DMMR Tutorial sheet 4

Induction

October 10th, 2019

1. Use strong induction to show that every positive integer n can be written as a sum of distinct powers of two, that is, as a sum of a subset of the integers $2^0 = 1$, $2^1 = 2$, $2^2 = 4$, and so on.

   Before beginning your proof, state the property (the one you are asked to prove for every integer n) in completely formal notation with all quantifiers.

2. What is wrong with this "proof"?

   "Theorem" For every positive integer n, if x and y are positive integers with $\max(x, y) = n$, then $x = y$.

   Base case: Suppose that $n = 1$. If $\max(x, y) = 1$ and x and y are positive integers, we have $x = 1$ and $y = 1$.

   Induction hypothesis: Let k be a positive integer. Assume that whenever $\max(x, y) = k$ and x and y are positive integers, then $x = y$. Now let $\max(x, y) = k + 1$, where x and y are positive integers.

   Induction step: Then $\max(x - 1, y - 1) = k$, so by the induction hypothesis, $x - 1 = y - 1$. It therefore follows that $x = y$, completing the induction step.

3. Let $n \geq 0$ be an integer. Prove by induction:

   (a) 8 divides $3^{2n+2} + 7$

   (b) 64 divides $3^{2n+2} + 56n + 55$

4. A finite continued fraction is either an integer n or of the form $n + (1/F)$ where F is a finite continued fraction. For example, $7/9 = 0 + 1/(9/7)$, $9/7 = 1 + 1/(7/2)$, $7/2 = 3 + 1/2$; so, $7/9 = 0 + 1/(1 + 1/(3 + 1/2))$. Similarly, $17/14 = 1 + 1/(4 + 1/(1 + 1/2))$. What you have to prove is that every rational can be expressed as a finite continued fraction. Let $P(k)$ be "any rational with denominator k can be expressed as a finite continued fraction". Prove by strong induction $\forall x \in \mathbb{Z}^+(P(x))$.

   In your proof you can use the division algorithm: if a is an integer and d a positive integer then there are unique integers q and r, with $0 \leq r < d$ such that $a = dq + r$.

5. Two sequences $\{a_n\}_{n\in\mathbb{Z}^+}$ and $\{b_n\}_{n\in\mathbb{Z}^+}$ are defined recursively as follows.

$$
\begin{array}{lll}
a_1 = 1 & \text{for } n \geq 1 & a_{n+1} = a_n + 2b_n \\
b_1 = 1 & \text{for } n \geq 1 & b_{n+1} = a_n + b_n
\end{array}
$$

   Prove by induction that for all $n \in \mathbb{Z}^+$, $a_n^2 - 2b_n^2 = (-1)^n$.