Categories and Quantum Informatics: Monoids and comonoids

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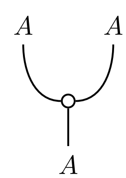
The tensor product of a monoidal category allows us to consider multiplications on its objects, leading to the notion of a monoid. In fact, this notion is so important, that one can almost say the entire reason for defining monoidal categories is that one can define monoids in them. We investigate such structures in Section 4.1, and their relation to dual objects. We also consider comonoids, whose operation is something like copying. Classical information can be copied and deleted, whereas quantum information cannot. This leads to big differences between classical and quantum information; we think of a classical system as a quantum one equipped with special morphisms that copy and delete the information it carries. We prove categorical no-deleting and no-cloning theorems in Sections 4.1 and 4.2, showing that if these structures are able to copy and delete every state of the system, then the category collapses. Finally, we characterize when a tensor product is a categorical product in Section 4.3.

4.1 Monoids and comonoids

Let's start by making the notions of copying and deleting more precise in our setting of monoidal categories.

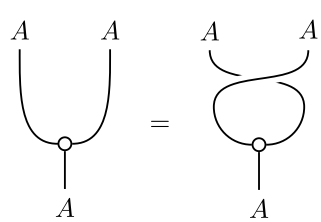
Comonoids

Clearly, copying should be an operation of type $A \xrightarrow{d} A \otimes A$. As we will be using this morphism a lot, we will draw it as follows rather than with a generic box:



(4.1)

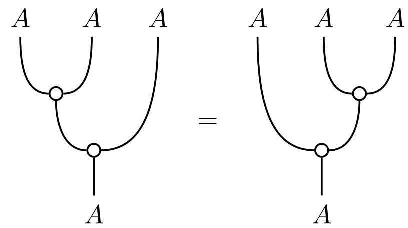
What does it mean that d copies information? First, it shouldn't matter if we switch both output copies, corresponding to the requirement that $d = \sigma_{A,A} \circ d$:



(4.2)

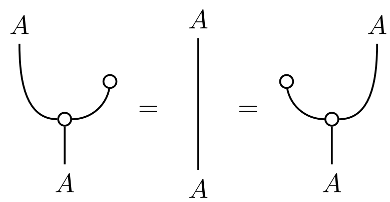
Note that it doesn't matter which braiding we choose here, because this equation is equivalent to the one in which we choose the other braiding.

Secondly, if we make a third copy, if shouldn't matter if we start from the first or the second copy. We can formulate this as $\alpha_{A,A,A} \circ (d \otimes \mathrm{id}_A) \circ d = (\mathrm{id}_A \otimes d) \circ d$, with the following graphical representation:



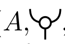
(4.3)

Finally, remember that we think of I as the empty system. So deletion should be an operation of type $A \xrightarrow{e} I$. With this in hand, we can formulate what it means that both output copies should equal the input: that $\rho_A \circ (\mathrm{id}_A \otimes e) \circ d = \mathrm{id}_A$ and $\mathrm{id}_A = \lambda_A \circ (e \otimes \mathrm{id}_A) \circ d$. Graphically:



(4.4)

These three properties together constitute the structure of a *cocommutative comonoid* on A.

Definition 4.1 (Comonoid)**.** A *comonoid* in a monoidal category is a triple (A,,) of an object A and morphisms : $A \longrightarrow A \otimes A$ and : $A \longrightarrow I$ satisfying equations (4.3) and (4.4). If the monoidal category is braided and equation (4.2) holds, the comonoid is called *cocommutative*.

The morphism  is called the *comultiplication*, and  is called the *counit*. Properties (4.3) and (4.4) are *coassociativity* and *counitality*.

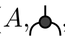
Example 4.2. Here are some comonoids in our example monoidal categories.

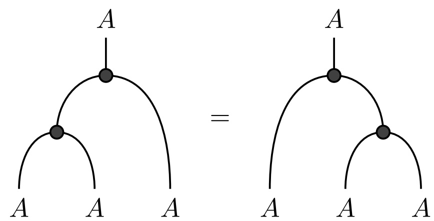
- In **Set**, the tensor product is in fact a Cartesian product. It therefore follows from counitality (4.4) that any object A carries a unique cocommutative comonoid structure with comultiplication $A \xrightarrow{d} A \times A$ given by $d(a) = (a, a)$, and the unique function $A \longrightarrow 1$ as counit.
- In **Rel**, any group G forms a comonoid with comultiplication $g \sim (h, h^{-1}g)$ for all $g, h \in G$, and counit $1 \sim \bullet$. To see counitality, for example, notice that the left-hand side of (4.4) is the relation $g \sim h$ where $h^{-1}g = 1$, and the right-hand side is $g \sim 1^{-1}g$; that is, both equal the identity $g \sim g$.

  The comonoid is cocommutative when the group is abelian. The left-hand side of (4.2) is $g \sim (h, h^{-1}g)$ for all $h \in G$, whereas the right-hand side is $g \sim (k, k^{-1}g)$ for all $k \in G$. But if $k = h^{-1}g$, then $k^{-1}g = g^{-1}hg = h$ when G is abelian, so that left and right-hand sides are equal.
- In **FHilb**, any choice of basis $\{e_i\}$ for a Hilbert space H provides it with cocommutative comonoid structure, with comultiplication $A \xrightarrow{d} A \otimes A$ defined by $e_i \mapsto e_i \otimes e_i$ and counit $A \xrightarrow{e} I$ defined by $e_i \mapsto 1$.

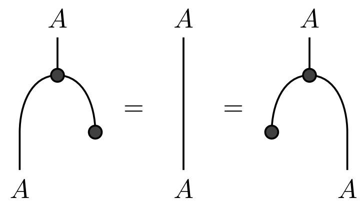
Monoids

Dualizing everything gives the better-known notion of a *monoid*.

Definition 4.3 (Monoid)**.** A *monoid* in a monoidal category is a triple (A, , ) of an object A, a morphism : $A \otimes A \to A$, and a state : $I \to A$, satisfying the following two equations called *associativity* and *unitality*:

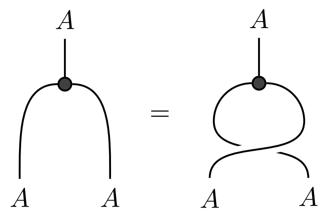


(4.5)



(4.6)

In a braided monoidal category, a monoid is *commutative* when the following equation holds:



(4.7)

Again, the choice of braid is arbitrary here: this condition is equivalent to the one using the inverse braiding.

Example 4.4. There are many examples of monoids:

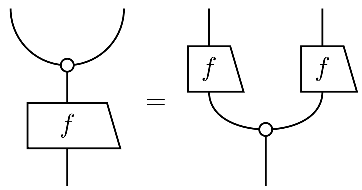
- The tensor unit I in any monoidal category can be equipped with the structure of a monoid, with $m = \rho_I\ (= \lambda_I)$ and $u = \mathrm{id}_I$.
- A monoid in **Set** gives the ordinary mathematical notion of a monoid. Any group is an example.
- A monoid in **Vect** is called an *algebra*. The multiplication is a linear function $A \otimes A \xrightarrow{m} A$, corresponding to a bilinear function $A \times A \to A$. Hence an algebra is a set where we can not only add vectors and multiply vectors with scalars, but also multiply vectors with each other in a bilinear way. For example, $\mathbb{C}^n$ forms an algebra under pointwise multiplication; the unit is the vector $(1, 1, \ldots, 1)$. For another example, the vector space of complex n-by-n matrices $\mathbb{M}_n$ forms an algebra under matrix multiplication.

We have used a black dot for the comonoid structures and a white dot for the monoid structures, but that is not essential: we will just make sure to use different colours to differentiate structures as the need arises. Later on we will use monoids and comonoids for which the multiplication is the adjoint of the comultiplication, and the unit is the adjoint of the counit, and in that case we will use the same colour dots for all of these structures.

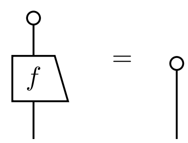
Combining monoids

Given a monoidal category, we can build a new category whose objects are comonoids, with the following morphisms.

Definition 4.5. A *comonoid homomorphism* from a comonoid (A, d, e) to a comonoid (A', d', e') is a morphism $A \xrightarrow{f} A'$ such that $(f \otimes f) \circ d = d' \circ f$ and $e' \circ f = e$. These equations have the following graphical representations:



(4.8)



(4.9)

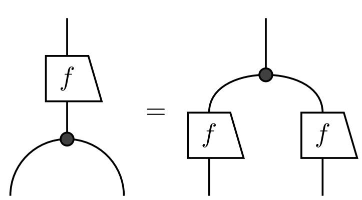
The visual impression is that the morphism f is copied by d', and deleted by e'. Comonoid homomorphisms compose associatively, and the identity morphism is always a comonoid homomorphism, so comonoids and comonoid homomorphisms form a valid category.

Example 4.6. Consider again the comonoids of Example 4.2.

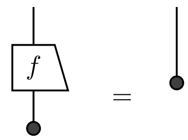
- In **Set**, any function $f\colon A \to B$ is a comonoid homomorphism: by definition $(f \times f)(a, a) = \big(f(a), f(a)\big)$, and $A \xrightarrow{f} B \to I$ equals the unique function $A \to I$.
- In **Rel**, any surjective homomorphism $f\colon G \to H$ of groups is a comonoid homomorphism. The left-hand side of (4.8) is the relation $g \sim (h, h^{-1}f(g))$ for $h \in H$, and the right-hand side is $g \sim (f(g'), f(g')^{-1}f(g))$. Since f is surjective, any $h \in H$ is of the form $f(g')$ for some $g' \in G$, making both sides equal. Similarly, both sides of (4.9) come down to the relation $1 \sim f(1) = 1$.
- In **FHilb**, any function $f\colon \{d_i\} \to \{e_j\}$ between bases extends linearly to a comonoid homomorphism between the Hilbert spaces they span. Almost by definition $d(f(d_i)) = f(d_i) \otimes f(d_i)$ and $e(f(d_j)) = 1 = e(d_j)$.

We can define a monoid homomorphism in a similar way.

Definition 4.7 (Monoid homomorphism)**.** In a monoidal category, a *monoid homomorphism* from a monoid (A, m, u) to a monoid (A', m', u') is a morphism $A \xrightarrow{f} A'$ such that $f \circ m = m' \circ (f \otimes f)$ and $u' = f \circ u$. These equations have the following graphical representations:



(4.10)



(4.11)

Again we can use this notion to form a category, whose objects are monoids and whose morphisms are monoid homomorphisms.

In a braided monoidal category we can combine two comonoids to give a single comonoid on the tensor product object, as the following lemma shows.

Lemma 4.8 (Product comonoid)**.** *In a braided monoidal category, given a pair of comonoids, we can produce a new comonoid with the following comultiplication and counit:*

(4.12)

Proof. The two comonoid structures are just sitting on top of each other, and the coassociativity and counitality properties of the original comonoids are inherited by the new composite structure. □

In the case that the braiding is a symmetry, this gives the actual categorical product of comonoids in the category of cocommutative comonoids and comonoid homomorphisms.

We can form the product of two monoids in a very similar way.

Example 4.9. Products of the comonoids of Example 4.2 are as follows.

- The product comonoid on sets A and B in **Set** is simply the unique comonoid on $A \times B$.
- The product comonoid of groups G and H in **Rel** is the comonoid of the product group $G \times H$ with multiplication $(g_1, h_1)(g_2, h_2) = (g_1 g_2, h_1 h_2)$.
- The product of comonoids on Hilbert spaces H and K in **FHilb** that copy orthonormal bases $\{d_i\}$ and $\{e_j\}$ is the comonoid that copies the orthonormal basis $\{d_i \otimes e_j\}$ of $H \otimes K$.

In a monoidal dagger category, there is a duality between monoids and comonoids.

Lemma 4.10. *If (A, d, e) is a comonoid in a monoidal dagger category, then $(A, d^\dagger, e^\dagger)$ is a monoid.*

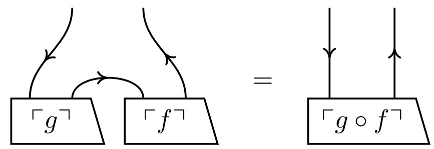
Proof. Equations (4.5) and (4.6) are just (4.3) and (4.4) vertically reflected. □

The previous lemma shows that Examples 4.2 and 4.4 are related by taking daggers in **Rel**. Taking daggers in **Rel** constructs converse relations, and applying this to Example 4.2 turns the comultiplication $G \xrightarrow{d} G \times G$ given by $g \sim (h, h^{-1}g)$ for a group G into the multiplication $G \times G \xrightarrow{m} G$ given by $(g, h) \sim gh$.

Monoids of operators

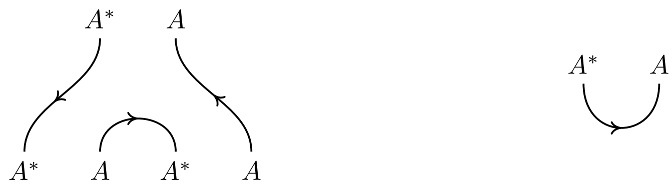
One of the most important features of matrices is that they can be multiplied. In other words, linear maps $\mathbb{C}^n \to \mathbb{C}^n$ can be composed. Using the closure properties of the previous subsection we can *internalize* this, to see that the vector space $\mathbb{M}_n$ of Example 4.4 is actually a monoid that lives in the same category as $\mathbb{C}^n$.

More generally, if an object A in a monoidal category has a dual A^*, then operators $A \xrightarrow{f} A$ correspond bijectively to states $I \xrightarrow{\ulcorner f \urcorner} A^* \otimes A$. Composition $A \xrightarrow{g \circ f} A$ of operators transfers to states $I \xrightarrow{\ulcorner g \circ f \urcorner} A^* \otimes A$:



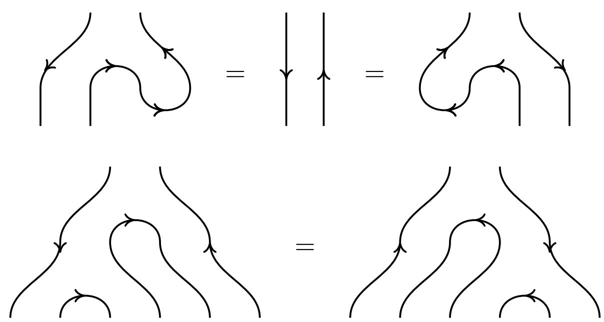
Thus the object $A^* \otimes A$ canonically becomes a monoid. We will call it the *pair of pants* monoid.

Lemma 4.11. *If $A \dashv A^*$ are dual objects in a monoidal category, then $A^* \otimes A$ is a monoid, with multiplication and unit defined as follows:*



(4.13)

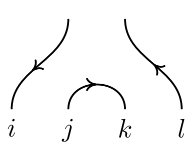
Proof. Straightforward graphical manipulation shows:



Hence this definition satisfies unitality and associativity. □

Example 4.12. The pair of pants algebra on the object $\mathbb{C}^n$ in the category **FHilb** is the algebra $\mathbb{M}_n$ of n-by-n matrices under matrix multiplication.

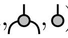
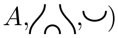
Proof. Fix an orthonormal basis $\{|i\rangle\}$ for $A = \mathbb{C}^n$, so that an orthonormal basis of $A^* \otimes A$ is given by $\{\langle j| \otimes |i\rangle\}$. Define a linear function $A^* \otimes A \longrightarrow \mathbb{M}_n$ by mapping $\langle j| \otimes |i\rangle$ to the matrix e_{ij}, which has a single entry 1 on row i and column j and zeroes elsewhere. This is clearly a bijection. Furthermore, it respects multiplication; using the decorated notation from Section 3.3:

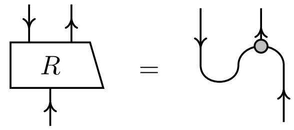


$$= \begin{bmatrix} \langle i| \otimes |l\rangle & \text{if } j = k, \\ 0 & \text{if } j \neq k \end{bmatrix} \quad \longmapsto \quad \begin{bmatrix} e_{il} & \text{if } j = k, \\ 0 & \text{if } j \neq k \end{bmatrix} = e_{ij} e_{kl}.$$

Similarly, it respects units, and is therefore a monoid homomorphism. □

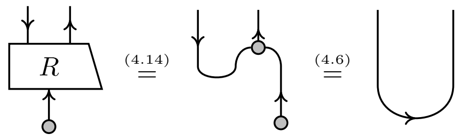
Pair of pants monoids are universal, in the sense that any monoid embeds into a pair of pants monoid.

Proposition 4.13. *In a monoidal category, for a monoid* (*A*,) *and a duality* $A \dashv A^*$*, there is a monoid homomorphism* $R : $ (*A*,) $\to$ ($A^* \otimes A$,) *with a retraction.*

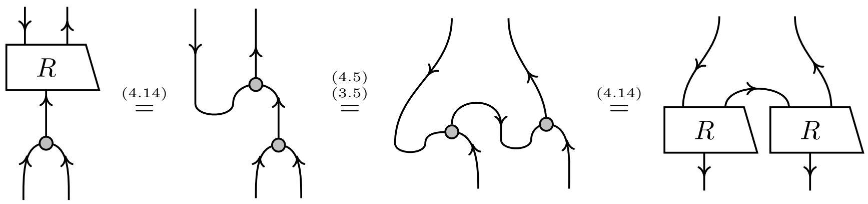


(4.14)

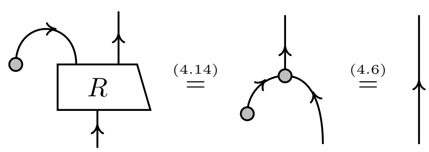
Proof. The morphism R preserves units:



It also preserves multiplication:



where the middle equation uses associativity and the snake equation. Finally, R has a left inverse:



This finishes the proof. □

4.2 Uniform deleting and copying

Uniform deleting

The counit $A \xrightarrow{e} I$ of a comonoid A tells us we can 'forget' about A if we want to. In other words, we can delete the information contained in A. It is perfectly possible to delete individual systems like this. The no-deleting theorem only prohibits a systematic way of deleting arbitrary systems.

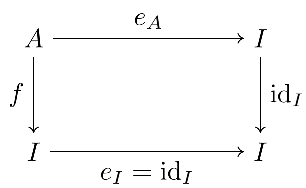
What happens when *every* object in our category can be deleted *systematically*? In our setting, deleting systematically means that the deleting operations respect the categorical structure. This means that deleting is *uniform*, in the sense that it doesn't matter if we delete something right away, or first process it for a while and then delete the result. In that case, we can say something quite dramatic. Let us first make uniform deleting precise.

Definition 4.14 (Uniform deleting)**.** A category has *uniform deleting* if there is a natural transformation $A \xrightarrow{e_A} I$ with $e_I = \mathrm{id}_I$.

Naturality of e_A here means that $e_B \circ f = e_A$ for any morphism $A \xrightarrow{f} B$. This is already strong enough to imply that any monoidal category whose tensor unit I is terminal, such as **Set**, has uniform deleting.

Proposition 4.15. *A category* **C** *has uniform deleting if and only if* I *is terminal.*

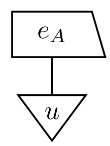
Proof. Uniform deleting gives a morphism $A \xrightarrow{e_A} I$ for each object A. Naturality and $e_I = \mathrm{id}_I$ then imply that any morphism $A \xrightarrow{f} I$ must equal e_A:



Conversely, if I is terminal, we can define $A \xrightarrow{e_A} I$ as the unique morphism of that type. This will automatically satisfy naturality as well as $e_I = \mathrm{id}_I$. □

To further justify calling the notion of Definition 4.14 deleting, we now observe that it deletes states.

Definition 4.16 (Deletable state)**.** A state $I \xrightarrow{u} A$ of an object A in a monoidal category with a deleting map $A \xrightarrow{e_A} I$ is *deletable* when:



$$= \tag{4.15}$$

Corollary 4.17. *Consider a monoidal category with maps $A \xrightarrow{e_A} I$ for each object A. If the maps e_A provide uniform deleting, then any state is deletable. The converse holds when the category is well-pointed.*

Proof. If there is uniform deleting, then $e_A \circ u = \mathrm{id}_I$ for each state $I \xrightarrow{u} A$ by Proposition 4.15.

Now suppose that the category is well-pointed, and let $A \xrightarrow{f} I$ and $A \xrightarrow{g} I$ be morphisms. By Proposition 4.15 it suffices to show that $f \circ u = g \circ u$ for any state $I \xrightarrow{u} A$. Both are states of I, so $e_I \circ f \circ u = \mathrm{id}_I = e_I \circ g \circ u$. That is, both scalars $f \circ u$ and $g \circ u$ are inverse to the scalar e_I, and hence must be equal: $f \circ u = g \circ u \circ e_I \circ f \circ u = g \circ u$. □

The no-deleting theorem below will show that uniform deleting has significant effects in a compact category. Namely, the category must collapse, in the following sense.

Definition 4.18 (Preorder)**.** A *preorder* is a category that has at most one morphism $A \longrightarrow B$ for any pair of objects A, B.

From our viewpoint, preorders are degenerate categories; they are uninteresting, as there is only one way to process a system – there is no dynamics.

Theorem 4.19 (No deleting)**.** *If a compact category has uniform deleting, then it must be a preorder.*

Proof. By Proposition 4.15, the tensor unit I is terminal. So any two parallel morphisms $A \xrightarrow{f,g} B$ must have the same coname $\llcorner f \lrcorner = \llcorner g \lrcorner$, whence $f = g$. □

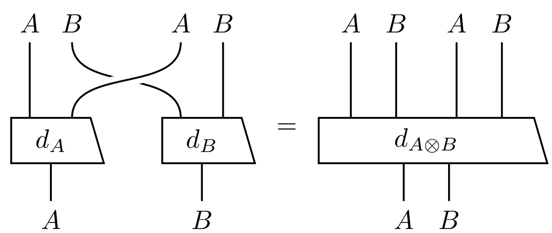
Uniform copying

We now move to uniform copying. The comultiplication $A \xrightarrow{d} A \otimes A$ of a comonoid lets us copy the information contained in one object A. What happens if we have this ability for all objects, systematically? In this section we will prove a categorical no-cloning theorem, showing that compact categories with uniform copying must degenerate.

Uniform deleting meant deleting something straight away is the same as processing it for a while first and then deleting the result. We want a similar definition to say that a copying procedure is uniform. It shouldn't matter whether we copy something first and then process both copies, or process the original first and then copy the result. This amounts to naturality of the comultiplication: it must respect composition.

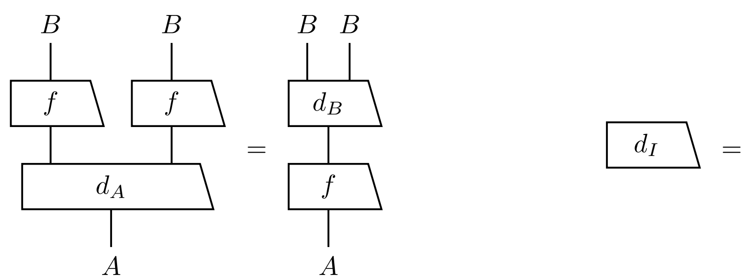
Moreover, we want these copying maps to respect the tensor product: copying a compound object should be the same as copying both constituents. The following definition makes this precise, using Lemma 4.8 for compound objects.

Definition 4.20 (Uniform copying)**.** A braided monoidal category has *uniform copying* if there is a natural transformation $A \xrightarrow{d_A} A \otimes A$ with $d_I = \rho_I^{-1}$, satisfying equations (4.2) and (4.3), and making the following diagram commute for all objects A, B.



(4.16)

Naturality and $d_I = \rho_I^{-1}$ graphically look like this for arbitrary $A \xrightarrow{f} B$:



(4.17)

Example 4.21. The monoidal category **Set** has uniform copying. The copying maps $A \xrightarrow{d_A} A \times A$ given by $a \mapsto (a, a)$ fit the bill: $d_1(\bullet) = (\bullet, \bullet) = \rho_1(\bullet)$, and both sides of (4.16) are the function $A \times B \longrightarrow A \times B \times A \times B$ given by $(a, b) \mapsto (a, b, a, b)$.

Here are some more examples.

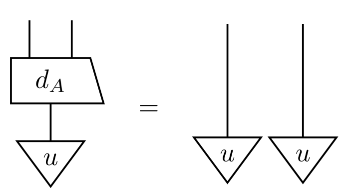
Definition 4.22 (Discrete category, indiscrete category)**.** A category is *discrete* when the only morphisms are identities. A category is *indiscrete* when there is a unique morphism $A \longrightarrow B$ for each two objects A and B. Such categories are completely determined by their set of objects.

Notice that discrete and indiscrete categories are automatically dagger categories, and in fact groupoids.

Example 4.23. A braided monoidal category that is discrete generally cannot have uniform copying: unless $A \otimes A = A$, there is no morphism $A \longrightarrow A \otimes A$ whatsoever. A braided monoidal category that is indiscrete always has uniform copying: any equation between well-typed morphisms holds because there is only a single morphism of that type, and the copying map d_A can simply be taken to be the unique morphism $A \longrightarrow A \otimes A$.

To justify calling the notion of Definition 4.20 copying, we now observe that it actually copies states.

Definition 4.24 (Copyable state)**.** A state $I \xrightarrow{u} A$ of an object A in a braided monoidal category with a copying map $A \xrightarrow{d_A} A \otimes A$ is *copyable* when:

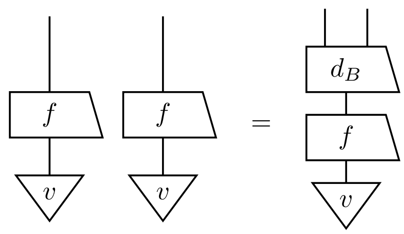


(4.18)

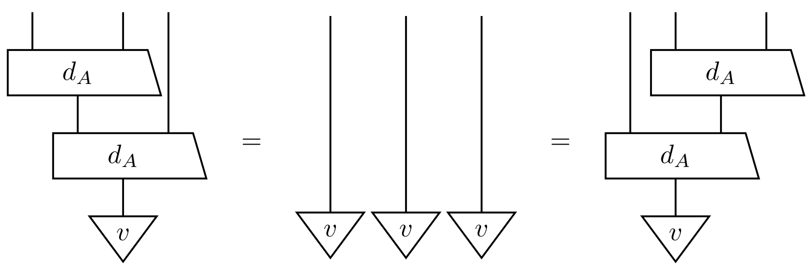
Proposition 4.25. *Consider a braided monoidal category equipped with maps $A \xrightarrow{d_A} A \otimes A$ for each object A. If the maps d_A provide uniform copying, then any state is copyable. The converse holds when the category is monoidally well-pointed.*

Proof. If there is uniform copying, then, by naturality of the copying maps, we have $d_A \circ u = (u \otimes u) \circ \rho_I^{-1}$ for each state $I \xrightarrow{u} A$.

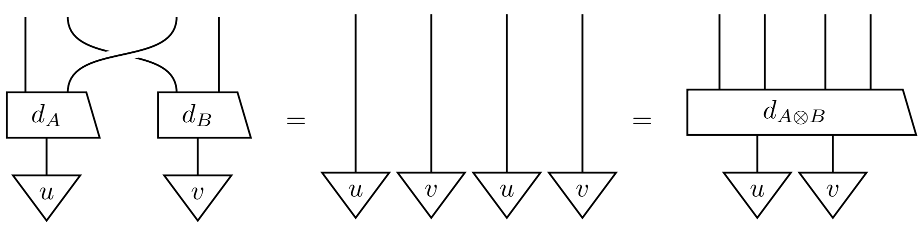
Now suppose the category is monoidally well-pointed and any state is copyable. In particular, the state $I \xrightarrow{\mathrm{id}_I} I$ is then copyable, which means $d_I = \rho_I^{-1}$. To see that d_A is natural, let $A \xrightarrow{f} B$ be any morphism. By monoidal well-pointedness, it suffices to show for any state $I \xrightarrow{v} A$ that



But that is just copyability of the state $I \xrightarrow{f \circ v} B$. Associativity (4.5) and commutativity (4.7) similarly follow from well-pointedness. For example:



because any state $I \xrightarrow{v} A$ is copyable. Finally, we have to verify equation (4.16). This is where we need monoidal well-pointedness, rather than mere well-pointedness:



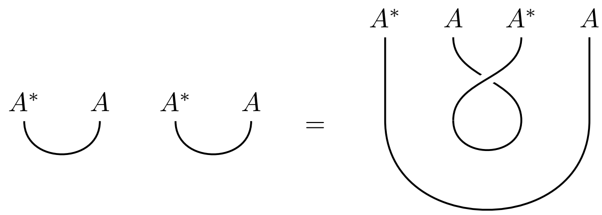
for all states $I \xrightarrow{u} A$ and $I \xrightarrow{v} B$. □

Hence our definition of uniform copying coincides with the usual one in monoidally well-pointed categories such as **Set**, **Rel**, and **Hilb**. Definition 4.20 is more general and makes sense for non-well-pointed categories, too.

No-cloning

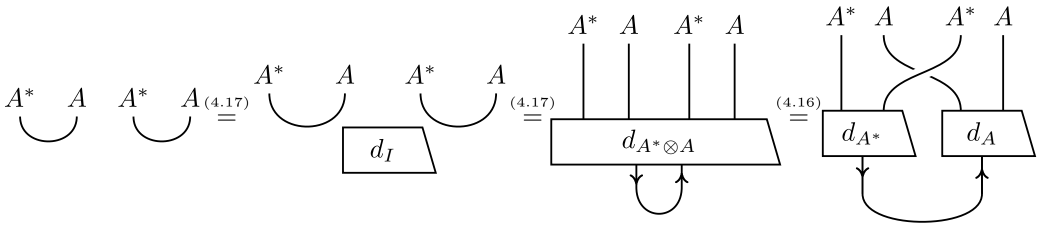
You might have expected Example 4.21: in classical physics, as modeled in **Set**, you *can* uniformly copy states. The no-cloning theorem says something about quantum physics, which we have modeled by compact categories, which **Set** is not. Uniform copying on a compact category turns out to be a drastic restriction. It means that the category degenerates: it must have trivial dynamics, in the sense that up to scalars there is only one operator $A \to A$ on each object A. To prove this categorical no-cloning theorem, we start with a preparatory lemma.

Lemma 4.26. *If a braided monoidal category with duals has uniform copying, then the following holds:*



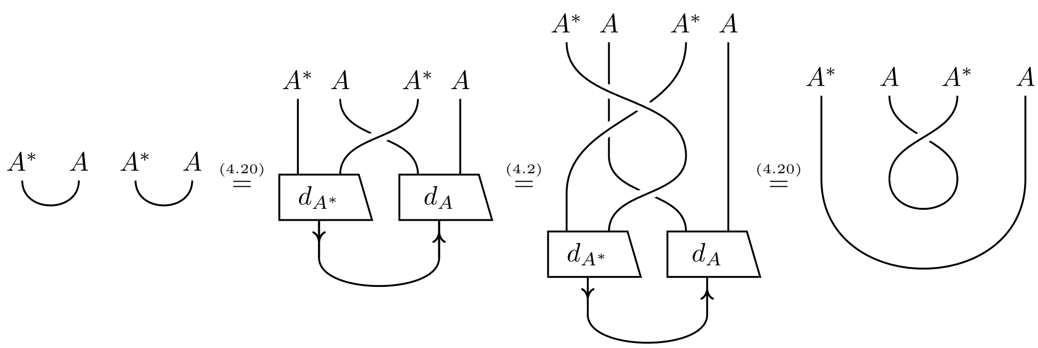
(4.19)

Proof. First, consider the following equalities:



(4.20)

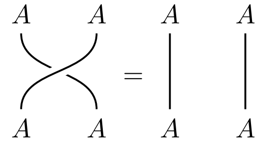
Then we can use this as follows:



This completes the proof. □

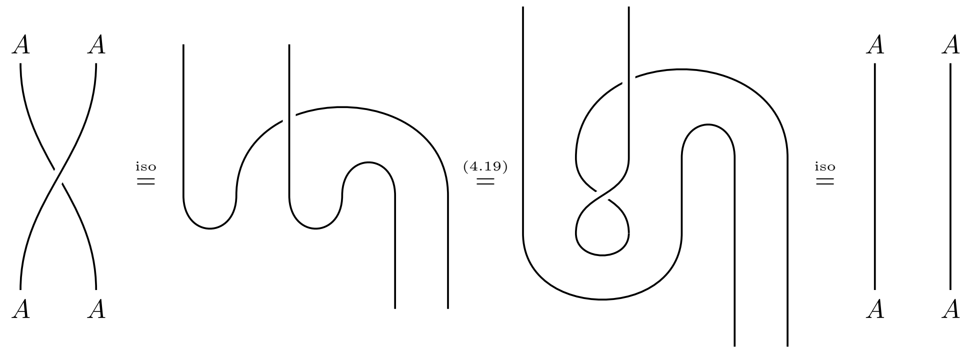
The previous lemma already shows the core of the degeneracy, as it equates two morphisms with different connectivity. We can now prove the no-cloning theorem.

Proposition 4.27. *In a braided monoidal category with duals and uniform copying, the braiding is the identity:*



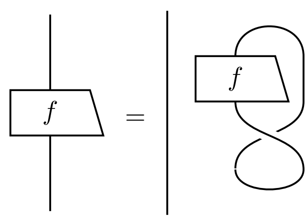
(4.21)

Proof. We show this as follows:



This completes the proof. □

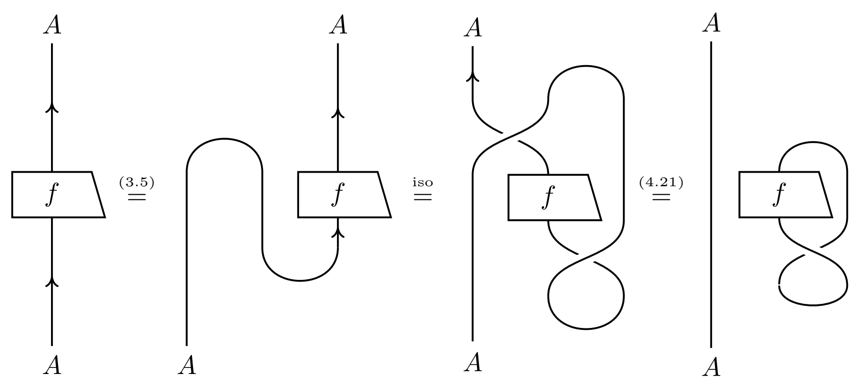
Theorem 4.28 (No cloning)**.** *If a braided monoidal category with duals has uniform copying, then every endomorphism is a scalar multiple of the identity:*



(4.22)

Notice that the scalar is $\mathrm{Tr}(f)$.

Proof. Perform the following calculation:



This completes the proof. □

While highly degenerate, such categories are not necessarily trivial.

4.3 Products

Let's forget about duals for this section. What happens when a symmetric monoidal category has both uniform copying and deleting? When the copying and deletion operations form comonoids, it turns out that the tensor product is an actual categorical product.

Definition 4.29. A category that has a terminal object and products for all pairs of objects is called *cartesian.*

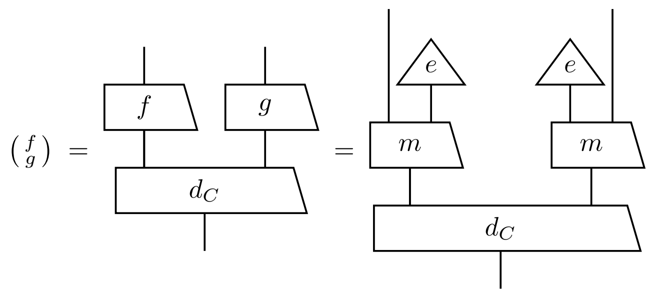
Theorem 4.30. *The following are equivalent for a symmetric monoidal category:*

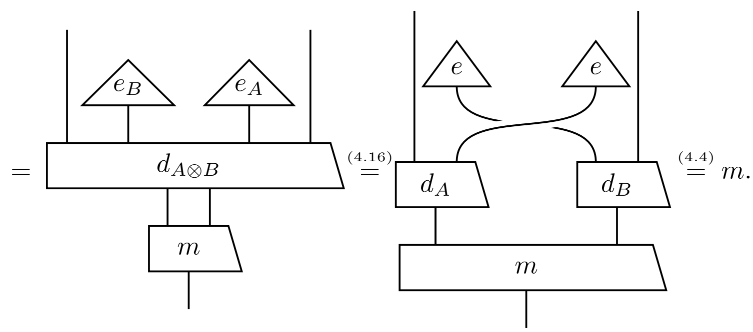
- *it is cartesian: tensor products are products and the tensor unit is terminal;*
- *it has uniform copying and deleting, and equation* (4.4) *holds.*

Proof. If the category is Cartesian, then $d_A = \left(\begin{smallmatrix} \mathrm{id}_A \\ \mathrm{id}_A \end{smallmatrix}\right)$ and the unique morphism $A \xrightarrow{e_A} I$ provide uniform copying and deleting operators that moreover satisfy (4.4).

For the converse, we need to prove that $A \otimes B$ is a product of A and B. Define $p_A = \rho_A \circ (\mathrm{id}_A \otimes e_B) \colon A \otimes B \longrightarrow A$ and $p_B = \lambda_B \circ (e_A \otimes \mathrm{id}_B) \colon A \otimes B \longrightarrow B$. For given $C \xrightarrow{f} A$ and $C \xrightarrow{g} B$, define $\left(\begin{smallmatrix} f \\ g \end{smallmatrix}\right) = (f \otimes g) \circ d$.

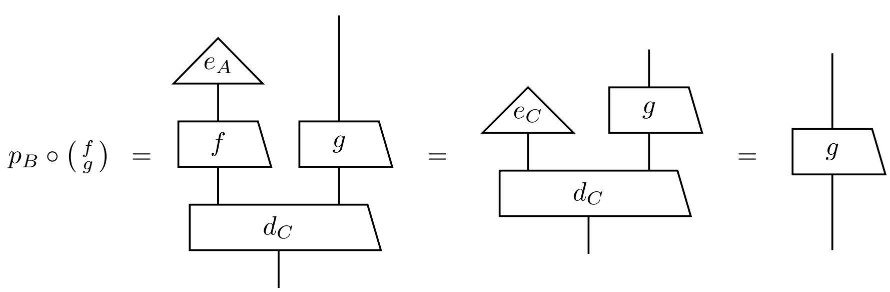
First, suppose $C \xrightarrow{m} A \otimes B$ satisfies $p_A \circ m = f$ and $p_B \circ m = g$. Then:





The second equality is our assumption, and the third equality is naturality of d, the fourth equality follows from the definition of uniform copying, and the last equality uses counitality. Hence mediating morphisms, if they exist, are unique: they are all equal $\left(\begin{smallmatrix} f \\ g \end{smallmatrix}\right)$.

Finally, we show that $\left(\begin{smallmatrix} f \\ g \end{smallmatrix}\right)$ indeed satisfies $p_A \circ \left(\begin{smallmatrix} f \\ g \end{smallmatrix}\right) = f$ and $p_B \circ \left(\begin{smallmatrix} f \\ g \end{smallmatrix}\right) = g$.



The first equality holds by definition, the second equality is naturality of e, and the last equality is equation (4.4). Similarly $p_A \circ \left(\begin{smallmatrix} f \\ g \end{smallmatrix}\right) = f$. □