# DMMR Tutorial sheet 6

Basic Counting, Permutations and Combinations, Binomial Coefficients

October 25, 2019

1. (a) How many bit sequences (bit strings) of length 11 are there which start with one of the two bit sequences 101 or 010, or end with one of the bit sequences 111 or 000, or both?

   (b) Consider the following statement: "In a class with 185 students, there are at least $x$ students all of whose first names start with the same letter of the English alphabet.". What is the maximum integer value of $x$ for which this statement is *always* true?

   **Solution:**

   (a) Let $A$ (respectively, $B$) denote the set of bit sequences of length 11 that start with either 101 or 010 (respectively, that end with either 111 or 000). There are $2^8$ bit sequences of length 11 that start with 101, and likewise $2^8$ bit sequences that start with 010. Since these two sets are disjoint, we have $|A| = 2^8 + 2^8 = 2^9$. Similarly, we have $|B| = 2^9$. Our goal is to calculate $|A \cup B|$. Using the subtraction rule, we have $|A \cup B| = |A| + |B| - |A \cap B|$. So, $|A \cup B| = 2^9 + 2^9 - |A \cap B| = 2^{10} - |A \cap B|$. So, we need to calculate the size of $A \cap B$. This is the set of bit strings of length 11 that start with either 101 or 010, and that end with either 111 or 000. There are $2 \cdot 2^5 \cdot 2 = 2^7$ such strings, because there are 2 choices for the first 3 bits, $2^5$ choices for the middle 5 bits, and 2 choices for the last 3 bits.
   So, the overall answer is $|A \cup B| = 2^{10} - |A \cap B| = 2^{10} - 2^7 = 2^7 \cdot (2^3 - 1) = 2^7 \cdot 7$.

   (b) There are 26 letters in the English alphabet. We apply the generalized pigeonhole principal, which tells us that that there must be at least $\lceil \frac{185}{26} \rceil = 8$ students whose names start with the same letter. If all strings are possible as first names, we can not do better 8, because if the first letter is "as evenly distributed as possible" among the 26 letters there will at most 8 names with the same first letter. (But of course in reality there are likely to be many more with the same first letter.)

   $\square$

2. How many different strings can be formed by reordering the letters of the word: ABRACADABRA ?

   **Solution:**
   There are 11 letters in ABRACADABRA. Of these, 5 are A's, 2 are B's, 1 is C, 1 is D, and 2 are R. So, the number of ways to reorder this string is given by the multinomial coefficient:

$$\binom{11}{5,2,1,1,2} = \frac{11!}{5!2!1!1!2!} = \frac{11 \cdot 10 \cdot 9 \cdot 8 \cdot 7 \cdot 6}{4}.$$

   $\square$

3. Prove that for all integers $k, n$, such that $1 \leq k \leq n$, the following inqualities hold (where $e = 2.71828\ldots$ is the base of the natural logarithm):

$$\left(\frac{n}{k}\right)^k \leq \binom{n}{k} \leq \left(\frac{n \cdot e}{k}\right)^k$$

(Hint: for the upper bound, use "Stirling's approximation with lower and upper bounds", given in the lecture notes.)

**Solution:**
First, observe that $\left(\frac{n}{k}\right)^k \leq \binom{n}{k}$,because $\binom{n}{k} = \frac{n}{k} \cdot \frac{n-1}{k-1} \ldots \frac{n-k+1}{1} \geq (\frac{n}{k})^k$, because $\frac{n}{k} \leq \frac{n-i}{k-i}$, for $i < k$.

Next, observe that $\binom{n}{k} = \frac{n \cdot (n-1) \ldots (n-k+1)}{k!} \leq \frac{n^k}{k!}$. By the lower bound of Stirling's approximation, we have $(k/e)^k \leq k!$, and thus $\frac{n^k}{k!} \leq \frac{n^k}{(k/e)^k} = \left(\frac{n \cdot e}{k}\right)^k$. $\square$

4. Prove the following identity holds for all non-negative integers $n$,$r$ and $k$, such that $r \leq n$, and $k \leq r$. (Try to give two different proofs: one based on a combinatorial argument, and another by manipulating the formulas that define binomial coeffients.)

$$\binom{n}{r} \cdot \binom{r}{k} = \binom{n}{k} \cdot \binom{n-k}{r-k}$$

**Solution:**
We show two different proofs:

- Combinatorial: $\binom{n}{r}$ is the number of possibilities to choose $r$ elements from a set of $n$ elements.

  Now consider a set $S$ of $n$ elements. We split $S$ into three disjoint subsets:

  - $S_1$ contains $k$ elements
  - $S_2$ contains $r - k$ elements
  - $S_3$ contains $n - r$ elements

  In total all elements are distributed into one of the sets since $(n - r) + (r - k) + k = n$.

  Now the possibilities to choose $S_1$,$S_2$ and $S_3$ can be calculated in two different ways: We could split $S$ into $S_3$ and $S_1 \cup S_2$ first. For this we need to choose the $r$ elements for $S_1 \cup S_2$ from $S$ which calculates to $\binom{n}{r}$. Then we split $S_1 \cup S_2$ into $S_1$ and $S_2$. This calculates to $\binom{r}{k}$. In total we get $\binom{n}{r} \cdot \binom{r}{k}$ possibilities to choose $S_1, S_2, S_3$.
  On the other hand we can split $S$ into $S_1$ and $S_2 \cup S_3$ first. There are $\binom{n}{k}$ possibilities. Then splitting $S_2 \cup S_3$ into $S_2$ and $S_3$ is possible in $\binom{n-k}{r-k}$, since $S_2 \cup S_3$ has $n - r + r - k = n - k$ elements and $r - k$ are chosen for $S_2$. So in total there are $\binom{n}{k} \cdot \binom{n-k}{r-k}$ possibilities to split $S$ in this way.

  Since the two procedures produce the same partitions, the number of possible outcomes have to be equal and we get $\binom{n}{r} \cdot \binom{r}{k} = \binom{n}{k} \cdot \binom{n-k}{r-k}$

- By the formula:

$$\begin{aligned}
\binom{n}{r} \cdot \binom{r}{k} &= \frac{n!}{r! \cdot (n-r)!} \cdot \frac{r!}{k!(r-k)!} \\
&= \frac{n!}{(n-r)! \cdot k! \cdot (r-k)!} \cdot \frac{r!}{r!} \\
&= \frac{n!}{(n-r)! \cdot k! \cdot (r-k)!} \cdot \frac{(n-k)!}{(n-k)!} \\
&= \frac{n!}{k! \cdot (n-k)!} \cdot \frac{(n-k)!}{(r-k)! \cdot (n-r)!} \\
&= \frac{n!}{k! \cdot (n-k)!} \cdot \frac{(n-k)!}{(r-k)! \cdot (n-k-(r-k))!} \\
&= \binom{n}{k} \cdot \binom{n-k}{r-k}
\end{aligned}$$

□

5. How many different solutions does the following inequality have, in which $x_1$ and $x_2$ must be non-negative integers and $x_3$ must be a positive integer?

$$x_1 + x_2 + x_3 \leq 13$$

(Show how you have calculated your answer.)

**Solution:**

By adding an extra non-negative variable that "takes up the slack" in the inequality, we can rewrite this as finding the number of different solutions to

$$x_1 + x_2 + x_3 + x_4 = 13$$

when $x_1$, $x_2$, and $x_4$ are non-negative integers and $x_3$ is a positive integer.

However, we can then notice that if we let $x_3' = (x_3 - 1)$, then the above is equivalent to the number of solutions of:

$$x_1 + x_2 + x_3' + 1 + x_4 = 13$$

or equivalently, to the number of solutions of:

$$x_1 + x_2 + x_3' + x_4 = 12$$

where $x_1$, $x_2$, $x_3'$ and $x_4$ are all non-negative integers.

Finally, as also described in lectures, the number of solutins to this last question is given by the number of 12-combinations from a 4 element set, which is given by:

$$\binom{12 + (4-1)}{(4-1)} = \frac{15!}{3!12!} = \frac{15 \cdot 14 \cdot 13}{6}$$

□