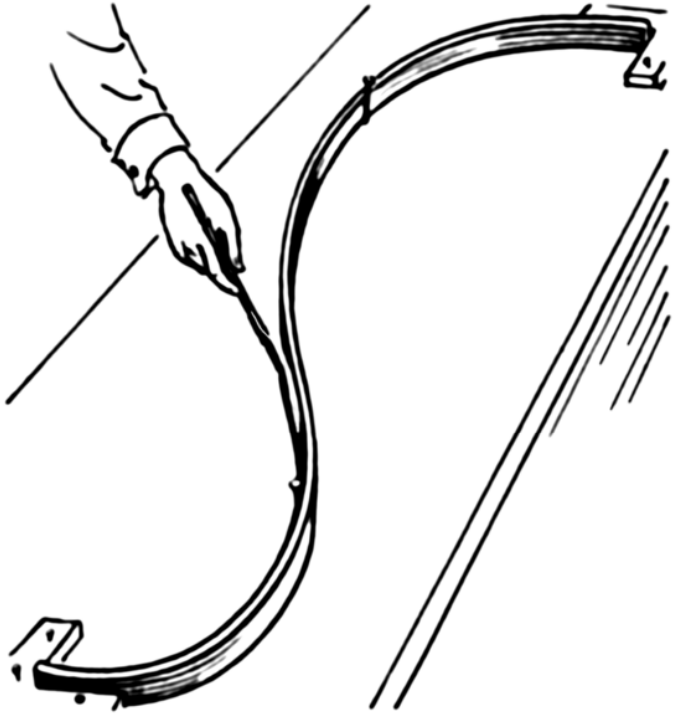
Computer Graphics

Lecture 16

Curves and Surfaces II

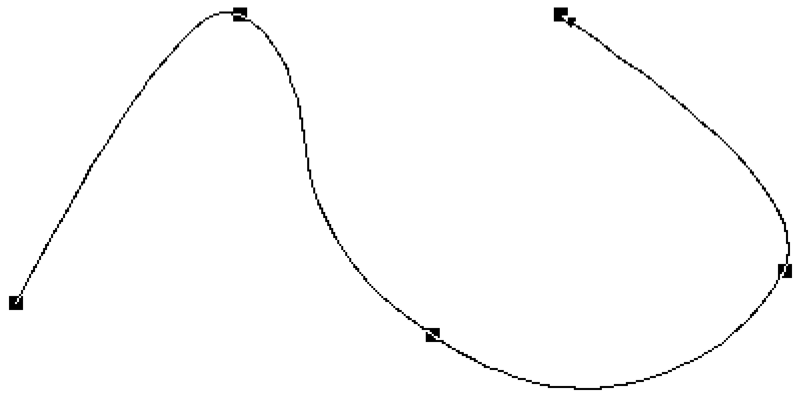
Spline

- A long flexible strips of metal used by draftspersons to lay out the surfaces of airplanes, cars and ships
- Ducks weights attached to the splines were used to pull the spline in differen directions
- The metal splines had second order continuity



Interpolating Splines

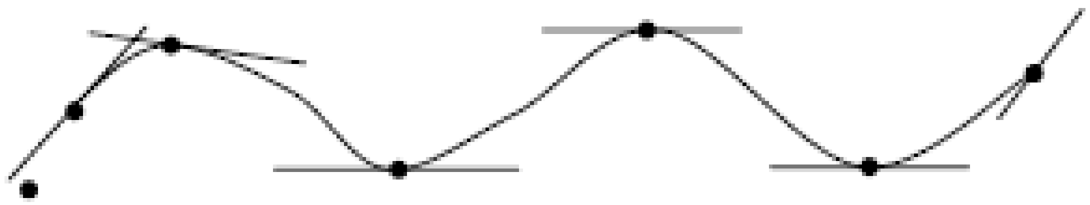
- When drawing a long curve with many control points, it will be convenient if the curve passes through the control curves



Catmull-Rom Spline

- Think of the Hermite curve
- We set the tangent vectors at the endpoints such that they are decided by the two surrounding control points

Hermite Specification

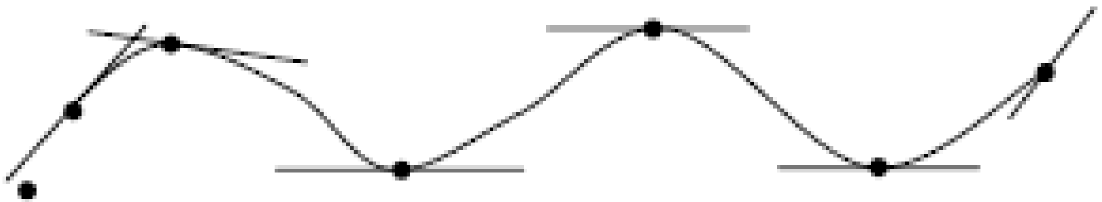


Catmull-Rom Spline

- Catmull-Rom spline interpolates control points. The gradient at each control point is the vector between adjacent control points.
- C1 continuity

$$P^i(t) = T \cdot M_{CR} \cdot G_B$$

$$= \frac{1}{2} \cdot T \cdot \begin{bmatrix} -1 & 3 & -3 & 1 \\ 2 & -5 & 4 & -1 \\ -1 & 0 & 1 & 0 \\ 0 & 2 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} P_{i-3} \\ P_{i-2} \\ P_{i-1} \\ P_i \end{bmatrix}$$

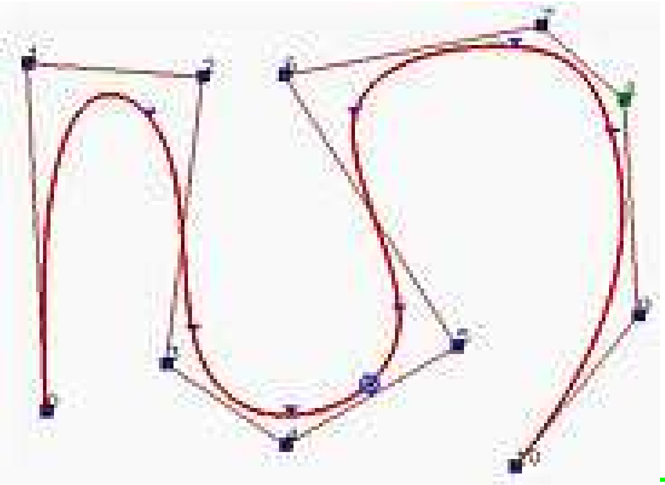


C2 continuity?

- What if we want C2 continuity
- For example when representing the trajectories of the body
- We may want to use the acceleration to compute the force
- The curve does not necessarily have to pass through the control points

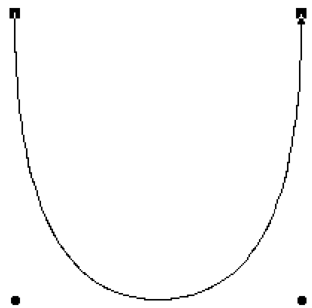
B-Splines (for basis splines)

- B-Splines
  - Another polynomial curve for modelling curves and surfaces
  - Consists of curve segments whose polynomial coefficients only depend on just a few control points
    - Local control
  - Segments joined at *knots*



B-splines

- The curve does not necessarily pass through the control points
- The shape is constrained to the convex hull made by the control points
- Uniform cubic b-splines has C_2 continuity
  - Higher than Hermite or Bezier curves



The basic one: Uniform Cubic B-Splines

- Cubic B-splines with uniform *knot-vector* is the most commonly used form of B-splines

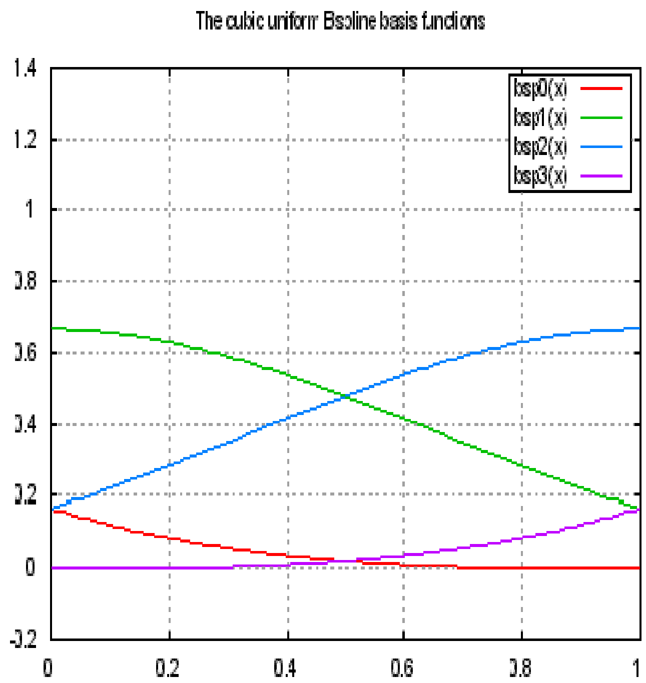
$$X(t) = \mathbf{t}^{\mathrm{T}}\mathbf{M}\mathbf{Q}^{(\mathbf{i})} \qquad for \quad t_i \le t \le t_{i+1}$$

$$where: \qquad \mathbf{Q}^{(\mathbf{i})} = (x_{i-3}, \dots, x_i)$$

$$\mathbf{M} = \frac{1}{6}\begin{bmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 0 & 3 & 0 \\ 1 & 4 & 1 & 0 \end{bmatrix},$$

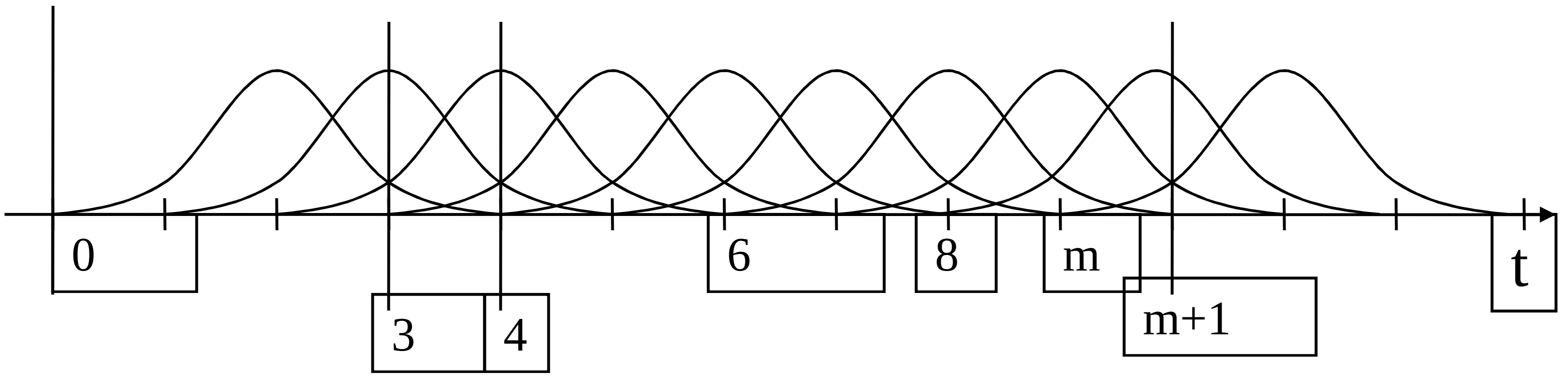
$$\mathbf{t}^{\mathrm{T}} = \left((t-t_i)^3, (t-t_i)^2, t-t_i, 1\right)$$

$$t_i : \text{knots}, \qquad 3 \le i$$

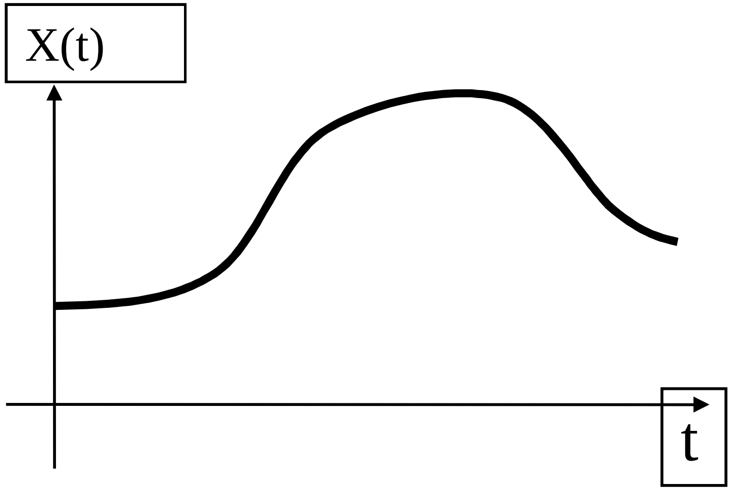


Longer curves

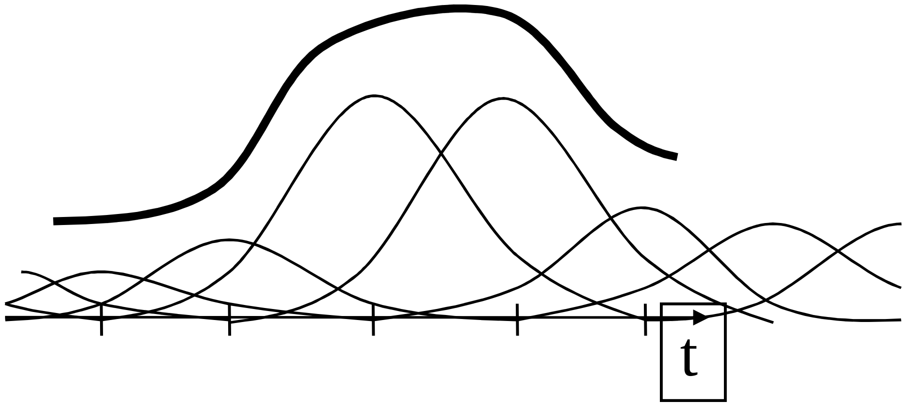
- We can have a list of control points and use the uniform cubic spline to define a long C2 continuous curve
- The unweighted cubic B-Splines have been shown for clarity.
- These are weighted and summed to produce a curve of the desired shape



Generating a curve



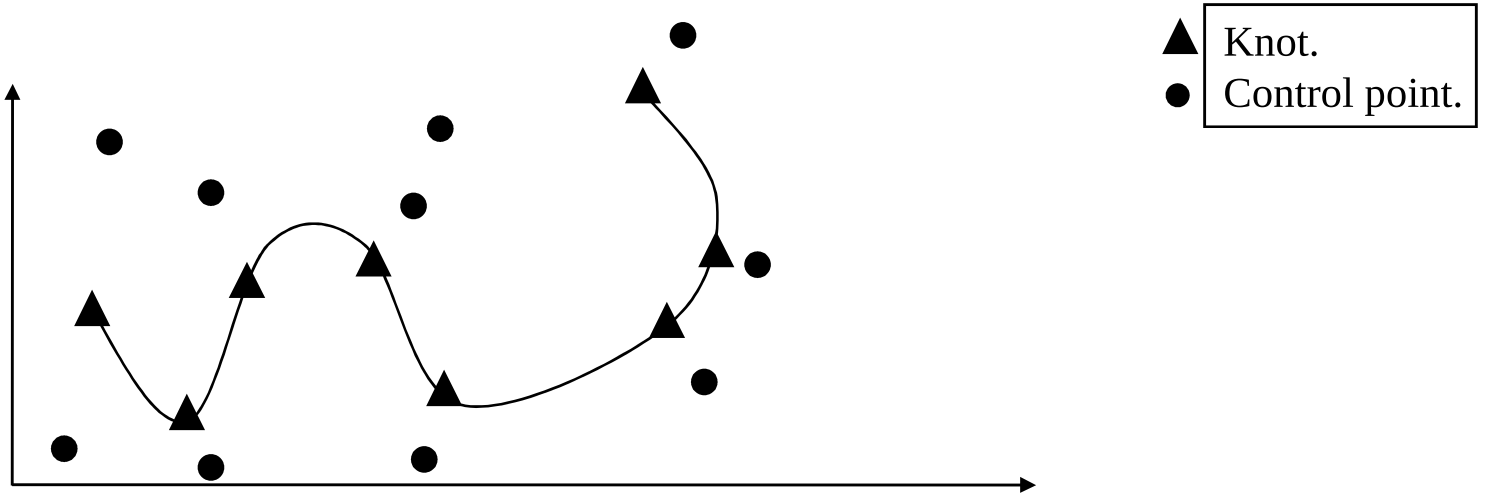
Opposite we see an example of a shape to be generated.



Here we see the curve again with the *weighted* B-Splines which generated the required shape.

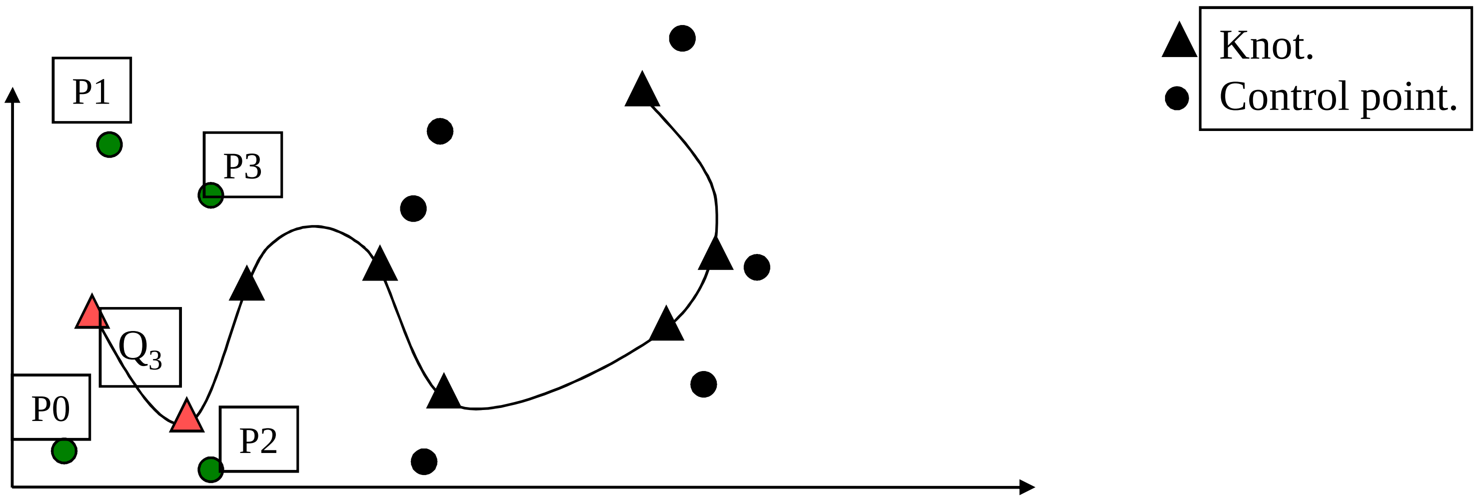
Cubic Uniform B-Spline 2D example

- For each $i \geq 4$, there is a knot between Q_{i-1} and Q_i at $t = t_i$.
- Initial points at t_3 and t_{m+1} are also knots. The following illustrates an example with control points set $P_0 \dots P_9$:



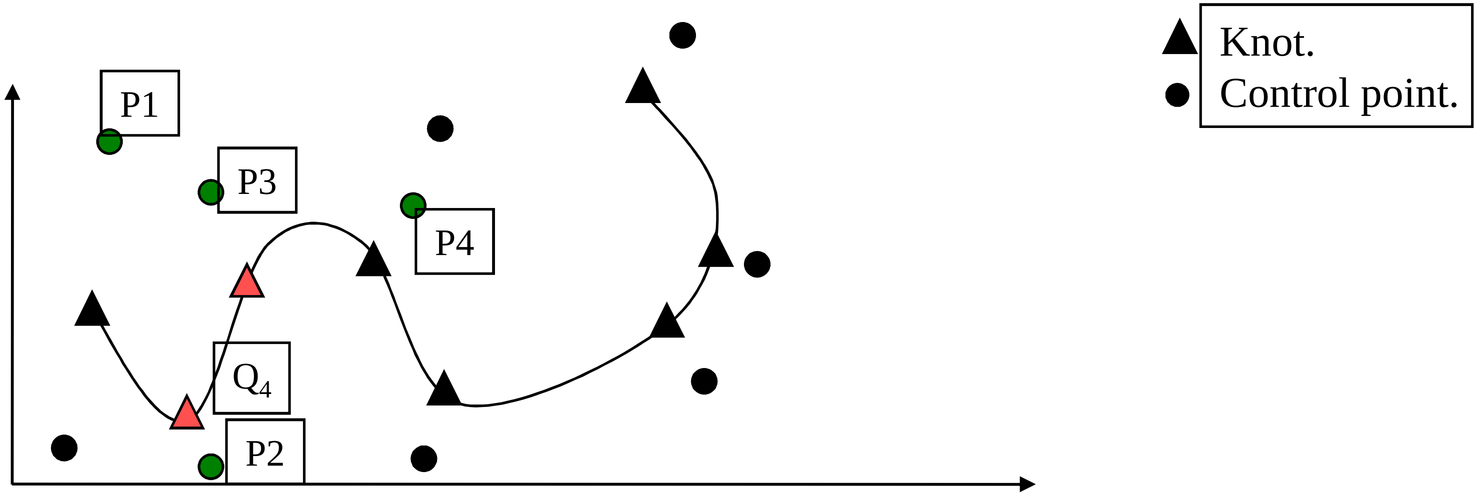
Uniform Non-rational B-Splines.

- First segment Q_3 is defined by point P_0 through P_3 over the range $t_3 = 0$ to $t_4 = 1$. So m at least 3 for cubic spline.



Uniform Non-rational B-Splines.

- Second segment Q_4 is defined by point P_1 through P_4 over the range $t_4 = 1$ to $t_5 = 2$.



An example of using a uniform cubic B-spline

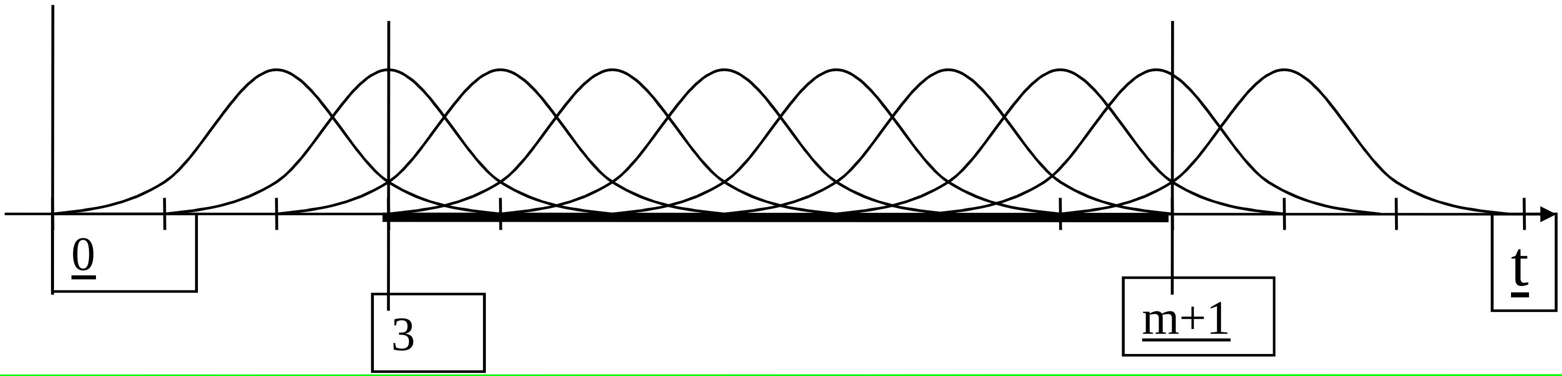
- Representing trajectories of characters
- Representing the joint angles of the characters
- Need more control points to represent a longer continuous movement

An example of using a uniform cubic B-spline

- We may need to compute the amount of torque produced at the joints
- Or the amount of force exerted at end-effectors
- Then, need a C2 continuous curve

Domain of the function

- Order k, Degree k-1
- Control points $\boldsymbol{P_i}$ **(**$i=0,\ldots,m$**)**
- Knots : t_j, ($j=0,\ldots, k + m$)
- The domain of the function $t_{k-1} \leqq t \leqq t_{m+1}$
  - Below, k = 4, m = 9, domain, $t_3 \leqq t \leqq t_{10}$



B-Spline : A more general definition

A Bspline of order *k* is a parametric curve composed of a linear combination of basis B-splines $B_{i,n}$

P_i ($i=0,\ldots,m$**)** the control points

Knots: t_j, $j=0,\ldots, k + m$

$$p(t) = \sum_{i=0}^{m} P_i B_{i,n}(t)$$

The B-spline basis functions can be defined recursively by

$$B_{i,1}(t) = \begin{cases} 1, \mathrm{t_i \le t < t_{i+1}} \\ 0, \text{otherwise} \end{cases}$$

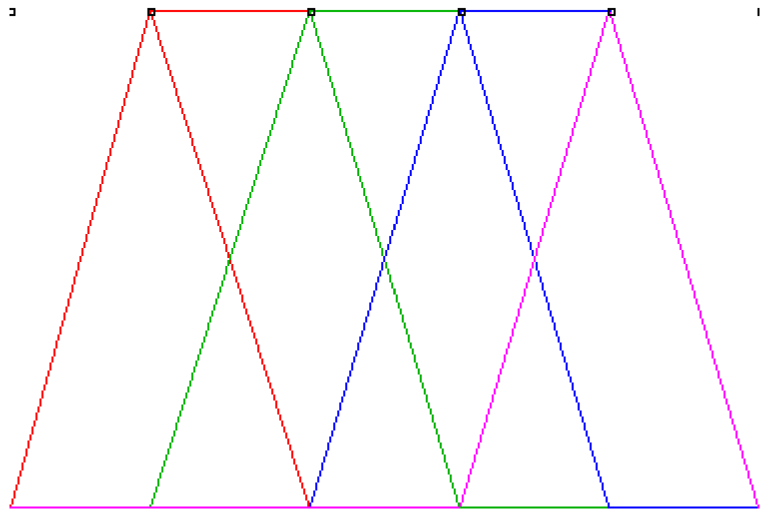
$$B_{i,k}(t) = \frac{t - t_i}{t_{i+k-1} - t_i} B_{i,k-1}(t) + \frac{t_{i+k} - t}{t_{i+k-1} - t_i} B_{i+1,k-1}(t)$$

The shape of the basis functions

$B_{i,2}$: linear basis functions

Order = 2, degree = 1

C0 continuous



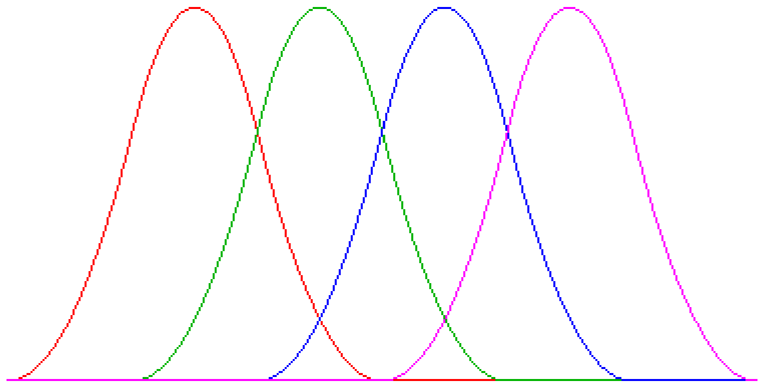
http://www.ibiblio.org/e-notes/Splines/Basis.htm

The shape of the basis functions

$B_{i,3}$: Quadratic basis functions

Order = 3, degree = 2

C1 continuous



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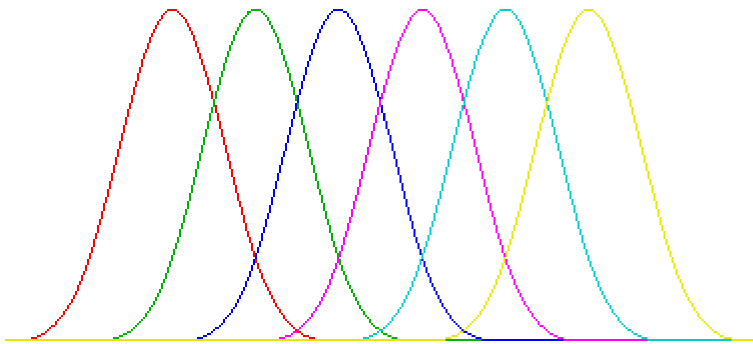
The shape of the basis functions

$B_{i,4}$: Cubic basis functions

Order = 4, degree = 3

C2 continuous



http://www.ibiblio.org/e-notes/Splines/Basis.htm

Uniform / non-uniform B-splines

- Uniform B-splines
  - The knots are equidistant / non-equidistant
  - The previous examples were uniform B-splines

$t_0, t_1, t_2, \ldots, t_m$ were equidistant, same interval

- Parametric interval between knots does not have to be equal.

$\longrightarrow$ Non-uniform B-splines

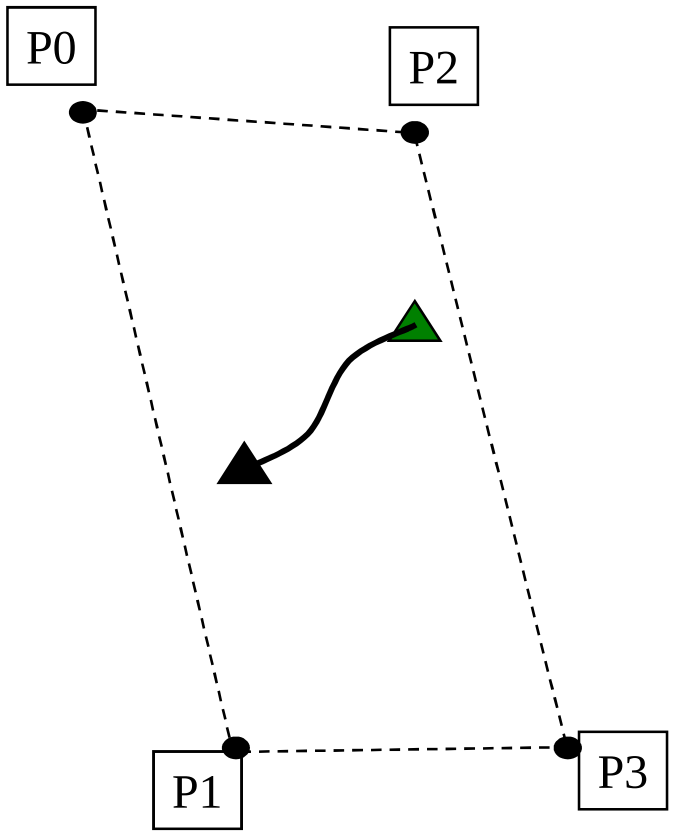
Non-uniform B-splines.

- Blending functions no longer the same for each interval.
- Advantages
  - Continuity at selected control points can be reduced to C_1 or lower – allows us to interpolate a control point without side-effects.
  - Can interpolate start and end points.
  - **Easy to add extra knots and control points.**
    - **Good for shape modelling !**

Controlling the shape of the curves

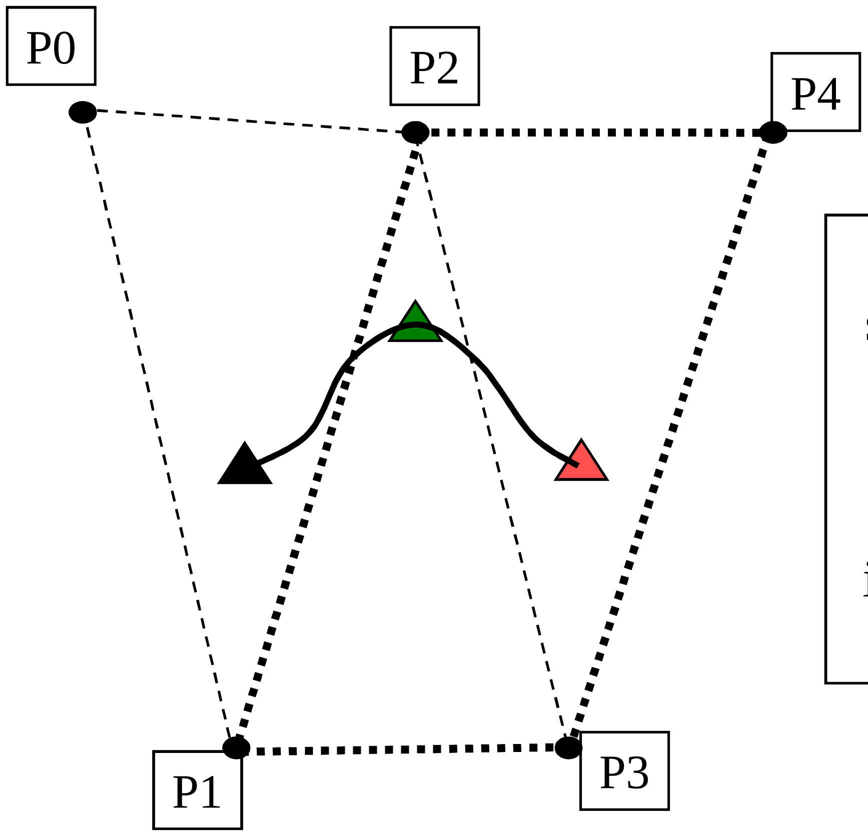
- Can control the shape through
  - Control points
    - Overlapping the control points to make it pass through a specific point
  - Knots
    - Changing the continuity by increasing the multiplicity at some knot (non-uniform bsplines)

Controlling the shape through control points



First knot shown with 4 control points, and their convex hull.

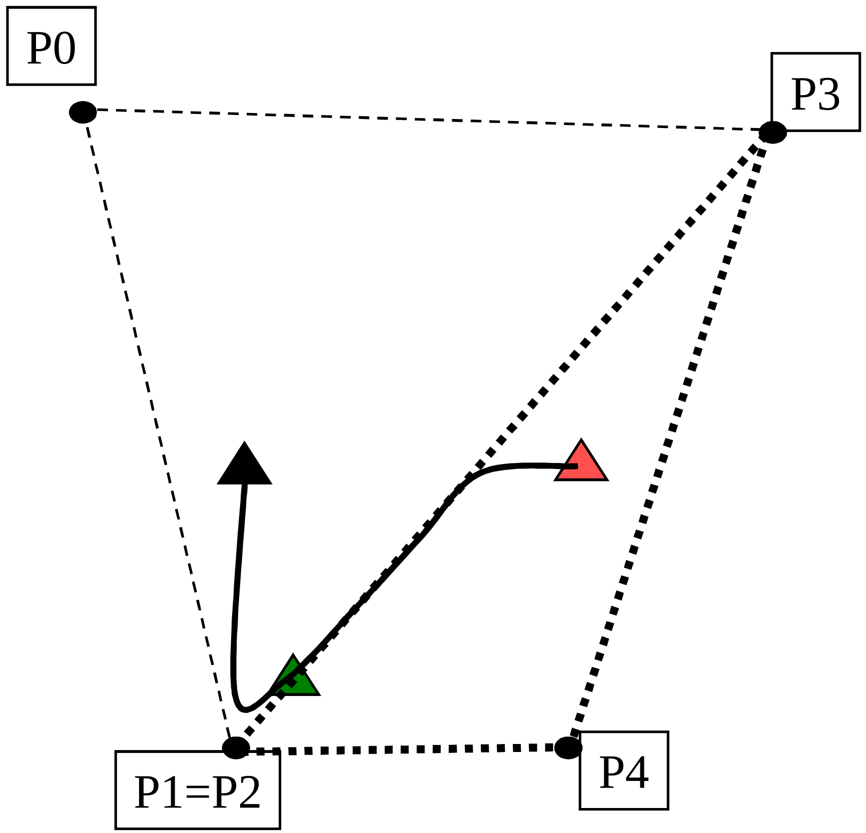
Controlling the shape through control points



First two curve segments shown with their respective convex hulls.

Centre Knot must lie in the intersection of the 2 convex hulls.

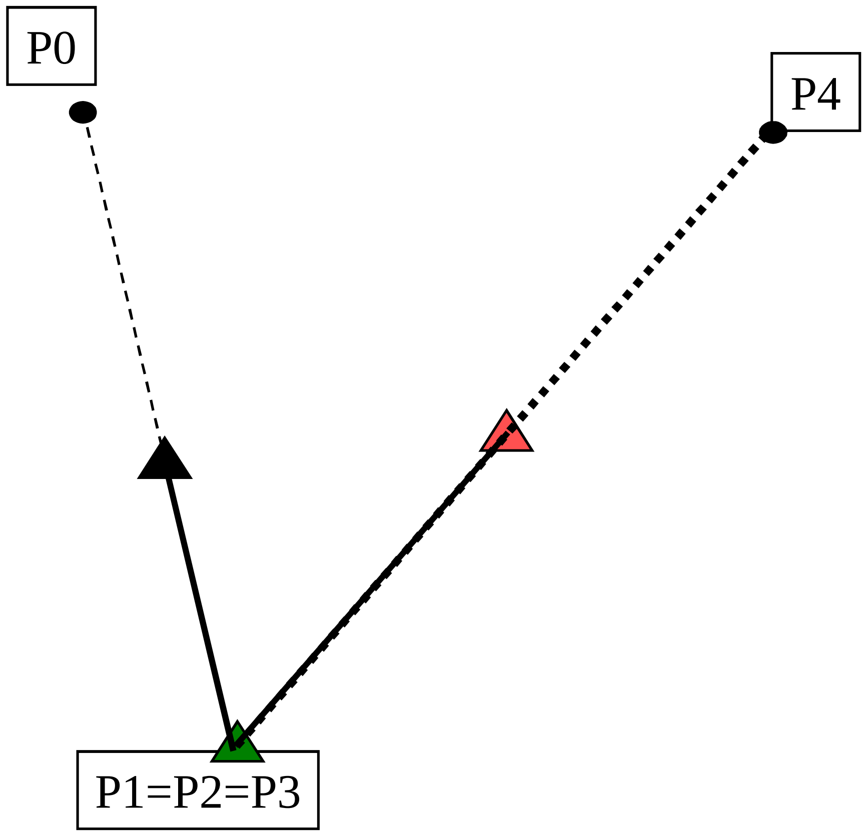
Repeated control point.



First two curve segments shown with their respective convex hulls.

The curve is forced to lie on the line that joins the 2 convex hulls.

Triple control point.



First two curve segments shown with their respective convex hulls.

Both convex hulls collapse to straight lines – all the curve must lie on these lines.

Controlling the shape through knots

- Smoothness increases with order *k* in $B_{i,k}$
  - Quadratic, $k = 3$, gives up to C_1 continuity.
  - Cubic, $k = 4$ gives up to C_2 continuity.
- However, we can lower continuity order too with *Multiple Knots*, ie. $t_i = t_{i+1} = t_{i+2} = \dots$ Knots are coincident and so now we have non-uniform knot intervals.
- A knot with multiplicity *p* is continuous to the *(k-1-p)th* derivative.
- A knot with multiplicity *k* has no continuity at all, i.e. the curve is broken at that knot.

$$B_{i,1}(t) = \begin{cases} 1, & t_i \le t < t_{i+1} \\ 0, & \text{otherwise} \end{cases}$$

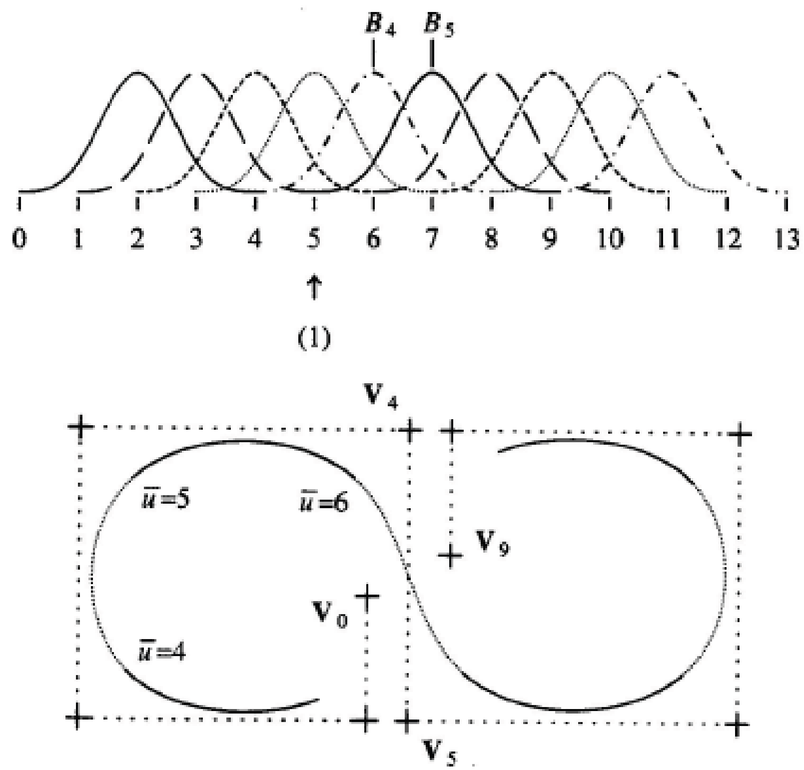
$$B_{i,k}(t) = \frac{t - t_i}{t_{i+k-1} - t_i} B_{i,k-1}(t) + \frac{t_{i+k} - t}{t_{i+k-1} - t_i} B_{i+1,k-1}(t)$$

B-Splines at multiple knots

- Cubic B-spline
- Multiplicities are indicated

Knot multiplicity

- Consider the uniform cubic (n=4) B-spline curve, t={0,1, … ,13}, m=9 , n=4, 7 segments



Knot multiplicity

- Double knot at 5,
- knot ={0,1,2,3,4,5,5,6,7,8,9,10,11,12}
- 6 segments, continuity = 1

Knot multiplicity

- Triple knot at 5
- knot={0,1,2,3,4,5,5,5,6,7,8,9,10,11}
- 5 segments

Knot multiplicity

- Quadruple knot at 5
- 4 segments

Summary of B-Splines.

- Functions that can be manipulated by a series of control points with C_2 continuity and local control.
- Don't pass through their control points, although can be forced.
- Uniform
  - Knots are equally spaced in t.
- Non-Uniform
  - Knots are unequally spaced
  - Allows addition of extra control points anywhere in the set.

Summary cont.

- Do not have to worry about the continuity at the join points
- For interactive curve modelling
  - B-Splines are very good.

Reading for this lecture

- Foley at al., Chapter 11, sections 11.2.3, 11.2.4, 11.2.9, 11.2.10, 11.3 and 11.5.
- Introductory text, Chapter 9, sections 9.2.4, 9.2.5, 9.2.7, 9.2.8 and 9.3.
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