# Cognitive Modeling

## Lecture 5: Models of Arithmetic

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Reading: Cooper (2002, Ch. 3)





## Why study models of arithmetic?

Task is an example of a *cognitive skill* – acquired through conscious practice.

- driving (vs. walking)
- reading/writing (vs. understanding/speaking)

Model is an example of a *production system*.

- Often used to model cognitive skills.
- Useful in explaining the how humans perform the task correctly by integrating many smaller subskills.
- Failure of individual subskills may help explain systematic failures in main skill.





## Multi-column subtraction

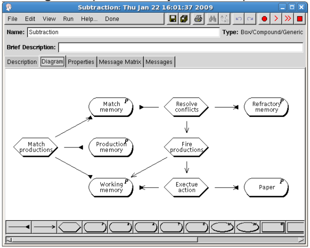
How do skilled students perform this task?
What types of errors are made by learners?

- random errors versus systematic errors.
- factual (arithmetic) errors versus procedural errors.
- incorrect subskills versus failure to apply subskills.

Young and O'Shea (1981) hypothesized that many errors are caused by failing to apply a sub-component of the skill.

# Basic architecture: Subtraction

Architecture is general production system, not specific to task:








# Basic architecture: Subtraction

- **Working memory**: holds current goals for task (multi-column subtraction) and subtasks (e.g. borrow).
- **Production memory**: holds production rules encoding *when* and *how* to perform subtasks (condition-action pairs) including arithmetic facts.
- **Match memory**: holds any production rules whose conditions are currently met.
- **Conflict resolution**: determines which rule in **Match memory** to fire.
- **Refractory memory**: keeps track of rules that have fired to prevent them firing again unless later reintroduced into **Match memory**.





# Comparison to ACT-R

- **Working memory**: similar to ACT-R Goal module.
- **Production memory**: combines ACT-R production system and Declarative module.
- **Match memory**: Similar to ACT-R Retrieval buffer.
- **Conflict resolution**: Here, based on recency. ACT-R: based on subsymbolic activation levels.





# Diagnosing Student Models

If teacher believes a student has a different model from their own (correct) one:

- make list (bug catalog) and match to it;
- reason about what student would have to believe in order to exhibit behavior indicating this.

*Student model:* representation of student's current state of knowledge.

*Diagnosis:* process of inferring the student model.

## Interlude: Skilled examples

First, we need to understand the skill children are learning (maybe not the way all of you learned).





## Examples of children's work

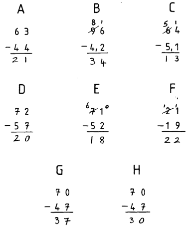



Figure 1. Examples of subtraction errors.

Figure from Young and O'Shea (1981)





## Problems with children's work

- A: always subtract smaller digit from larger.
- B: always borrow.
- C: both A and B.
- D: subtracting larger number from smaller equals zero.
- E: borrowing makes 10.
- F: add instead of subtract.
- G,H: errors only with subtracting from zero.

Note that only *patterns* of errors distinguish G,H from A,D.
Finding flaws in the underlying procedure (rather than specific errors) requires looking at multiple problems.





## Young and O'Shea's Model

Production rule model of multi-column subtraction:

- contains a fairly small number of simple production rules.
- children's errors are modeled by deleting production rules from a model that works correctly.
- accounts for a large percentage of errors found in practice.
- supports hypothesis that many errors arise from forgetting a sub-component of the skill.

## A Simple Production Rule Model

| Condition | | Action |
|---|---|---|
| S1: goal = process column & minuend greater than or equal to subtrahend | ⟶ | Take absolute difference of minuend and subtrahend and write in the answer space |
| S2: goal = process column & minuend less than subtrahend | ⟶ | Push goal 'borrow' onto stack |
| S3: goal = borrow | ⟶ | Decrement next minuend by 1, add 10 to current minuend and delete the current goal |





## Example

goal stack: [process column]

| | | | |
|---|---|---|---|
| | 4 | 9 | minuend |
| –1 | | 8 | subtrahend |
| | | | |
| | | * | |

*S1 is the only applicable production, so it fires.*

goal stack: [process column]

| | | | |
|---|---|---|---|
| | 4 | 9 | minuend |
| –1 | | 8 | subtrahend |
| | | 1 | |
| | | * | |

*Now S1 is still the only applicable production! We need a fix...*

* indicates current column





## A Revised Subtraction Model

| Condition | | Action |
|---|---|---|
| S1: goal = subtract & all answer spaces empty | ⟶ | Place marker on rightmost column & push goal 'process column' |
| S2: goal = process column & minuend greater than or equal to subtrahend | ⟶ | Take absolute difference of minuend and subtrahend and write in the answer space |
| S3: goal = process column & minuend less than subtrahend | ⟶ | Push goal 'borrow' onto stack |
| S4: goal = process column & answer space filled in | ⟶ | Move one column left |
| S5: goal = borrow | ⟶ | Decrement next minuend by 1, add 10 to current minuend and delete the current goal |





## Example

goal stack: [subtract]

| | | | |
|---|---|---|---|
| | 4 | 9 | minuend |
| –1 | | 8 | subtrahend |
| | | | |
| | | * | |

*S1 is the only applicable production, so it fires. The marker is placed, the new goal put on the stack and S2 fires.*

goal stack: [process column, subtract]

| | | | |
|---|---|---|---|
| | 4 | 9 | minuend |
| –1 | | 8 | subtrahend |
| | | 1 | |
| | | * | |

*S2 and S4 both satisfy the conditions but recency rules out S2.*

## Example

| | | | |
|---|---|---|---|
| | 4 | 9 | minuend |
| | −1 | 8 | subtrahend |
| | | 1 | |

*

subtract

goal stack

*S2's conditions are satisfied so it fires, then S4 will fire.*

| | | |
|---|---|---|
| 4 | 9 | minuend |
| −1 | 8 | subtrahend |
| 3 | 1 | |

*

process column
subtract

goal stack

*Now no rules are satisfied so the system halts.*





## Revised Model Reconsidered

| Condition | | Action |
|---|---|---|
| S1: goal = subtract & all answer spaces empty | ⟶ | Place marker on rightmost column & push goal 'process column' |
| S2: goal = process column & minuend greater than or equal to subtrahend | ⟶ | Take absolute difference of minuend and subtrahend and write in the answer space |
| S3: goal = process column & minuend less than subtrahend | ⟶ | Push goal 'borrow' onto stack |
| S4: goal = process column & answer space filled in | ⟶ | Move one column left |
| S5: goal = borrow | ⟶ | Decrement next minuend by 1, add 10 to current minuend and delete the current goal |

| | | |
|---|---|---|
| 4 | 9 | |
| −1 | 8 | |
| | | OK |

| | | | |
|---|---|---|---|
| 4 | 0 | 7 | |
| −1 | 0 | 8 | |
| | | | not OK |





## Young and O'Shea's rules

| Condition | | Action |
|---|---|---|
| `Init`: goal = subtract & all answer spaces empty | ⟶ | Place marker on rightmost column & push goal 'process column' |
| `Read`: goal = process column & no M or S in working memory | ⟶ | Read M and S |
| `Compare`: M and S in working memory | ⟶ | Compare M and S |
| `FindDiff`: M and S in working memory | ⟶ | push goal 'find difference', push goal 'next column' |
| `Borr2a`: M < S | ⟶ | Push goal 'borrow' |
| `BorrS1`: goal = borrow | ⟶ | Decrement next minuend by 1 |
| `BorrS2`: goal = borrow | ⟶ | Add 10 to current minuend |
| `AbsDiff`: goal = find difference | ⟶ | Take absolute difference between M and S as result |
| `Write`: result in working memory | ⟶ | Write result |
| `Next`: goal = process column & answer space filled in | ⟶ | Move one column left |
| `Carry`: result is (1,X) | ⟶ | Carry 1 and take X as result |





## Analysis of rules

- Why absolute difference?

  `AbsDiff`: goal = find difference ⟶ Take absolute difference between M and S as result

- What is the carry rule doing here?

  `Carry`: result is (1,X) ⟶ Carry 1 and take X as result

## Faulty Models

Leaving out specific rules leads to many common errors.

- `Compare`: M and S in working memory ⟶ Compare M and S. If missing, *take smaller from larger*.
- `BorrS1`: goal = borrow ⟶ Decrement next minuend by 1. If missing, *borrow freely, no payback*.

But not all:

- Always borrow.
- Zero errors.





## Additional rules: borrowing

Replace

`Borr2a`: M < S ⟶ Push goal 'borrow'

with one of these:

`Borr2b`: M > S ⟶ Push goal 'borrow'

`Borr1`: M and S in working memory ⟶ Push goal 'borrow'

- accounts for *always borrow* behavior.
- Young and O'Shea suggest teaching methods are to blame: students given only examples without borrowing, then only examples with borrowing. Never learn *conditions* for borrowing.





## Additional rules: zeros

| **Condition** | | **Action** |
|---|---|---|
| `Nmin00`: M=N, S=0 | ⟶ | result is 0 |
| `0minNN`: M=0, S=N | ⟶ | result is N |
| `0minN0`: M=0, S=N | ⟶ | result is 0 |
| `NminNN`: M=N, S=N | ⟶ | result is N |

- Treated as additional production rules.
- Are these really procedural errors or arithmetic (factual) errors? Do students require more training in multi-column subtraction or arithmetic facts?





## Summary

- Arithmetic (multicolumn subtraction) as example of a cognitive skill;
- Using general architecture of a production system, subtraction can be modeled using specific production rules;
- Missing rules lead to degraded behavior similar to patterns of student errors;
- Diagnosis: inferring which skills (and subskills) students have mastered (or failed to master);

# References

Cooper, Richard P. 2002. *Modelling High-Level Cognitive Processes*. Lawrence Erlbaum Associates, Mahwah, NJ.

Young, R. M. and T. O'Shea. 1981. Errors in children's subtraction. *Cognitive Science* 5(2):153–177.