# Communication and Concurrency
# Lectures 10 & 11

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## Showing bisimilarity

To establish $E \sim F$

1. Present a candidate relation $R$ with $(E, F) \in R$
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$$\begin{array}{lcl} A & \stackrel{\text{def}}{=} & a.\overline{c}.A \\ B & \stackrel{\text{def}}{=} & c.\overline{b}.B \\ C_0 & \stackrel{\text{def}}{=} & \overline{b}.C_1 + a.C_2 \\ C_1 & \stackrel{\text{def}}{=} & a.C_3 \\ C_2 & \stackrel{\text{def}}{=} & \overline{b}.C_3 \\ C_3 & \stackrel{\text{def}}{=} & \tau.C_0 \end{array}$$

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$R$ below is a bisimulation

$$\{((A|B)\backslash c, C_1), ((\overline{c}.A|B)\backslash c, C_3)$$
$$((A|\overline{b}.B)\backslash c, C_0), ((\overline{c}.A|\overline{b}.B)\backslash c, C_2)\}$$

## Showing Bisimilarity II

Same sort of argument establishes that $\sim$ is a congruence.

1. if $E \sim F$ then $G|E \sim G|F$
2. Proof: Assume that $E \sim F$, so there is a bisimulation $B$ with $(E, F) \in B$.
3. Let $C$ be the relation

$$\{(H|E', H|F') \,:\, (E', F') \in B\}$$

4. Show that $C$ is a bisimulation ...

## Some Results

$$\begin{array}{lcl} Id & = & \{(E,E)\} \\ B^{-1} & = & \{(E,F) \,:\, (F,E) \in B\} \\ B_1 B_2 & = & \{(E,G) \,:\, \text{there is } F.\ (E,F) \in B_1 \\ & & \text{and } (F,G) \in B_2\} \end{array}$$

Proposition Assume $B_i$ $(i = 1, 2, \ldots)$ is a bisimulation. Then the following are bisimulations:

1. $Id$
2. $B_i^{-1}$
3. $B_1 B_2$
4. $\bigcup\{B_i \,:\, i \geq 1\}$

Corollary $\sim$ is the largest bisimulation

## A bigger example: $\texttt{Cnt} \sim \texttt{Ct}'_0$

$$\begin{array}{lcl} \texttt{Cnt} & \stackrel{\text{def}}{=} & \texttt{up}.(\texttt{Cnt} \mid \texttt{down}.0) \\ \texttt{Ct}'_0 & \stackrel{\text{def}}{=} & \texttt{up}.\texttt{Ct}'_1 \\ \texttt{Ct}'_{i+1} & \stackrel{\text{def}}{=} & \texttt{up}.\texttt{Ct}'_{i+2} + \texttt{down}.\texttt{Ct}'_i \quad i \geq 0. \end{array}$$

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$$\begin{array}{lcl} P_0 & = & \{\texttt{Cnt} \mid 0^j \,:\, j \geq 0\} \\ P_{i+1} & = & \{E \mid 0^j \mid \texttt{down}.0 \mid 0^k \,:\, E \in P_i \text{ and } j \geq 0 \text{ and } k \geq 0\} \end{array}$$

where $F \mid 0^0 = F$ and $F \mid 0^{i+1} = F \mid 0^i \mid 0$ and brackets are dropped between parallel components.

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$B = \{(E, \texttt{Ct}'_i) \,:\, i \geq 0 \text{ and } E \in P_i\}$ is a bisimulation

## More Properties I

Proposition

1. $E + F \sim F + E$
2. $E + (F + G) \sim (E + F) + G$
3. $E + 0 \sim E$
4. $E + E \sim E$

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## More Properties II

Proposition

1. $(E + F)\backslash K \sim E\backslash K + F\backslash K$
2. $(a.E)\backslash K \sim 0$ if $a \in K \cup \overline{K}$
3. $(a.E)\backslash K \sim a.(E\backslash K)$ if $a \notin K \cup \overline{K}$

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- Example

$$\begin{aligned} x_1 &\sim a.x_{11} + b.x_{12} + a.x_{13} \\ x_2 &\sim \bar{a}.x_{21} + c.x_{22}, \end{aligned}$$

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- $$\begin{aligned} x_1 | x_2 \sim \ & a.(x_{11}|x_2) + b.(x_{12}|x_2) + a.(x_{13}|x_2) + \\ & \bar{a}.(x_1|x_{21}) + \\ & c.(x_1|x_{22}) + \tau.(x_{11}|x_{21}) + \tau.(x_{13}|x_{21}). \end{aligned}$$

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- Two processes $E$ and $F$ are weak bisimulation equivalent (or weakly bisimilar) if there is a weak bisimulation relation $B$ such that $(E, F) \in B$. We write $E \approx F$ if $E$ and $F$ are weakly bisimilar

## Exercise

Which of the following are weakly bisimilar?

| | | Y/N |
|---|---|---|
| $a.\tau.b.0$ | $a.b.0$ | |
| $a.(b.0 + \tau.c.0)$ | $a.(b.0 + c.0)$ | |
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| $a.0 + b.0 + \tau.b.0$ | $a.0 + \tau.b.0$ | |
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| $a.(b.0 + \tau.c.0)$ | $a.(b.0 + \tau.c.0) + a.c.0$ | Y |
| $a.0 + b.0 + \tau.b.0$ | $a.0 + \tau.b.0$ | Y |
| $a.0 + b.0 + \tau.b.0$ | $a.0 + b.0$ | N |
| $a.(b.0 + \tau.b.0)$ | $a.b.0$ | Y |

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4. $A_0 \approx B_1$

$$\{(A_0, B_1), (A_1, B_1), (A_2, B_2)\}$$

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  2. $\approx \; = \; \approx'$

## Protocol that may lose messages

$$\begin{array}{lcl}
\texttt{Sender} & \stackrel{\text{def}}{=} & \texttt{in}(x).\overline{\texttt{sm}}(x).\texttt{Send1}(x) \\
\texttt{Send1}(x) & \stackrel{\text{def}}{=} & \texttt{ms}.\overline{\texttt{sm}}(x).\texttt{Send1}(x) + \texttt{ok}.\texttt{Sender} \\
\texttt{Medium} & \stackrel{\text{def}}{=} & \texttt{sm}(y).\texttt{Med1}(y) \\
\texttt{Med1}(y) & \stackrel{\text{def}}{=} & \overline{\texttt{mr}}(y).\texttt{Medium} + \tau.\overline{\texttt{ms}}.\texttt{Medium} \\
\texttt{Receiver} & \stackrel{\text{def}}{=} & \texttt{mr}(x).\overline{\texttt{out}}(x).\overline{\texttt{ok}}.\texttt{Receiver} \\
 & & \\
\texttt{Protocol} & \equiv & (\texttt{Sender} \mid \texttt{Medium} \mid \texttt{Receiver}) \backslash \{\texttt{sm}, \texttt{ms}, \texttt{mr}, \texttt{ok}\} \\
 & & \\
\texttt{Cop} & \stackrel{\text{def}}{=} & \texttt{in}(x).\overline{\texttt{out}}(x).\texttt{Cop}
\end{array}$$

## $\texttt{Protocol} \approx \texttt{Cop}$

Let $B$ be the following relation

$$\begin{array}{l}
\{(\texttt{Protocol}, \texttt{Cop})\} \cup \\
\{((\texttt{Send1}(m) \mid \texttt{Medium} \mid \overline{\texttt{ok}}.\texttt{Receiver}) \backslash J, \\
\qquad \texttt{Cop}) \; : \; m \in D\} \cup \\
\{((\overline{\texttt{sm}}(m).\texttt{Send1}(m) \mid \texttt{Medium} \mid \texttt{Receiver}) \backslash J, \\
\qquad \overline{\texttt{out}}(m).\texttt{Cop}) \; : \; m \in D\} \cup \\
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\end{array}$$

$B$ is a weak bisimulation

## Properties of weak bisimulation

$$\begin{array}{lcl} Id & = & \{(E,E)\} \\ B^{-1} & = & \{(E,F) : (F,E) \in B\} \\ B_1B_2 & = & \{(E,G) : \text{there is } F.\ (E,F) \in B_1 \\ & & \text{and } (F,G) \in B_2\} \end{array}$$

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Proposition Assume $B_i$ ($i = 1, 2, \ldots$) is a weak bisimulation. Then the following are weak bisimulations:

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Proposition If $E \sim F$ then $E \approx F$

## Tau laws

1. $a.\tau.E \approx a.E$
2. $E + \tau.E \approx \tau.E$
3. $a.(E + \tau.F) + a.F \approx a.(E + \tau.F)$

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