Categories and Quantum Informatics: Coherence

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In this section we prove the coherence theorem for monoidal categories. To do so, we first discuss *strict* monoidal categories, which are easier to work with. Then we rigorously introduce the notion of monoidal equivalence, which encodes when two monoidal categories 'behave the same'. This puts us in a position to prove the Strictification Theorem 1.33, which says that any monoidal category is monoidally equivalent to a strict one. From there we prove the Coherence Theorem.

1.8 Strictness

Some types of monoidal category have no data encoded in their unit and associativity morphisms. In this section we prove that in fact, every monoidal category can be made into a such a *strict* one.

Definition 1.24. A monoidal category is *strict* if the natural isomorphisms $\alpha_{A,B,C}$, λ_A and ρ_A are all identities.

The category $\mathbf{Mat}_{\mathbb{C}}$ can be given strict monoidal structure.

Definition 1.25. The following structure makes $\mathbf{Mat}_{\mathbb{C}}$ strict monoidal:

- **tensor product** $\otimes\colon \mathbf{Mat}_{\mathbb{C}} \times \mathbf{Mat}_{\mathbb{C}} \to \mathbf{Mat}_{\mathbb{C}}$ is given on objects by multiplication of numbers $n \otimes m = nm$, and on morphisms by Kronecker product of matrices;
- **the monoidal unit** is the natural number 1;
- **associators, left unitors and right unitors** are the identity matrices.

The Strictification Theorem 1.33 below will show that any monoidal category is monoidally equivalent to a strict one. Sometimes this is not as useful as it sounds. For example, you often have some idea of what you want the objects of your category to be, but you might have to abandon this to construct a strict version of your category. In particular, it's often useful for categories to be *skeletal*. Every monoidal category is equivalent to a skeletal monoidal category, and skeletal categories are often particularly easy to work with. However, *not* every monoidal category is *monoidally* equivalent to a strict, skeletal category. If you have to choose, it often turns out that skeletality is the more useful property to have. However, sometimes you *can* have both properties: for example, the monoidal category $\mathbf{Mat}_{\mathbb{C}}$ of Definition 1.25 is both strict and skeletal.

1.9 Monoidal functors

Monoidal functors are functors that preserve monoidal structure; they have to satisfy some coherence properties of their own.

Definition 1.26. A *monoidal functor* $F\colon \mathbf{C} \to \mathbf{C}'$ between monoidal categories is a functor equipped with natural isomorphisms

$$(F_2)_{A,B}\colon F(A) \otimes' F(B) \to F(A \otimes B) \tag{1.26}$$
$$F_0\colon I' \to F(I) \tag{1.27}$$

making the following diagrams commute:

$$\begin{tikzcd}
\big(F(A) \otimes' F(B)\big) \otimes' F(C) \arrow[rr, "\alpha'_{F(A),F(B),F(C)}"] \arrow[d, "(F_2)_{A,B} \otimes' \mathrm{id}_{F(C)}"'] & & F(A) \otimes' \big(F(B) \otimes' F(C)\big) \arrow[d, "\mathrm{id}_{F(A)} \otimes' (F_2)_{B,C}"] \\
F(A \otimes B) \otimes' F(C) \arrow[d, "(F_2)_{A\otimes B,C}"'] & & F(A) \otimes' F(B \otimes C) \arrow[d, "(F_2)_{A,B\otimes C}"] \\
F\big((A \otimes B) \otimes C\big) \arrow[rr, "F(\alpha_{A,B,C})"'] & & F\big(A \otimes (B \otimes C)\big)
\end{tikzcd} \tag{1.28}$$

$$\begin{tikzcd}
F(A) \otimes' I' \arrow[r, "\rho'_{F(A)}"] \arrow[d, "\mathrm{id}_{F(A)} \otimes' F_0"'] & F(A) \arrow[d, "F(\rho_A^{-1})"'] \\
F(A) \otimes' F(I) \arrow[r, "(F_2)_{A,I}"'] & F(A \otimes I)
\end{tikzcd}
\qquad
\begin{tikzcd}
I' \otimes' F(A) \arrow[r, "\lambda'_{F(A)}"] \arrow[d, "F_0 \otimes' \mathrm{id}_{F(A)}"] & F(A) \arrow[d, "F(\lambda_A^{-1})"] \\
F(I) \otimes' F(A) \arrow[r, "(F_2)_{I,A}"'] & F(I \otimes A)
\end{tikzcd} \tag{1.29}$$

We can ask for monoidal functors to be compatible with a braided monoidal structure.

Definition 1.27. A *braided monoidal functor* is a monoidal functor $F : \mathbf{C} \to \mathbf{C}'$ between braided monoidal categories such that the following diagram commutes:

$$\begin{tikzcd}
F(A) \otimes' F(B) \arrow[r, "\gamma'_{F(A),F(B)}"] \arrow[d, "(F_2)_{A,B}"'] & F(B) \otimes' F(A) \arrow[d, "(F_2)_{B,A}"] \\
F(A \otimes B) \arrow[r, "F(\gamma_{A,B})"'] & F(B \otimes A)
\end{tikzcd} \tag{1.30}$$

We can also ask for a monoidal functor to be compatible with a symmetric monoidal structure, but this does not lead to any further algebraic conditions.

Definition 1.28. A *symmetric monoidal functor* is a braided monoidal functor $F : \mathbf{C} \to \mathbf{C}'$ between symmetric monoidal categories.

The only reason to introduce this definition is because it sounds a bit strange to talk about braided monoidal functors between symmetric monoidal categories.

Definition 1.29. A *monoidal equivalence* is a monoidal functor that is an equivalence as a functor.

Example 1.30. The equivalence $R\colon \mathbf{Mat}_{\mathbb{C}} \to \mathbf{FHilb}$ is a monoidal equivalence.

Proof. Set $R_0 = \mathrm{id}_{\mathbb{C}} \colon \mathbb{C} \to \mathbb{C}$, and define $(R_2)_{m,n} \colon \mathbb{C}^m \otimes \mathbb{C}^n \to \mathbb{C}^{mn}$ by $|i\rangle \otimes |j\rangle \mapsto |ni+j\rangle$ for the computational basis. Then $(R_2)_{m,1} = \rho_{\mathbb{C}^m}$ and $(R_2)_{1,n} = \lambda_{\mathbb{C}^n}$, satisfying (1.29). Equation (1.28) is also satisfied by this definition. Thus R is a monoidal functor. □

We can also ask for natural transformations between monoidal functors to satisfy some equations.

Definition 1.31 (Monoidal natural transformation)**.** Let $F, G \colon \mathbf{C} \to \mathbf{C}'$ be monoidal functors between monoidal categories. A *monoidal natural transformation* is a natural transformation $\mu \colon F \Rightarrow G$ making the following diagrams commute:

$$\begin{tikzcd}
F(A) \otimes' F(B) \arrow[r, "(F_2)_{A,B}"] \arrow[d, "\mu_A \otimes' \mu_B"'] & F(A \otimes B) \arrow[d, "\mu_{A\otimes B}"] \\
G(A) \otimes' G(B) \arrow[r, "(G_2)_{A,B}"'] & G(A \otimes B)
\end{tikzcd}
\qquad
\begin{tikzcd}
 & F(I) \arrow[dd, "\mu_I"] \\
I' \arrow[ur, "F_0"] \arrow[dr, "G_0"'] & \\
 & G(I)
\end{tikzcd}
\tag{1.31}$$

The further notions of braided or symmetric monoidal natural transformation do not give any further algebraic conditions, as with the definition of symmetric monoidal functor, but for similar reasons they give useful terminology.

Definition 1.32. A *braided monoidal natural transformation* is a monoidal natural transformation $\mu : F \Rightarrow G$ between braided monoidal functors. A *symmetric monoidal natural transformation* is a monoidal natural transformation between symmetric monoidal functors.

We can use the notion of monoidal natural transformation to give an alternative definition of monoidal equivalence, as a pair of monoidal functors $F \colon \mathbf{C} \to \mathbf{D}$ and $G \colon \mathbf{D} \to \mathbf{C}$ such that both $F \circ G$ and $G \circ F$ are naturally monoidally isomorphic to the identity functors. This turns out to be equivalent to Definition 1.29.

1.10 Strictification

We now prove the Strictification Theorem by 'going one level up' from monoidal categories to monoidal functors, using the 'higher' coherence properties of their own that monoidal functors have to satisfy.

Theorem 1.33 (Strictification)**.** *Every monoidal category is monoidally equivalent to a strict monoidal category.*

Proof. We will emulate Cayley's theorem, which states that any group G is isomorphic to the group of all permutations $G \to G$ that commute with right multiplication, by sending g to left-multiplication with g.

Let $\mathbf{C}$ be a monoidal category, and define $\mathbf{D}$ as follows. Objects are pairs (F, γ) consisting of a functor $F \colon \mathbf{C} \to \mathbf{C}$ and a natural isomorphism

$$F(A) \otimes B \xrightarrow{\gamma_{A,B}} F(A \otimes B).$$

We can think of γ as witnessing that F commutes with right multiplication. A morphism $(F, \gamma) \to (F', \gamma')$ is a natural transformation $\theta \colon F \Rightarrow F'$ making the following square commute for all objects A, B of $\mathbf{C}$:

$$\begin{tikzcd}
F(A) \otimes B \arrow[r, "\gamma_{A,B}"] \arrow[d, "\theta_A \otimes \mathrm{id}_B"'] & F(A \otimes B) \arrow[d, "\theta_{A\otimes B}"] \\
F'(A) \otimes B \arrow[r, "\gamma'_{A,B}"'] & F'(A \otimes B)
\end{tikzcd}
\tag{1.32}$$

Composition is given by $(\theta' \circ \theta)_A = \theta'_A \circ \theta_A$. The tensor product of objects in $\mathbf{D}$ is given by $(F, \gamma) \otimes (F', \gamma') := (F \circ F', \delta)$, where δ is the composition

$$F(F'(A)) \otimes B \xrightarrow{\gamma_{F'(A),B}} F(F'(A) \otimes B) \xrightarrow{F(\gamma'_{A,B})} F(F'(A \otimes B));$$

the tensor product of morphisms $\theta\colon F \to F'$ and $\theta'\colon G \to G'$ is the composite

$$F(G(A)) \xrightarrow{\theta_{G(A)}} F'(G(A)) \xrightarrow{F'(\theta'_A)} F'(G'(A)),$$

which again satisfies (1.32). It can be checked that $((F,\gamma)\otimes(F',\gamma'))\otimes(F'',\gamma'') = (F,\gamma)\otimes((F',\gamma')\otimes(F'',\gamma''))$, and that the category accepts a strict monoidal structure, with unit object given by the identity functor on $\mathbf{C}$.

Now consider the functor $L\colon \mathbf{C} \to \mathbf{D}$ given by

$$L(A) = (A \otimes -, \alpha_{A,-,-}), \qquad L(f) = f \otimes -.$$

We can think of this functor as "multiplying on the left". We will show that L is a full, faithful monoidal functor. For faithfulness, if $L(f) = L(g)$ for morphisms f, g in $\mathbf{C}$, that means $f \otimes \mathrm{id}_I = g \otimes \mathrm{id}_I$, and so $f = g$ by naturality of ρ. For fullness, let $\theta\colon L(A) \to L(B)$ be a morphism in $\mathbf{D}$, and define $f\colon A \to B$ as the composite

$$A \xrightarrow{\rho_A^{-1}} A \otimes I \xrightarrow{\theta_I} B \otimes I \xrightarrow{\rho_B} B.$$

Then the following diagram commutes:

$$\begin{tikzcd}
A \otimes C \arrow[r, "\rho_A^{-1} \otimes \mathrm{id}_C"] \arrow[d, "f \otimes \mathrm{id}_C"'] & (A \otimes I) \otimes C \arrow[r, "\alpha_{A,I,C}"] \arrow[d, "\theta_I \otimes \mathrm{id}_C"'] & A \otimes (I \otimes C) \arrow[r, "\mathrm{id}_A \otimes \lambda_C"] \arrow[d, "\theta_{I \otimes C}"] & A \otimes C \arrow[d, "\theta_C"] \\
B \otimes C \arrow[r, "\rho_B^{-1} \otimes \mathrm{id}_C"'] & (B \otimes I) \otimes C \arrow[r, "\alpha_{B,I,C}"'] & B \otimes (I \otimes C) \arrow[r, "\mathrm{id}_B \otimes \lambda_C"'] & B \otimes C
\end{tikzcd}$$

The left square commutes by definition of f, the middle square by (1.32), and the right square by naturality of θ. Moreover, the rows both equal the identity by the triangle identity. Hence $\theta_C = f \otimes \mathrm{id}_C$, and so $\theta = L(f)$.

We now show that L is a monoidal functor. Define the isomorphism $L_0\colon I \to L(I)$ to be λ^{-1}, and define $(L_2)_{A,B}\colon L(A) \otimes L(B) \to L(A \otimes B)$ by

$$\alpha^{-1}_{A,B,-}\colon \big(A \otimes (B \otimes -), (A \otimes \alpha_{B,-,-}) \circ \alpha_{A,B\otimes -,-}\big) \to \big((A \otimes B) \otimes -, \alpha_{A\otimes B,-,-}\big).$$

These form a well-defined morphism in $\mathbf{D}$, because equation (1.32) is just the pentagon identity of $\mathbf{C}$. Verifying equations (1.29) comes down to the fact that $\lambda_I = \rho_I$ and the triangle identity. Because $\mathbf{D}$ is strict, equation (1.28) comes down the pentagon identity of $\mathbf{C}$.

Finally, let $\mathbf{C_s}$ be the subcategory of $\mathbf{D}$ encompassing all objects that are isomorphic to those of the form $L(A)$, and all morphisms between them. Then $\mathbf{C_s}$ is still a strict monoidal category, and L restricts to a functor $L\colon \mathbf{C} \to \mathbf{C_s}$ that is still monoidal, full and faithful, but is additionally essentially surjective on objects by construction. Thus $L\colon \mathbf{C} \to \mathbf{C_s}$ is a monoidal equivalence. $\square$

The Strictification Theorem means that, if you prefer, you can always strictify your monoidal category to obtain an equivalent one for which α, λ and ρ are all identities.

1.11 The coherence theorem

We now derive the Coherence Theorem from the Strictification Theorem. To state the former, we have to be more precise about what we have coyly called 'well-typed' equations. To do so, we will talk about different ways to parenthesize a finite list of things. More precisely, define *bracketings* in $\mathbf{C}$ inductively: () is the empty bracketing; for any object A in $\mathbf{C}$ there is a bracketing A; and if v and w are bracketings, then so is $(v \otimes w)$. For example, $v = (A \otimes B) \otimes (C \otimes D)$ and $((A \otimes B) \otimes C) \otimes D$ are different bracketings, even though they could denote the same object of $\mathbf{C}$. Actually, a bracketing is independent of the objects and even of the category. So for example, we may use the notation $v(W, X, Y, Z) = (W \otimes X) \otimes (Y \otimes Z)$ to define a procedure v that operates on any quartet of objects. Thus it also makes sense to talk about transformations $\theta\colon v \Rightarrow w$ built from coherence isomorphisms.

Theorem 1.34 (Coherence for monoidal categories)**.** *Let $v(A, \ldots, Z)$ and $w(A, \ldots, Z)$ be bracketings in a monoidal category $\mathbf{C}$. Any two transformations $\theta, \theta'\colon v \Rightarrow w$ built from α, α^{-1}, λ, λ^{-1}, ρ, ρ^{-1}, id, $\otimes$, and $\circ$ are equal.*

Proof. Let $L\colon \mathbf{C} \longrightarrow \mathbf{C_s}$ be the monoidal equivalence from Theorem 1.33. Inductively define a morphism $L_v\colon v(L(A), \ldots, L(Z)) \longrightarrow L(v(A, \ldots, Z))$ in $\mathbf{C_s}$ by setting $L_{()} = L_0$, $L_A = L(A)$, and $L_{(x \otimes y)} = L_2 \circ (L_x \otimes L_y)$. Define L_w similarly. Then the following diagram in $\mathbf{C_s}$ commutes:

$$\begin{tikzcd}
v(L(A), \ldots, L(Z)) \arrow[r, "\theta_{(L(A), \ldots, L(Z))}"] \arrow[d, "L_v"'] & w(L(A), \ldots, L(Z)) \arrow[d, "L_w"] \\
L(v(A, \ldots, Z)) \arrow[r, "L(\theta_{(A, \ldots, Z)})"'] & L(w(A, \ldots, Z))
\end{tikzcd}$$

The same diagram for θ' commutes similarly. But as $\mathbf{C_s}$ is a strict monoidal category, and θ and θ' are built from coherence isomorphisms, we must have $\theta_{(L(A), \ldots, L(Z))} = \theta'_{(L(A), \ldots, L(Z))} = \mathrm{id}$. Since L_v and L_w are by construction isomorphisms, it follows from the diagram above that $L(\theta_{(A, \ldots, Z)}) = L(\theta'_{(A, \ldots, Z)})$. Finally, L is an equivalence and hence faithful, so $\theta_{(A, \ldots, Z)} = \theta'_{(A, \ldots, Z)}$. $\square$

Notice that the transformations θ, θ' in the previous theorem have to go from a single bracketing v to a single bracketing w. Suppose we have an object A for which $A \otimes A = A$. Then $A \xrightarrow{\mathrm{id}_A} A$ and $A \xrightarrow{\alpha_{A,A,A}} A$ are both well-defined morphisms. But equating them does not give a well-formed equation, as they do not give rise to transformations from the same bracketing to the same bracketing.

1.12 Braided monoidal functors

Versions of the Strictification Theorem 1.33 and the Coherence Theorem still hold for braided monoidal categories and symmetric monoidal categories. In fact, they link nicely with the Correctness Theorems for the graphical calculus. We will not go into details of the proofs, but just record the statements here.

Definition 1.35 (Braided monoidal functor, symmetric monoidal functor)**.** A *braided monoidal functor* is a monoidal functor $\mathbf{C} \xrightarrow{F} \mathbf{D}$ between braided monoidal categories, that additionally makes the following diagrams commute:

$$\begin{tikzcd}
F(A) \otimes F(B) \arrow[r, "\sigma_{F(A),F(B)}"] \arrow[d, "(F_2)_{A,B}"'] & F(B) \otimes F(A) \arrow[d, "(F_2)_{B,A}"] \\
F(A \otimes B) \arrow[r, "F(\sigma_{A,B})"'] & F(B \otimes A)
\end{tikzcd} \tag{1.33}$$

A *symmetric monoidal functor* is a braided monoidal functor between symmetric monoidal categories.

We call a braided monoidal category *strict* when the underlying monoidal category is strict. This does not mean that the braiding should be the identity.

Theorem 1.36 (Strictification for braided monoidal categories)**.** *Every braided monoidal category has a braided monoidal equivalence to a strict braided monoidal category. Every symmetric monoidal category has a symmetric monoidal equivalence to a strict symmetric monoidal category.* □

To state the coherence theorem in the braided and symmetric case, we again have to be precise about what 'well-formed' equations are. Consider a morphism f in a braided monoidal category that is built from the coherence isomorphisms and the braiding, and their inverses, using identities and tensor products. Using the graphical calculus of braided monoidal categories, we can always draw it as a *braid*. By the correctness of the graphical calculus for braided monoidal categories, this picture defines a morphism g built from coherence isomorphisms and the braiding in a canonical bracketing, say with all brackets to the left. Moreover, up to isotopy of the picture this is the unique such morphism g. We call that morphism g, or equivalently the isotopy class of its picture, the *underlying braid* of the original morphism f. Since the underlying braid is merely about the connectivity, and not about the actual objects, it lifts from morphisms to bracketings.

Corollary 1.37 (Coherence for braided monoidal categories)**.** *Let $v(A,\dots,Z)$ and $w(A,\dots,Z)$ be bracketings in a braided monoidal category $\mathbf{C}$. Any two transformations $\theta,\theta'\colon v \Rightarrow w$ built from α, α^{-1}, λ, λ^{-1}, ρ, ρ^{-1}, σ, σ^{-1}, id, $\otimes$, and $\circ$, are equal if and only if they have the same underlying braid.*

Proof. By definition of the underlying braid, this follows immediately from the Coherence Theorem 1.34 for monoidal categories and the Correctness Theorem of the graphical calculus for braided monoidal categories. □

For symmetric monoidal categories, we can simplify the underlying braid to an *underlying permutation*. It is a bijection between the set $\{1,\dots,n\}$ and itself, where n is the number of objects in the bracketing, namely precisely the bijection that is indicated by the graphical calculus when we draw the bracketing.

Corollary 1.38 (Coherence for symmetric monoidal categories)**.** *Let $v(A,\dots,Z)$ and $w(A,\dots,Z)$ be bracketings in a symmetric monoidal category $\mathbf{C}$. Any two transformations $\theta,\theta'\colon v \Rightarrow w$ built from α, α^{-1}, λ, λ^{-1}, ρ, ρ^{-1}, σ, σ^{-1}, id, $\otimes$, and $\circ$, are equal if and only if they have the same underlying permutation.*

Proof. By definition of the underlying permutation, this follows immediately from Corollary 1.37, and the correctness of the graphical calculus for symmetric monoidal categories. □