DMMR Tutorial 9

Discrete Probability

1. Suppose a biased coin, which lands heads with probability $1/3$ (and tails with probability $2/3$) each time it is flipped, is flipped 10 times.

   What is the conditional probability that the number of times the coin lands heads is exactly 3, given that (i.e., conditioned on the event that) the number of times it lands heads is divisible by 3? (Give an expression for this probability. You do not need to compute its exact value.)

2. Suppose that a fair coin is flipped three times consecutively.

   Let E_1 and E_2 denote the events that "the first flip comes up heads", and "the second flip comes up heads", respectively.

   Let B denote the event that exactly one of the first and second coin tosses comes up heads.

   Prove that E_1, E_2, and B are pairwise-independent events, but that they are NOT mutually independent events.

   (Thus, observe that pairwise independence of a collection of events does not imply mutual independence.)

3. Suppose that a pharmaceutical company has developed a new non-invasive test for a type of cancer. Their studies show that this new test has the following properties.

   (a) If the test is performed on a (random) person who has this type of cancer, then there is an 88% chance that the test result will be positive.

   (b) If, on the other hand, the test is performed on a (random) person who does not have this cancer, then there is a 9% chance that the test result will be positive.

   (c) Approximately 1 in 1000 persons in the entire population have this type of cancer.

   Suppose that this new test is performed on a (random) person in the population. What is the probability that the person actually has this cancer, given that their test result was positive?

4. Let us suppose that the number of crisp packets that the Walkers Crisp Company produces in a given day is a random variable. Suppose that the average number of crisp packets that Walkers produces in a day is 100,000.

   Prove that the probability that Walkers produces more than 1.5 million packets in a given day is at most $1/15$.

5. (This question is harder than the others. I offer some hints below.)

   Suppose you are a big fan of Star Wars.

   Suppose that Kellogg's Corn Flakes has made a deal with Disney (the company that owns the rights to Star Wars), allowing Kellogg's to place inside each Corn Flakes cereal box a small replica action figure for one of 25 Star Wars characters.

Suppose each of the 25 Star Wars action figures is equally likely to be placed in each Corn Flakes cereal box. (The box cover does not indicate which action figure is inside the cereal box.)

Suppose your goal is to collect all $n = 25$ Star Wars action figures.

What is the expected number of cereal boxes that you would have to buy in order to do this?

Solve this for general n, and then plug in $n = 25$ to get the specific solution.

(*Hints:* Consider the random variable X, which denotes the total number of boxes you bought until you obtained all n action figures. Consider also the random variables, X_i, $i = 1, \ldots, n$, denoting the number of boxes you bought after you had already collected $i - 1$ different action figures, up to and including the box which gave you the new i'th different action figure. Note that $X = \sum_{i=1}^{n} X_i$. Note also that X_i is a geometrically distributed random variable.)