

Asymmetric ciphers

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Introduction

So far: how two users can protect data using a shared secret key

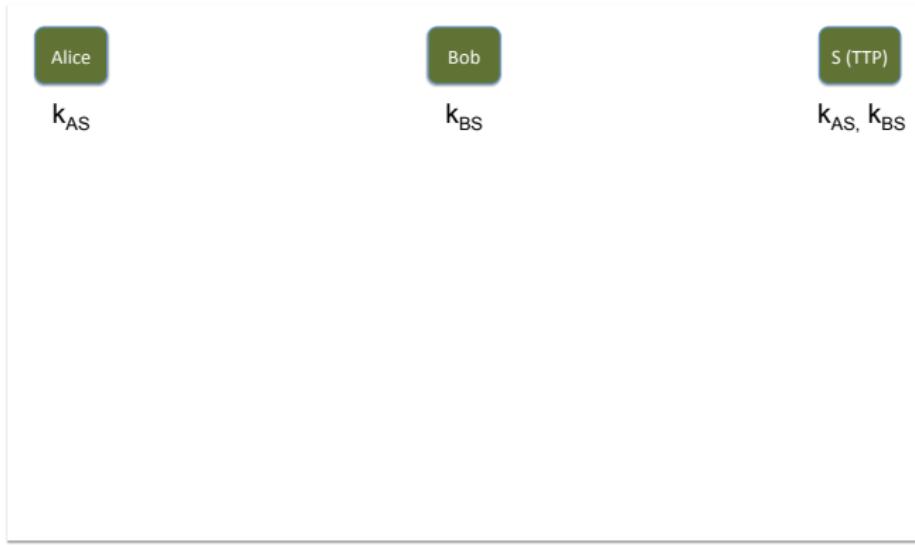
- ▶ One shared secret key per pair of users that want to communicate

Our goal now: how to establish a shared secret key to begin with?

- ▶ Trusted Third Party (TTP)
- ▶ Diffie-Hellman (DH) protocol
- ▶ RSA
- ▶ ElGamal (EG)

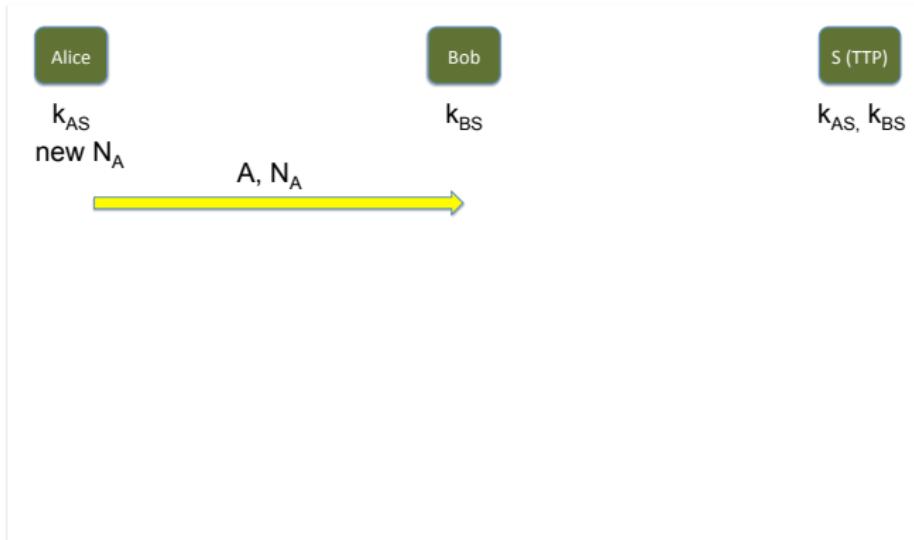
Online Trusted Third Party (TTP)

- ▶ Users $U_1, U_2, U_3, \dots, U_n, \dots$
- ▶ Each user U_i has a shared secret key K_i with the TTP
- ▶ U_i and U_j can establish a key $K_{i,j}$ with the help of the TTP
ex: using Paulson's variant of the Yahalom protocol



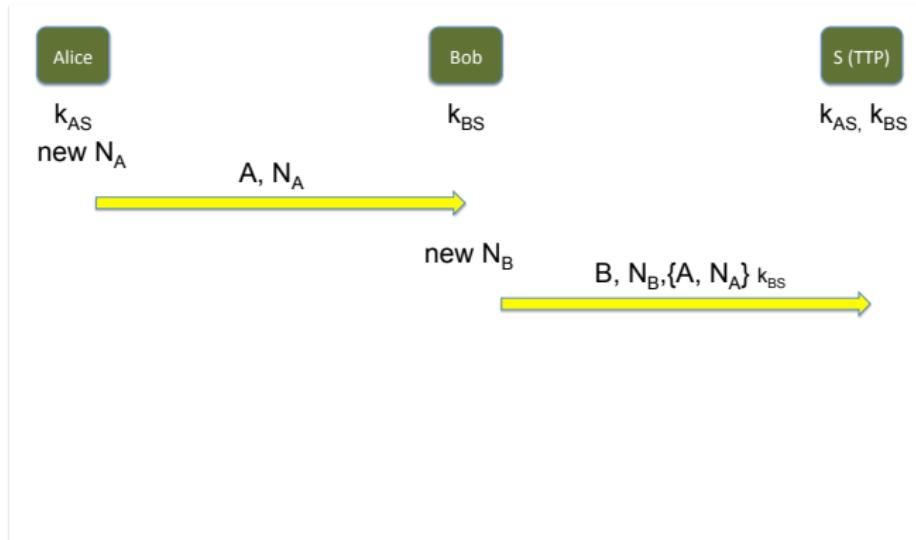
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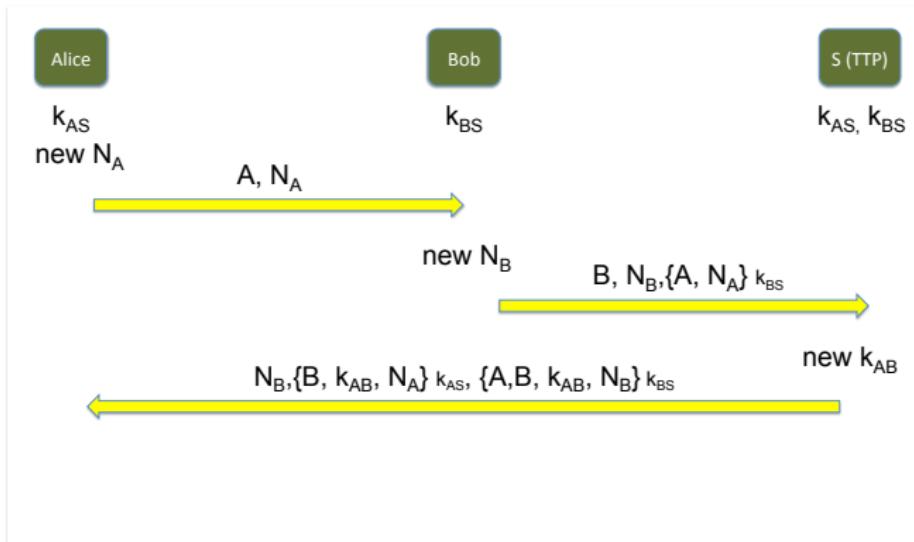
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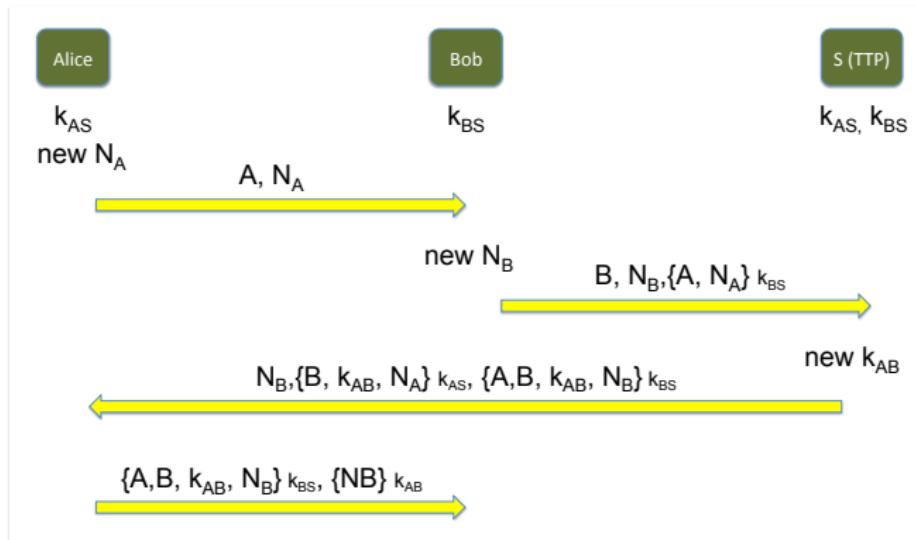
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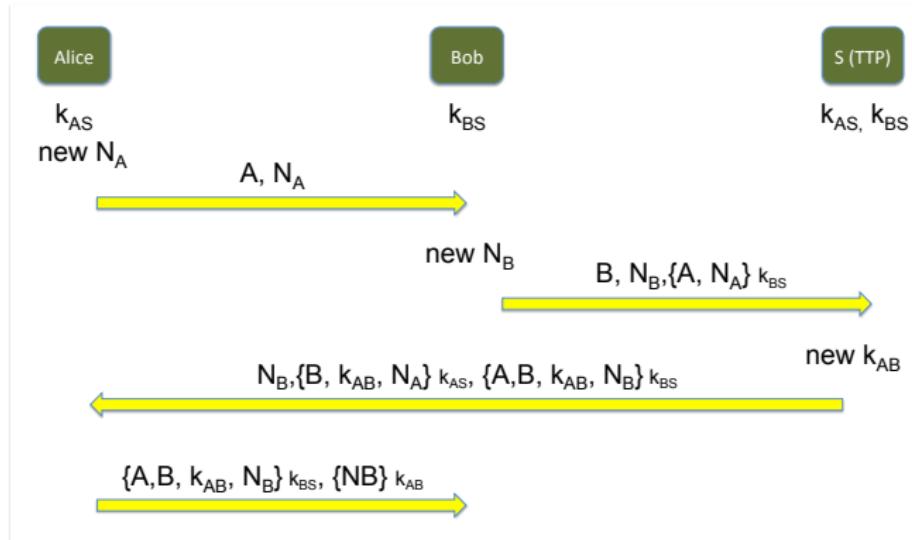
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Question: can we establish a shared secret key without a TTP?

Answer: Yes!

Public-key encryption in pictures



Alice

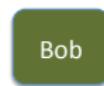
Bob



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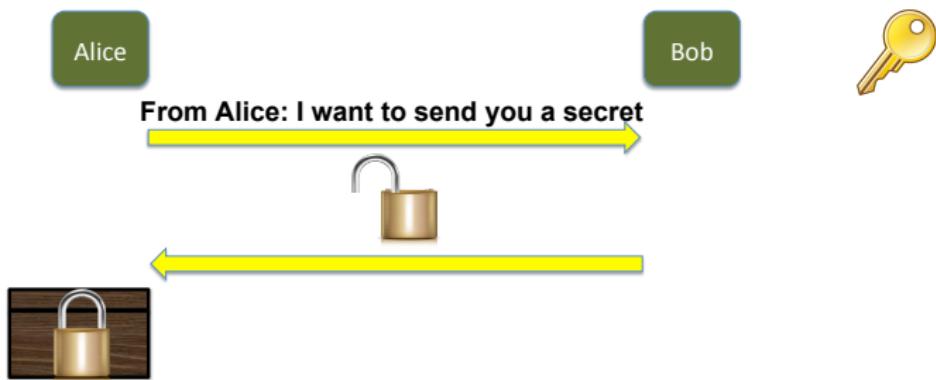
From Alice: I want to send you a secret



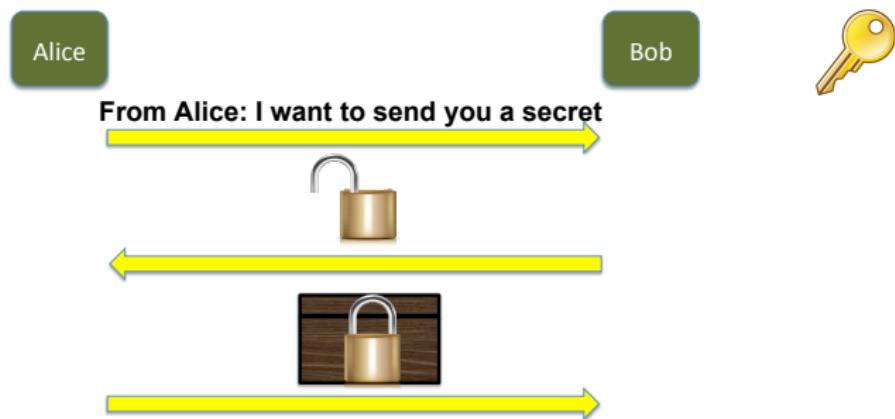
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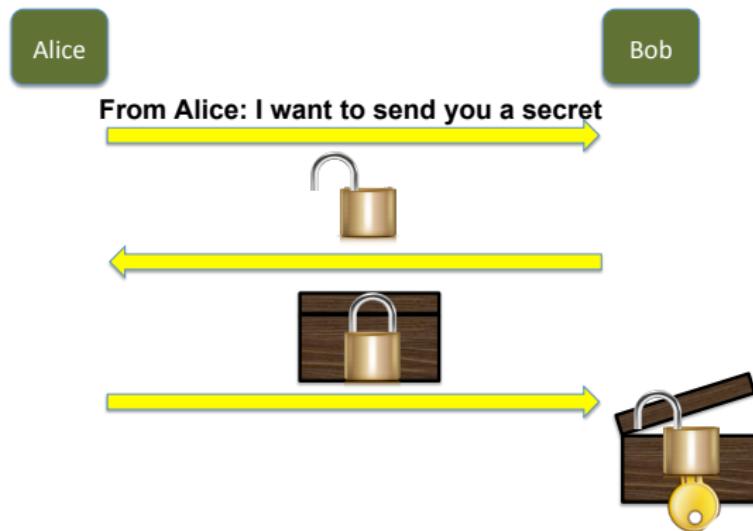
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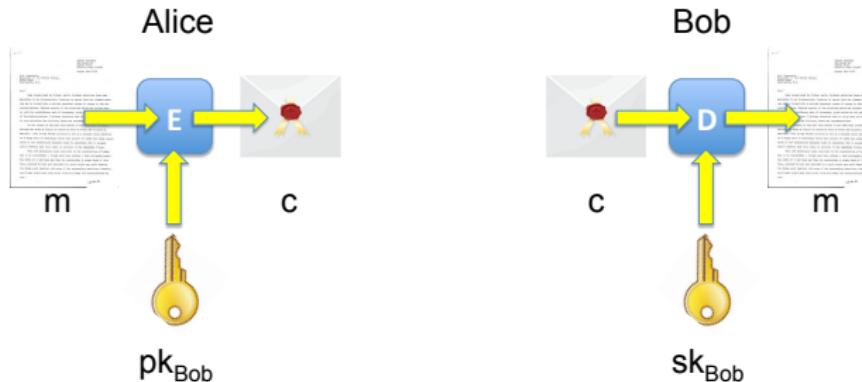


Public-key encryption in pictures



Public-key encryption

- ▶ key generation algorithm: $G : \mathcal{K} \times \mathcal{K}$
encryption algorithm $E : \mathcal{K} \times \mathcal{M} \rightarrow \mathcal{C}$
decryption algorithm $D : \mathcal{K} \times \mathcal{C} \rightarrow \mathcal{M}$
st. $\forall (sk, pk) \in G$, and $\forall m \in \mathcal{M}$, $D(sk, E(pk, m)) = m$



- ▶ the decryption key sk_{Bob} is secret (only known to Bob). The encryption key pk_{Bob} is known to everyone. And $sk_{Bob} \neq pk_{Bob}$

We need a bit of number theory now

Primes

Definition

$p \in \mathbb{N}$ is a **prime** if its only divisors are 1 and p

Ex: 2, 3, 5, 7, 11, 13, 17, 19, 23, 29

Theorem

*Every $n \in \mathbb{N}$ has a **unique factorization** as a product of prime numbers (which are called its factors)*

Ex: $23244 = 2 \times 2 \times 3 \times 13 \times 149$

Relative primes

Definition

a and b in \mathbb{Z} are **relative primes** if they have no common factors

Definition

The Euler function $\phi(n)$ is the number of elements that are relative primes with n :

$$\phi(n) = |\{m \mid 0 < m < n \text{ and } \gcd(m, n) = 1\}|$$

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- ▶ For p and q primes: $\phi(p \cdot q) = (p-1)(q-1)$

- ▶ Let $n \in \mathbb{N}$. We define $\mathbb{Z}_n = \{0, \dots, n-1\}$

$$\forall a \in \mathbb{Z}, \forall b \in \mathbb{Z}_n, a \equiv b \pmod{n} \Leftrightarrow \exists k \in \mathbb{N}. a = b + k \cdot n$$

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Theorem

Let $n \in \mathbb{N}$. Let $x \in \mathbb{Z}_n$. x has a inverse in \mathbb{Z}_n iff $\gcd(x, n) = 1$

$$(\mathbb{Z}_N)^*$$

- ▶ Let $n \in \mathbb{N}$. We define $(\mathbb{Z}_n)^* = \{x \in \mathbb{Z}_n \mid \gcd(x, n) = 1\}$
- Ex: $\mathbb{Z}_{12} = \{1, 5, 7, 11\}$

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Theorem (Euler)

$\forall p$ prime, $(\mathbb{Z}_p)^*$ is a cyclic group, i.e.

$$\exists g \in (\mathbb{Z}_p)^*, \{g, g^2, g^3, \dots, g^{p-2}\} = (\mathbb{Z}_p)^*$$

Intractable problems

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input: $n \in \mathbb{N}$

output: p_1, \dots, p_m primes st. $n = p_1 \cdot \dots \cdot p_m$

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- ▶ DISCRETE LOG:

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- ▶ DISCRETE LOG:
 - input: prime p , generator g of $(\mathbb{Z}_p)^*$, g^x
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- ▶ DHP:
 - input: prime p , generator g of $(\mathbb{Z}_p)^*$, $g^a \pmod{p}$, $g^b \pmod{p}$
 - output: $g^{ab} \pmod{p}$

We can now go back and see how to establish a key without a TTP

The Diffie-Hellman (DH) protocol

- ▶ Assumption: the DHP is hard in $(\mathbb{Z}_p)^*$
- ▶ Fix a very large prime p , and $g \in \{1, \dots, p-1\}$

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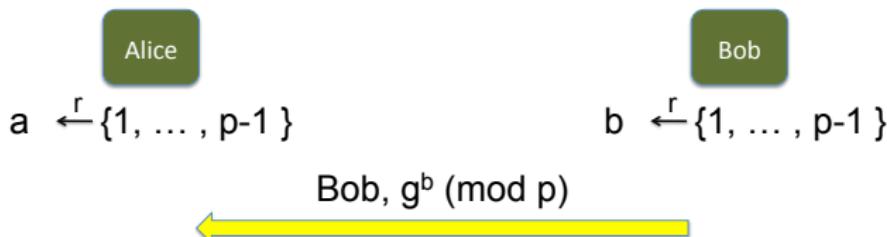
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Bob

$$b \xleftarrow{r} \{1, \dots, p-1\}$$

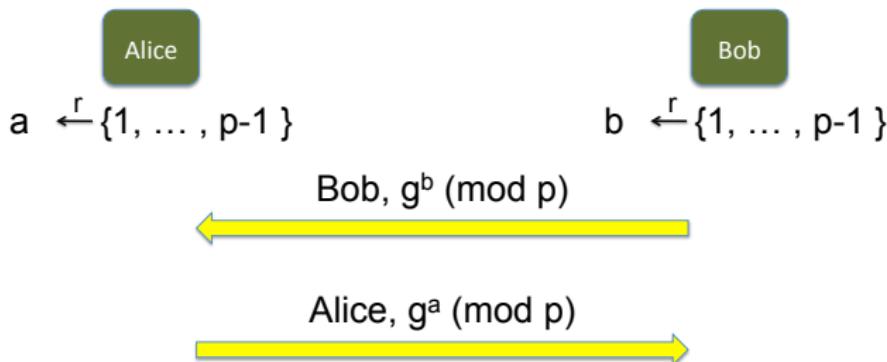
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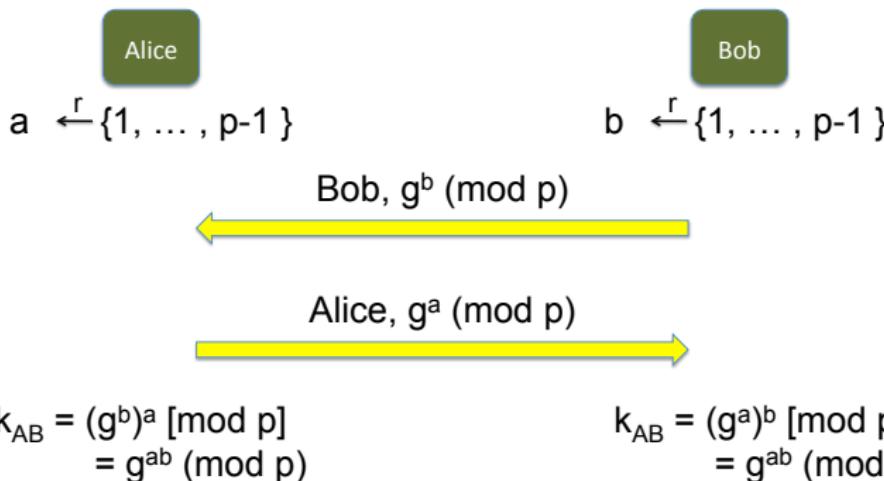
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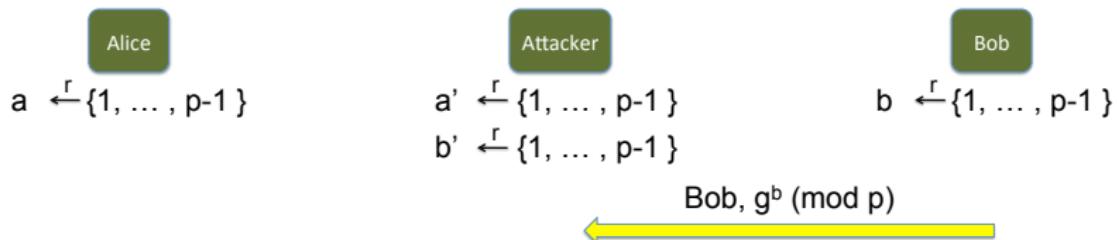
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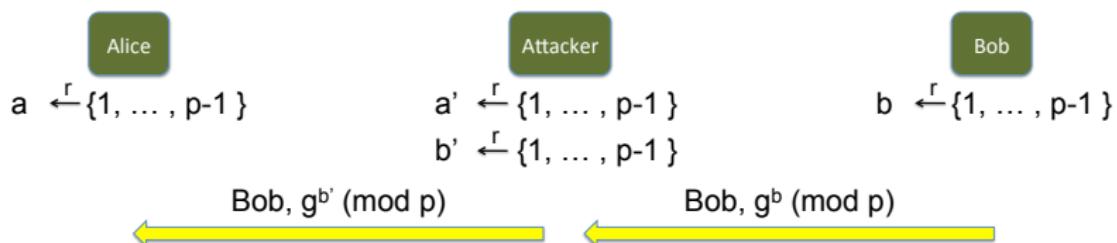
Man in the middle attack on DH

Alice	Attacker	Bob
$a \xleftarrow{r} \{1, \dots, p-1\}$	$a' \xleftarrow{r} \{1, \dots, p-1\}$ $b' \xleftarrow{r} \{1, \dots, p-1\}$	$b \xleftarrow{r} \{1, \dots, p-1\}$

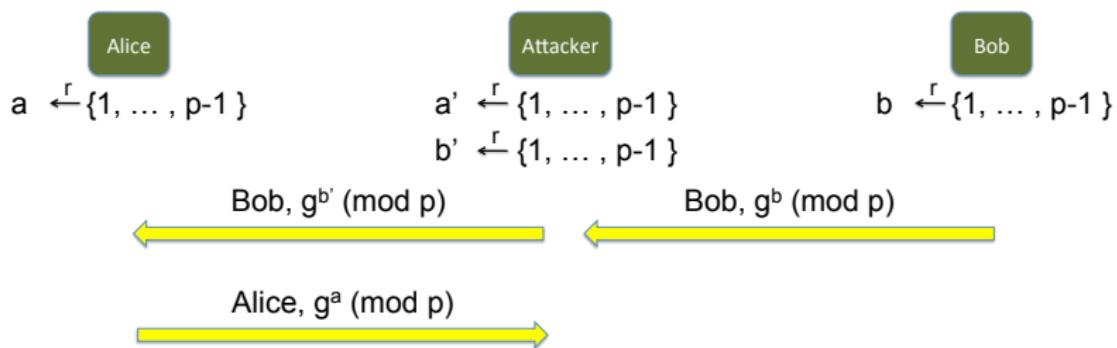
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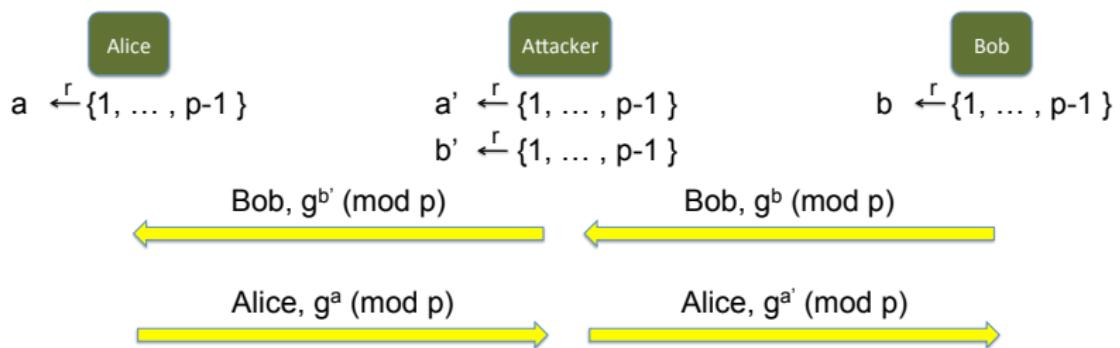
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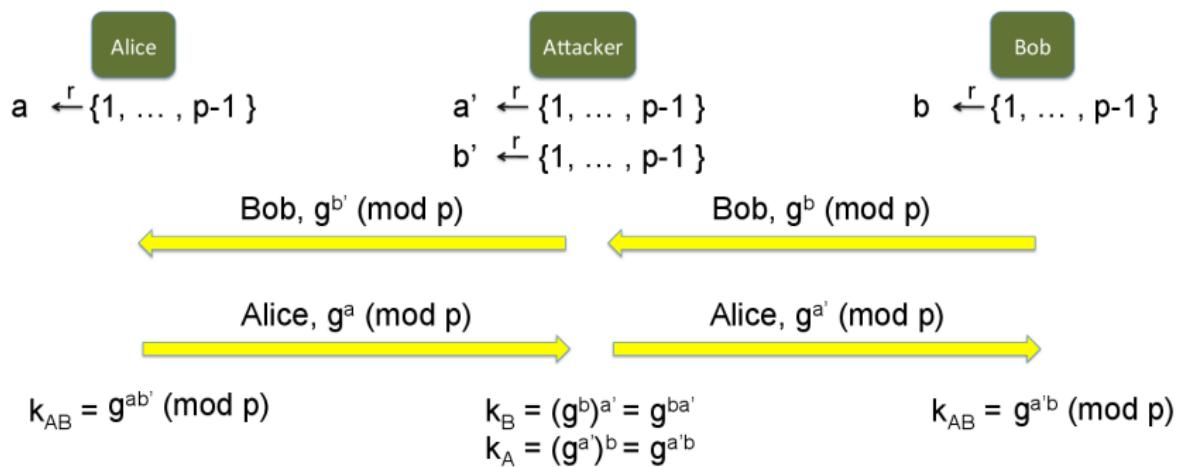
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RSA trapdoor permutation

- ▶ $G_{RSA}() = (pk, sk)$ where $pk = (N, e)$ and $sk = (N, d)$
and $N = p \cdot q$ with p, q random primes
and $e, d \in \mathbb{Z}$ st. $e \cdot d \equiv 1 \pmod{\phi(N)}$

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$$\begin{aligned} RSA^{-1}(sk, RSA(pk, x)) &= (x^e)^d \pmod{N} \\ &= x^{e \cdot d} \pmod{N} \\ &= x^{1+k\phi(N)} \pmod{N} \end{aligned}$$

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- ▶ $RSA(pk, x) = x^e \pmod{N}$ where $pk = (N, e)$
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- ▶ Consistency: $\forall (pk, sk) = G_{RSA}(), \forall x, RSA^{-1}(sk, RSA(pk, x)) = x$

Proof: Let $pk = (N, e)$, $sk = (N, d)$. and $x \in \mathbb{Z}_N$. Easy case where x and N are relatively prime

$$\begin{aligned} RSA^{-1}(sk, RSA(pk, x)) &= (x^e)^d \pmod{N} \\ &= x^{e \cdot d} \pmod{N} \\ &= x^{1+k\phi(N)} \pmod{N} \\ &= x \cdot x^{k\phi(N)} \pmod{N} \end{aligned}$$

RSA trapdoor permutation

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How **NOT** to use RSA

(G_{RSA}, RSA, RSA^{-1}) is called raw RSA. Do not use raw RSA directly as an asymmetric cipher!

RSA is deterministic \Rightarrow not secure against chosen plaintext attacks

(Details on the board)

ISO standard

Goal: build a CPA secure asymmetric cipher using (G_{RSA}, RSA, RSA^{-1})

Let (E_s, D_s) be a symmetric encryption scheme over $(\mathcal{M}, \mathcal{C}, \mathcal{K})$

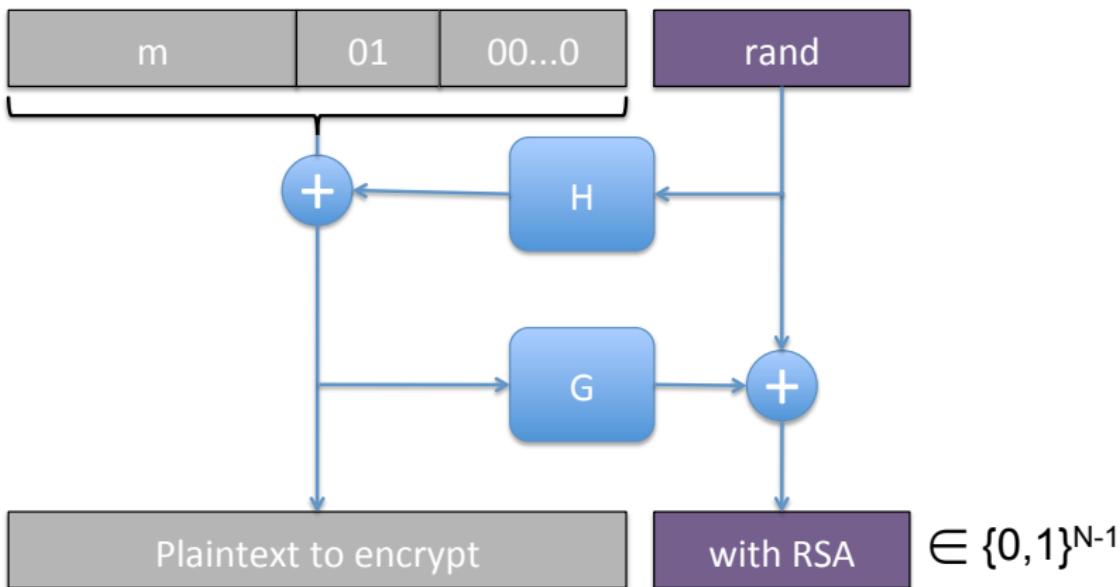
Let $H : (\mathbb{Z}_N)^* \rightarrow \mathcal{K}$

Build $(G_{RSA}, E_{RSA}, D_{RSA})$ as follows

- ▶ $G_{RSA}()$ as described above
- ▶ $E_{RSA}(pk, m)$:
 - ▶ pick random $x \in (\mathbb{Z}_N)^*$
 - ▶ $y \leftarrow RSA(pk, x) (= x^e)$
 - ▶ $k \leftarrow H(x)$
 - ▶ $E_{RSA}(pk, m) = y || E_s(k, m)$
- ▶ $D_{RSA}(pk, y || c) = D_s(H(RSA^{-1}(sk, y)), c)$

PKCS1 v2.0: RSA-OAEP

Goal: build a CPA secure asymmetric cipher using (G_{RSA}, RSA, RSA^{-1})



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