DMMR Coursework 2
Solutions

1. For any integer $n \geq 1$, let r_n denote the number of different ways that the set $[n] := \{1, \ldots, n\}$ can be partitioned into disjoint non-empty subsets, the union of which is the entire set $[n]$.
Let us define r_n more formally. For a set C, let $\mathsf{Pow}(C)$ denote the power set of C, i.e., the set of all subsets of C. By definition, r_n is:

$$r_n = |\{S \subseteq \mathsf{Pow}(\{1, \ldots, n\}) \mid \forall A \in S, A \neq \emptyset;\ \forall A, A' \in S, \text{ either } A \cap A' = \emptyset \text{ or } A = A'; \\ \& \ (\bigcup_{A \in S} A) = \{1, \ldots, n\}\}|.$$

For example $r_3 = 5$. This is because the set $\{1, 2, 3\}$ can be partitioned in precisely the following 5 distinct ways: $\{\{1\}, \{2\}, \{3\}\}$; $\{\{1\}, \{2, 3\}\}$; $\{\{2\}, \{1, 3\}\}$; $\{\{3\}, \{1, 2\}\}$; & $\{\{1, 2, 3\}\}$.

Prove that, for all *even* integers $n \geq 2$,

$$(n/2)^{(n/2)} \ \leq \ r_n \ \leq \ (n)^n.$$

(Hint: for establishing the lower bound, suppose you only count those partitions of $[n]$ that have exactly $n/2$ parts, and such that each of the numbers $1, 2, \ldots, (n/2)$ is in a separate part of the partition.) (10 Marks)

Solution:
To establish the lower bound, $r_n \geq (n/2)^{(n/2)}$, notice that if we only count those partitions where there are exactly $n/2$ parts, and each of the numbers $1, 2, \ldots, (n/2)$ is in a separate part of the partition, then we can count these partitions by counting the number of distinct functions, f, "assigning" to each of the remaining $(n/2)$ numbers in the set $D = \{(n/2) + 1, \ldots, n\}$, a "label" from [n/2] = {1,...,n/2}. This is because for each number $j \in D$, we use the the "label" $f(j)$ to indicate that j is in the same partition as $f(j) \in \{1, \ldots, n/2\} = [n/2]$. Since the elements of $[n/2]$ are all in separate paritithions, each such function describes a different such partition. Now simply recall that, since $|D| = n/2 = |[n/2]|$, the number of distinct functions $f : D \to [n/2]$ is $(n/2)^{(n/2)}$. Thus $r_n \geq (n/2)^{(n/2)}$.

To establish the upper bound $r_n \leq n^n$, let's observe that we can view a function $f : [n] \to [n]$ as specifying a partition of $[n]$, by mapping each $i \in [n]$ to the "name" $j = f(i)$ of its parition. More formally, f defines a partition of $[n]$ as follows: for each $j \in [n]$ such that $f^{-1}(j) \neq \emptyset$, let $f^{-1}(j)$ be one part of the partition of $[n]$. Clearly, every $i \in [n]$ does get assigned to some partition, and only one partition, namely the partition whose "name" is $f(i)$. It is clear that every possible partition can be described this way, so $r_n \leq n^n$. However, note that this is a (very) redundant way to represent partitions: many different functions will represent the same partition (just by "renaming" the different parts of the partition). This is why we can only conclude that $r_n \leq n^n$ from this argument.

Let us mention that the numbers r_n are called *Bell Numbers* in the combinatorics literature, and they are typically denoted by B_n. There is no simply formula known for the exact value of B_n. $\square$

2. Prove the following identity on binomial coefficients: for all integers $n \geq 1$,

$$\sum_{k=1}^{n} k \cdot \binom{n}{k} = n \cdot 2^{n-1}.$$

(You can prove this either combinatorially, or by using and manipulating formulas that define binomial coefficients.) (10 Marks)

Solution:

Let us first give a "combinatorial" proof: suppose we have a set of n people, $[n] = \{1, \ldots, n\}$. The left hand side of the formula counts the number of ways that we can, for each $k \in \{1, \ldots, n\} = [n]$, choose a k-member committee (i.e., a k-element subset) from $[n]$, and furthermore choose one of those k elements to be their "leader". The right hand side counts this same set in a different way: it counts the number of ways we can choose a leader from the set $[n]$ (there are n ways to do this), and then from the remaining $n-1$ elements choose *some* subset (including the empty subset), which will form all the non-leader members of the committee. Note that both the left and right hand side are counting the same thing.

Let us now prove this using the defining formula for binomial coefficients, and using the fact that $2^{n-1} = (1+1)^{n-1} = \sum_{j=0}^{n-1} \binom{n-1}{j}$, which follows directly from the binomial theorem, as seen in class.

$$\begin{aligned}
n \cdot 2^{n-1} &= n \cdot (1+1)^{n-1} \\
&= n \cdot \sum_{j=0}^{n-1} \binom{n-1}{j} \qquad \text{(by the Binomial Theorem)} \\
&= n \cdot \sum_{j=0}^{n-1} \frac{(n-1)!}{j!(n-1-j)!} \\
&= \sum_{j=0}^{n-1} \frac{n!}{j!(n-1-j)!} \\
&= \sum_{k=1}^{n} \frac{n!}{(k-1)!(n-k)!} \qquad \text{(by letting } k = j+1\text{)} \\
&= \sum_{k=1}^{n} k \cdot \frac{n!}{k!(n-k)!} \\
&= \sum_{k=1}^{n} k \cdot \binom{n}{k}
\end{aligned}$$

□

3. Let $G = (V, E)$ be a *directed* graph (digraph). For any subsets $X, Y \subseteq V$ of vertices, let $E(X, Y) := \{(u, v) \in E \mid u \in X \ \&\ v \in Y\}$ denote the set of edges going from a vertex in X to a vertex in Y. For a subset $A \subseteq V$ of vertices, let $d^{out}(A) = |E(A, V - A)|$ denote the total number of edges exiting A, and going to a vertex outside of A. Note that, by definition, $d^{out}(\emptyset) = 0$.

Prove that for any directed graph $G = (V, E)$, and for all subsets $A, B \subseteq V$ of its vertices, the following inequality holds:

$$d^{out}(A) + d^{out}(B) \geq d^{out}(A \cap B) + d^{out}(A \cup B).$$

(Hint: Consider the sets $E(A, V-A)$, $E(B, V-B)$, $E(A\cup B, V-(A\cup B))$, and $E(A\cap B, V-(A\cap B))$. Partition these sets into appropriate pieces, to show that the inequality holds.)

(10 Marks)

Solution:

We will prove the inequality by proving that:

$$d^{out}(A) + d^{out}(B) = d^{out}(A\cap B) + d^{out}(A\cup B) + |E(A-B, B-A)| + |E(B-A, A-B)|.$$

The inequality then follows immediately. To prove this, we proceed as described in the hint, partitioning sets like $E(A, V-A)$ into a disjoint union of subsets. As a consequence, note that the following equalities hold (can visualize using "Venn diagrams, with arrows" for edges):

$$\begin{aligned}
d^{out}(A) &= = |E(A\cap B, V-A)| + |E(A-B, V-A)| \\
d^{out}(B) &= = |E(A\cap B, V-B)| + |E(B-A, V-B)| \\
d^{out}(A\cap B) &= |E(A\cap B, V-(A\cup B))| + |E(A\cap B, A-B)| + |E(A\cap B, B-A)| \\
d^{out}(A\cup B) &= |E(A\cap B, V-(A\cup B))| + |E(A-B, V-(A\cup B))| + |E(B-A, V-(A\cup B))|
\end{aligned}$$

Note furthermore that $V-A-B = V-B-A = V-(A\cup B)$, and hence that:

$$\begin{aligned}
|E(A\cap B, V-A)| &= |E(A\cap B, V-(A\cup B))| + |E(A\cap B, B-A)| \\
|E(A\cap B, V-B)| &= |E(A\cap B, V-(A\cup B))| + |E(A\cap B, A-B)|
\end{aligned}$$

Likewise, we have:

$$\begin{aligned}
|E(A-B, V-A)| &= |E(A-B, V-(A\cup B))| + |E(A-B, B-A)| \\
|E(B-A, V-B)| &= |E(B-A, V-(A\cup B))| + |E(B-A, A-B)|
\end{aligned}$$

Hence, we have:

$$\begin{aligned}
d^{out}(A) + d^{out}(B) &= |E(A\cap B, V-A)| + |E(A-B, V-A)| + \\
&\quad |E(A\cap B, V-B)| + |E(B-A, V-B)| \\
&= |E(A\cap B, V-(A\cup B))| + |E(A\cap B, (B-A))| + \\
&\quad |E(A-B, V-(A\cup B)| + |E(A-B, B-A)| + \\
&\quad |E(A\cap B, V-(A\cup B)| + |E(A\cap B, A-B)| + \\
&\quad |E(B-A, V-(A\cup B))| + |E(B-A, A-B)| \\
&= d^{out}(A\cap B) + d^{out}(A\cup B) + |E(A-B, B-A)| + |E(B-A, A-B)|
\end{aligned}$$

This completes the proof of the claim. □

4. A bag contains 12 balls of the same shape and size. Of these, 9 balls are blue, and the remaining 3 balls are red.

   Suppose that you do the following iterative random experiment:

   In each iteration, 5 balls are removed randomly (without replacement) from the bag, in such a way that any 5 balls originally in the bag is equally likely to be the 5 balls that are removed from the bag.

   After doing this, you check whether among the 5 removed balls there are exactly 2 red balls. If so, then you STOP. Otherwise, you replace the 5 balls back into the bag, shake the bag up (to make sure it is randomly mixed again), and repeat the same experiment: random sample 5 balls from the bag, and check whether you have taken out exactly 2 red balls. You repeat this until the process STOPs (i.e., when the 5 removed balls in some iteration contain exactly 2 red balls among them). What is the expected number of times that you will sample 5 balls from this bag, in the above random experiment? Explain your calculation. (10 Marks)

Solution:

This expected value is

$$\frac{\binom{12}{5}}{\binom{3}{2}\binom{9}{3}}.$$

To see why, note that the probability of removing exactly 2 red balls, when 5 balls are removed from the bag containing 12 balls, 3 of which are red, is

$$p = \frac{\binom{3}{2}\binom{9}{3}}{\binom{12}{5}}.$$

This is because there are $\binom{12}{5}$ ways to choose 5 balls out of the bag (the denominator), and of these there are $\binom{3}{2}\binom{9}{3}$ ways of choosing exactly 2 red balls (and hence necessarily 3 blue balls.

Now, the random experiment that is described essentially repeats this same experiment until we get "success", i.e., until we draw exactly 2 red balls. This defines a geometrically distributed random variable with parameter p, and we learned in class that the expected value of such a random variable is $1/p$. Hence the expected value is $\frac{\binom{12}{5}}{\binom{3}{2}\binom{9}{3}}$.

$\square$

5. For a positive integer $n \geq 1$, let $\pi : [2n] \to [2n]$ denote some permutation of the set $[2n] = \{1, 2, \ldots, 2n\}$. In other words, π is a bijection from $[2n]$ to itself. For such a permuation, π, let $b_\pi = |\{i \in [2n] \mid \pi(i) > 2i\}|$ denote the number of indices, $i \in [2n]$, such that $\pi(i) > 2i$.

   Suppose that the permutation π is chosen uniformly at random from the set of all permutations of the set $[2n]$, meaning that each permutation of $[2n]$ is equally likely (has the same probability) to be chosen. What is the expected value of b_π? Explain your calculation. (10 Marks)

   Solution:
   For a randomly chosen permutation π, and for each $i \in [2n]$, let X_i be a random variable that is equal to 1 if $\pi(i) > 2i$, and is 0 otherwise. Note that if we let $X = \sum_{i=1}^{2n} X_i$, then $X = b_\pi$ is the number of indices i for which $\pi(i) > 2i$. We are interested in the expected value $E(X)$. By linearity of expectation, $E(X) = \sum_{i=1}^{2n} E(X_i)$.

   But what is $E(X_i)$? It is simply the probability that, for a randomly chosen permutation, π, we have $\pi(i) > 2i$. Note that for $i \geq n$, $E(X_i) = 0$, because we can not have $\pi(i) > 2i \geq 2n$. So, let us focus on $i \in [n-1]$; since π is chosen uniformly at random from all permutations, it is not hard to see that the probability $E(X_i)$ is $\frac{2n-2i}{2n} = \frac{n-i}{n}$. Thus

$$\begin{aligned} E(X) &= \sum_{i=1}^{2n} E(X_i) \\ &= \sum_{i=1}^{n-1} E(X_i) \\ &= \sum_{i=1}^{n-1} \frac{n-i}{n} \\ &= \sum_{j=1}^{n-1} \frac{j}{n} \qquad \text{(by letting } j = n-i\text{)} \\ &= \frac{1}{n} \cdot \sum_{j=1}^{n-1} j \\ &= \frac{1}{n} \cdot \frac{(n-1)n}{2} \\ &= \frac{n-1}{2} \end{aligned}$$

$\square$