection of the sets of vertices on the two paths is just $\{s, t\}$.

   A subset $A \subseteq V$ of the vertices is called a *directed s-t-cut* in G if $s, t \notin A$ and A intersects the set of vertices appearing on any directed s-t-path in G.

   Let $d_{s,t}$ be the maximum number of mutually vertex-disjoint directed s-t-paths in G. Let $c_{s,t}$ be the minimum size of any directed s-t-cut in G. Prove that $d_{s,t} \leq c_{s,t}$, for any directed graph $G = (V, E)$ and any $s, t \in V$.

   (Food for thought: can you think of any directed graph G where $d_{s,t} \neq c_{s,t}$? You are not expected to answer this, just think about it.)

   Solution:
   Let $D = \{P_1, \dots, P_d\}$ be any set of internally vertex-disjoint directed s-t-paths in a directed graph G. Let A be any directed s-t-cut in G. By definition, A must contain some vertex on every directed s-t-path $P_i \in D$, and since A contains neither s nor t, A must contain some internal vertex of every $P_i \in D$. Moreover, since any two distinct s-t-paths $P_i, P_k \in D$, $i \neq j$, are internally vertex disjoint, A must contain a internal vertex v_i from every $P_i \in D$, such that $v_i \neq v_j$ for all $i, j \in \{1, \dots, d\}$, $i \neq j$. Thus, $|A| \geq d = |D|$. Since this hold true for any directed s-t-cut A and any set D of internally vertex-disjoint directed s-t-paths, this implies that the smallest such set A is at least as large as the largest such set D, i.e., that $c_{s,t} \geq d_{s,t}$.

   Answer to the food for tought question: it turns out that for ALL graphs, G, and for all distict vertices s, t, we must have $d_{s,t} = c_{s,t}$. In other words, the minimum size of a directed s-t-cut is always equal to the maximum number of mutually vertex disjoint directed s-t-paths. This is a version of a theorem in graph theory called *Menger's Theorem*. (It is also related to a theorem about "network flows" called the "max-flow min-cut" theorem.) We will not prove it. It can be proved by induction on the number of edges of G, but the proof is a bit tricky. $\square$