DMMR Coursework 2

November 7, 2019

1. For any integer $n \geq 1$, let r_n denote the number of different ways that the set $[n] := \{1, \ldots, n\}$ can be partitioned into disjoint non-empty subsets, the union of which is the entire set $[n]$.
   Let us define r_n more formally, in case you are unsure what this means. For a set C, let $\mathsf{Pow}(C)$ denote the power set of C, i.e., the set of all subsets of C. By definition, r_n is:

   $$r_n = |\{S \subseteq \mathsf{Pow}(\{1, \ldots, n\}) \mid \forall A \in S, A \neq \emptyset; \; \forall A, A' \in S, \text{ if } A \neq A' \text{ then } A \cap A' = \emptyset; \; \& \; (\bigcup_{A \in S} A) = \{1, \ldots, n\}\}|.$$

   For example $r_3 = 5$. This is because the set $\{1, 2, 3\}$ can be partitioned in precisely the following 5 distinct ways: $\{\{1\}, \{2\}, \{3\}\}$; $\{\{1\}, \{2, 3\}\}$; $\{\{2\}, \{1, 3\}\}$; $\{\{3\}, \{1, 2\}\}$; & $\{\{1, 2, 3\}\}$.

   Prove that, for all *even* integers $n \geq 2$,

   $$(n/2)^{(n/2)} \;\leq\; r_n \;\leq\; n^n.$$

   (Hint: for establishing the lower bound, suppose you only count those partitions in which each of the numbers $1, 2, \ldots, n/2$ is in a separate part of the partition.) (10 Marks)

2. Prove the following identity on binomial coefficients: for all integers $n \geq 1$,

   $$\sum_{k=1}^{n} k \cdot \binom{n}{k} = n \cdot 2^{n-1}.$$

   (You can prove this either combinatorially, or by using and manipulating formulas that define binomial coefficients.) (10 Marks)

3. Let $G = (V, E)$ be a *directed* graph (digraph). For any subsets $X, Y \subseteq V$ of vertices, let $E(X, Y) := \{(u, v) \in E \mid u \in X \;\&\; v \in Y\}$ denote the set of edges going from a vertex in X to a vertex in Y. For a subset $A \subseteq V$ of vertices, let $d^{out}(A) = |E(A, V - A)|$ denote the total number of edges exiting A, and going to a vertex outside of A. Note that, by definition, $d^{out}(\emptyset) = 0$.

   Prove that for any directed graph $G = (V, E)$, and for all subsets $A, B \subseteq V$ of its vertices, the following inequality holds:

   $$d^{out}(A) + d^{out}(B) \geq d^{out}(A \cap B) + d^{out}(A \cup B).$$

   (Hint: Consider the sets $E(A, V - A)$, $E(B, V - B)$, $E(A \cup B, V - (A \cup B))$, and $E(A \cap B, V - (A \cap B))$. Partition these sets into appropriate pieces, to show that the inequality holds.)
   (10 Marks)

4. A bag contains 12 balls of the same shape and size. Of these, 9 balls are blue, and the remaining 3 balls are red.

   Suppose that you do the following iterative random experiment: In each iteration, 5 balls are removed randomly (without replacement) from the bag, in such a way that any 5 balls in the bag are equally likely to be the 5 balls that are removed.

   After doing this, you check whether among the 5 removed balls there are *exactly* 2 red balls. If so, then you STOP. Otherwise, you replace the 5 balls back into the bag, shake the bag up (to make sure it is randomly mixed again), and repeat the same experiment: random sample 5 balls from the bag, and check whether you have taken out exactly 2 red balls. You repeat this until the process STOPs (i.e., when the 5 removed balls in some iteration contain exactly 2 red balls among them).

   What is the expected number of times that you will sample 5 balls from this bag, in the above random experiment? Explain your calculation. (10 Marks)

5. For an integer $n \geq 1$, let $\pi : [2n] \to [2n]$ denote some permutation of the set $[2n] = \{1, 2, \ldots, 2n\}$. In other words, π is a bijection from $[2n]$ to itself. For such a permuation, π, let $b_\pi = |\{i \in [2n] \mid \pi(i) > 2i\}|$ denote the number of indices, $i \in [2n]$, such that $\pi(i) > 2i$.

   Suppose that the permutation π is chosen uniformly at random from the set of all permutations of the set $[2n]$, meaning that each permutation of $[2n]$ is equally likely (has the same probability) to be chosen. What is the expected value of b_π? Explain your calculation. (10 Marks)

Solutions should be handed in to the ITO before 3pm on Friday, 22nd of November. Don't forget to write your student number clearly on your solution sheet. No other method of submission (such as by email) will be accepted.