Cognitive Modeling

Lecture 5: Models of Arithmetic

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Reading: Cooper (2002, Ch. 3)

Why study models of arithmetic?

Task is an example of a *cognitive skill* – acquired through conscious practice.

- driving (vs. walking)
- reading/writing (vs. understanding/speaking)

Model is an example of a *production system*.

- Often used to model cognitive skills.
- Useful in explaining the how humans perform the task correctly by integrating many smaller subskills.
- Failure of individual subskills may help explain systematic failures in main skill.

Multi-column subtraction

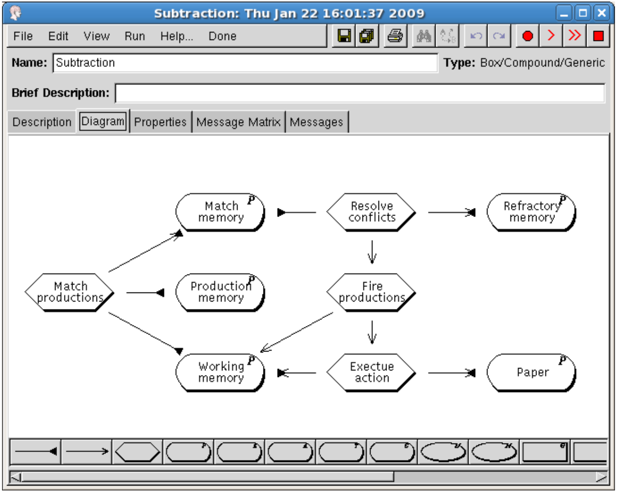
How do skilled students perform this task?
What types of errors are made by learners?

- random errors versus systematic errors.
- factual (arithmetic) errors versus procedural errors.
- incorrect subskills versus failure to apply subskills.

Young and O'Shea (1981) hypothesized that many errors are caused by failing to apply a sub-component of the skill.

Basic architecture: Subtraction

Architecture is general production system, not specific to task:



Basic architecture: Subtraction

- **Working memory**: holds current goals for task (multi-column subtraction) and subtasks (e.g. borrow).
- **Production memory**: holds production rules encoding *when* and *how* to perform subtasks (condition-action pairs) including arithmetic facts.
- **Match memory**: holds any production rules whose conditions are currently met.
- **Conflict resolution**: determines which rule in **Match memory** to fire.
- **Refractory memory**: keeps track of rules that have fired to prevent them firing again unless later reintroduced into **Match memory**.

Comparison to ACT-R

- **Working memory**: similar to ACT-R Goal module.
- **Production memory**: combines ACT-R production system and Declarative module.
- **Match memory**: Similar to ACT-R Retrieval buffer.
- **Conflict resolution**: Here, based on recency. ACT-R: based on subsymbolic activation levels.

Diagnosing Student Models

If teacher believes a student has a different model from their own (correct) one:

- make list (bug catalog) and match to it;
- reason about what student would have to believe in order to exhibit behavior indicating this.

Student model: representation of student's current state of knowledge.

Diagnosis: process of inferring the student model.

Interlude: Skilled examples

First, we need to understand the skill children are learning (maybe not the way all of you learned).

Examples of children's work

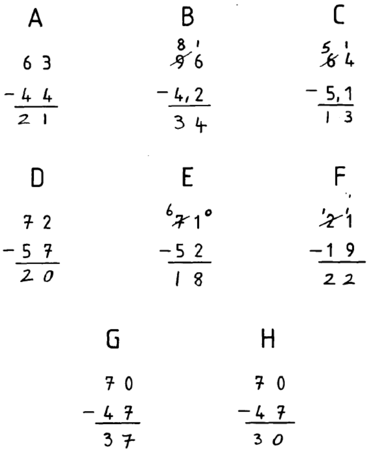


Figure 1. Examples of subtraction errors.

Figure from Young and O'Shea (1981)

Problems with children's work

- A: always subtract smaller digit from larger.
- B: always borrow.
- C: both A and B.
- D: subtracting larger number from smaller equals zero.
- E: borrowing makes 10.
- F: add instead of subtract.
- G,H: errors only with subtracting from zero.

Note that only *patterns* of errors distinguish G,H from A,D. Finding flaws in the underlying procedure (rather than specific errors) requires looking at multiple problems.

Young and O'Shea's Model

Production rule model of multi-column subtraction:

- contains a fairly small number of simple production rules.
- children's errors are modeled by deleting production rules from a model that works correctly.
- accounts for a large percentage of errors found in practice.
- supports hypothesis that many errors arise from forgetting a sub-component of the skill.

A Simple Production Rule Model

Condition		**Action**
S1: goal = process column & minuend greater than or equal to subtrahend	$\longrightarrow$	Take absolute difference of minuend and subtrahend and write in the answer space
S2: goal = process column & minuend less than subtrahend	$\longrightarrow$	Push goal 'borrow' onto stack
S3: goal = borrow	$\longrightarrow$	Decrement next minuend by 1, add 10 to current minuend and delete the current goal

Example

process column
goal stack

4	9	minuend
–1	8	subtrahend
	*	

S1 is the only applicable production, so it fires.

process column
goal stack

4	9	minuend
–1	8	subtrahend
	1	
	*	

Now S1 is still the only applicable production! We need a fix...

* indicates current column

A Revised Subtraction Model

Condition		Action
S1: goal = subtract & all answer spaces empty	⟶	Place marker on rightmost column & push goal 'process column'
S2: goal = process column & minuend greater than or equal to subtrahend	⟶	Take absolute difference of minuend and subtrahend and write in the answer space
S3: goal = process column & minuend less than subtrahend	⟶	Push goal 'borrow' onto stack
S4: goal = process column & answer space filled in	⟶	Move one column left
S5: goal = borrow	⟶	Decrement next minuend by 1, add 10 to current minuend and delete the current goal

Example

goal stack			
subtract	4	9	minuend
	−1	8	subtrahend
		*	

S1 is the only applicable production, so it fires. The marker is placed, the new goal put on the stack and S2 fires.

goal stack			
process column	4	9	minuend
subtract	−1	8	subtrahend
		1	
		*	

S2 and S4 both satisfy the conditions but recency rules out S2.

Example

goal stack			
subtract	4	9	minuend
	−1	8	subtrahend
		1	
	*		

S2's conditions are satisfied so it fires, then S4 will fire.

goal stack			
process column subtract	4	9	minuend
	−1	8	subtrahend
	3	1	
*			

Now no rules are satisfied so the system halts.

Revised Model Reconsidered

Condition		Action
S1: goal = subtract & all answer spaces empty	⟶	Place marker on rightmost column & push goal 'process column'
S2: goal = process column & minuend greater than or equal to subtrahend	⟶	Take absolute difference of minuend and subtrahend and write in the answer space
S3: goal = process column & minuend less than subtrahend	⟶	Push goal 'borrow' onto stack
S4: goal = process column & answer space filled in	⟶	Move one column left
S5: goal = borrow	⟶	Decrement next minuend by 1, add 10 to current minuend and delete the current goal

```
  4  9
 –1  8
 -----
 _____   OK
```

```
  4  0  7
 –1  0  8
 --------
 ________   not OK
```

Young and O'Shea's rules

Condition		**Action**
`Init`: goal = subtract & all answer spaces empty	$\longrightarrow$	Place marker on rightmost column & push goal 'process column'
`Read`: goal = process column & no M or S in working memory	$\longrightarrow$	Read M and S
`Compare`: M and S in working memory	$\longrightarrow$	Compare M and S
`FindDiff`: M and S in working memory	$\longrightarrow$	push goal 'find difference', push goal 'next column'
`Borr2a`: $M < S$	$\longrightarrow$	Push goal 'borrow'
`BorrS1`: goal = borrow	$\longrightarrow$	Decrement next minuend by 1
`BorrS2`: goal = borrow	$\longrightarrow$	Add 10 to current minuend
`AbsDiff`: goal = find difference	$\longrightarrow$	Take absolute difference between M and S as result
`Write`: result in working memory	$\longrightarrow$	Write result
`Next`: goal = process column & answer space filled in	$\longrightarrow$	Move one column left
`Carry`: result is (1,X)	$\longrightarrow$	Carry 1 and take X as result

Analysis of rules

- Why absolute difference?

  `AbsDiff`: goal = find difference $\longrightarrow$ Take absolute difference between M and S as result

- What is the carry rule doing here?

  `Carry`: result is (1,X) $\longrightarrow$ Carry 1 and take X as result

Faulty Models

Leaving out specific rules leads to many common errors.

- `Compare`: M and S in working memory $\longrightarrow$ Compare M and S. If missing, *take smaller from larger*.
- `BorrS1`: goal = borrow $\longrightarrow$ Decrement next minuend by 1. If missing, *borrow freely, no payback*.

But not all:

- Always borrow.
- Zero errors.

Additional rules: borrowing

Replace

`Borr2a`: M $<$ S $\longrightarrow$ Push goal 'borrow'

with one of these:

`Borr2b`: M $>$ S $\longrightarrow$ Push goal 'borrow'

`Borr1`: M and S in working memory $\longrightarrow$ Push goal 'borrow'

- accounts for *always borrow* behavior.
- Young and O'Shea suggest teaching methods are to blame: students given only examples without borrowing, then only examples with borrowing. Never learn *conditions* for borrowing.

Additional rules: zeros

Condition		**Action**
`Nmin00`: M=N, S=0	$\longrightarrow$	result is 0
`0minNN`: M=0, S=N	$\longrightarrow$	result is N
`0minN0`: M=0, S=N	$\longrightarrow$	result is 0
`NminNN`: M=N, S=N	$\longrightarrow$	result is N

- Treated as additional production rules.
- Are these really procedural errors or arithmetic (factual) errors? Do students require more training in multi-column subtraction or arithmetic facts?

Summary

- Arithmetic (multicolumn subtraction) as example of a cognitive skill;
- Using general architecture of a production system, subtraction can be modeled using specific production rules;
- Missing rules lead to degraded behavior similar to patterns of student errors;
- Diagnosis: inferring which skills (and subskills) students have mastered (or failed to master);

References

Cooper, Richard P. 2002. *Modelling High-Level Cognitive Processes*. Lawrence Erlbaum Associates, Mahwah, NJ.

Young, R. M. and T. O'Shea. 1981. Errors in children's subtraction. *Cognitive Science* 5(2):153–177.