DMMR Tutorial sheet 7

Graphs

1. Suppose there is a finite set A of job applicants and a finite set J of job openings, and that for some fixed positive integer $k \geq 1$, every job applicant $a \in A$ has applied to exactly k jobs in J, and every job opening $j \in J$ has received exactly k job applications from applicants in A.

   Prove that $|A| = |J|$, and that it is possible to match each job applicant $a \in A$ with a unique job $f(a) \in J$ which a has applied for, such that all applicants and all jobs are "matched", and no job (no applicant) is matched to more than one applicant (one job, respectively). In other words, prove that there is a bijective function $f : A \to J$, such that, for all $a \in A$, a has applied to $f(a)$.

   [Hint: apply the generalized pigeonhole principle to show that Hall's Theorem applies in this setting.]

2. How many non-isomorphic (simple, undirected) graphs are there with exactly 4 vertices? Justify your answer.

3. Suppose $G = (V, E)$ is a directed graph, and u and v are vertices of G. Show that either u and v are in the same strongly connected component of G, or they are in disjoint strongly connected components of G.

4. Recall that the n-dimensional **hypercube**, or n-cube, is the simple undirected graph whose nodes are bit strings of length n, and such that there is an edge between a pair of nodes if and only if their bit strings differ in exactly one bit position.

   (a) For what values of $n \geq 1$ does the n-cube have an Euler circuit?

   (b) Prove by induction that for all $n \geq 2$, the n-cube has a Hamiltonian circuit.

5. Consider a directed graph $G = (V, E)$, and let $s, t \in V$ be two distinct *and non-adjacent* vertices of G. A directed s-t-path in G is a sequence of vertices $s = v_0, v_1, \ldots, v_k = t$, such that $(v_{i-1}, v_i) \in E$, for all $i \in \{1, \ldots, k\}$. Two distinct directed s-t-paths are called "internally vertex-disjoint" if they share no vertex in common other than s and t themselves, i.e., the intersection of the sets of vertices on the two paths is just $\{s, t\}$.

   A subset $A \subseteq V$ of the vertices is called a *directed s-t-cut* in G if $s, t \notin A$ and A intersects the set of vertices appearing on any directed s-t-path in G.

   Let $d_{s,t}$ be the maximum number of mutually vertex-disjoint directed s-t-paths in G. Let $c_{s,t}$ be the minimum size of any directed s-t-cut in G. Prove that $d_{s,t} \leq c_{s,t}$, for any directed graph $G = (V, E)$ and any $s, t \in V$.

   (Food for thought: can you think of any directed graph G where $d_{s,t} \neq c_{s,t}$? You are not expected to answer this, just think about it.)