Curves

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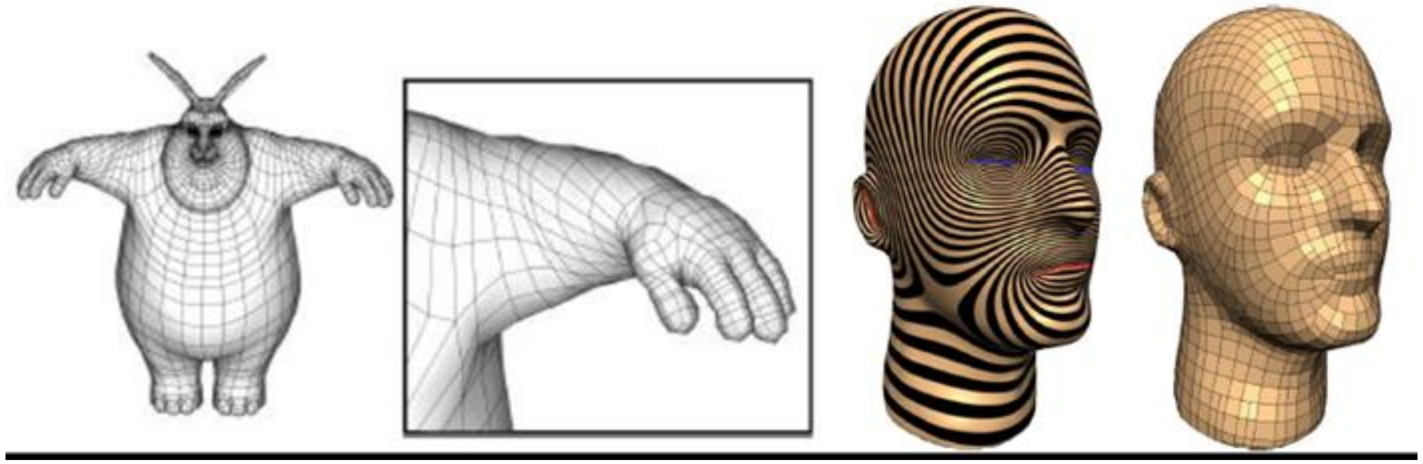
Some slides are courtesy of Steve Marschner and Taku Komura

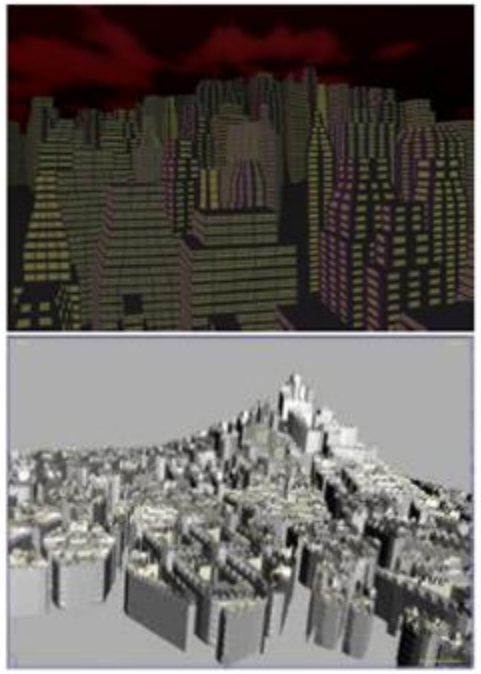
How to create a virtual world?

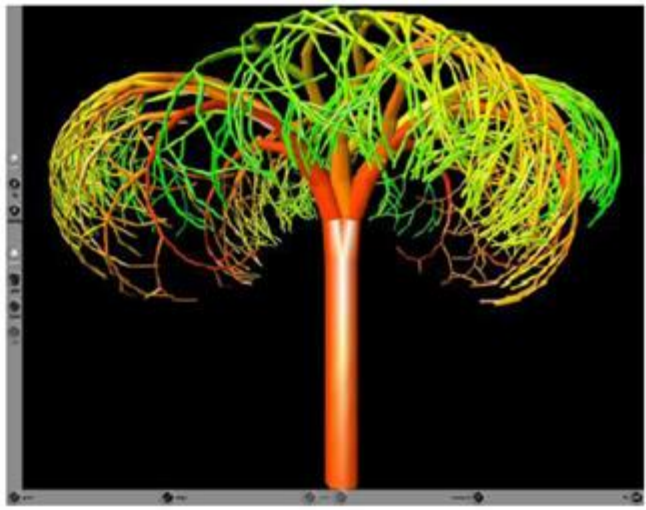
To compose scenes

We need to define objects

- Characters
- Terrains
- Objects (trees, furniture, buildings etc)

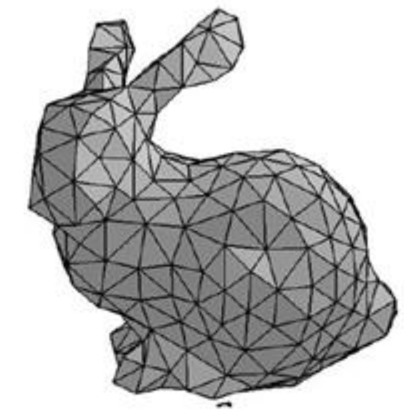


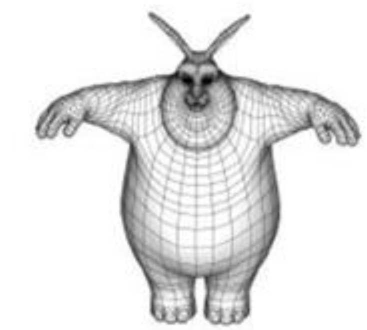




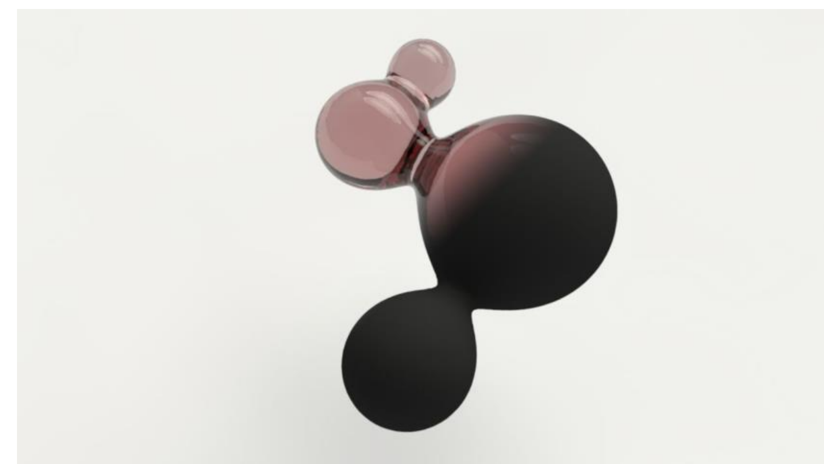
Geometric representations

- Meshes
    - Triangle, quadrilateral, polygon

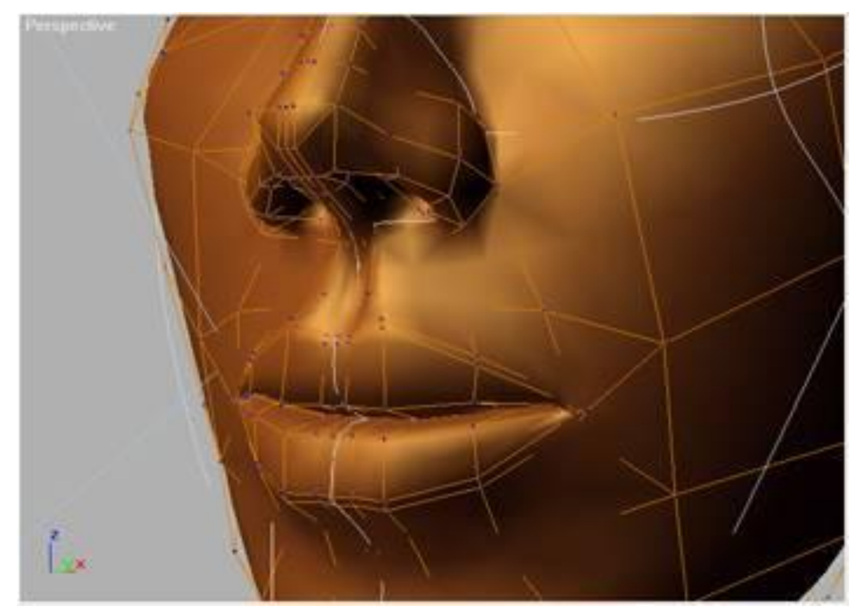




- Implicit surfaces
    - Blobs, metaballs



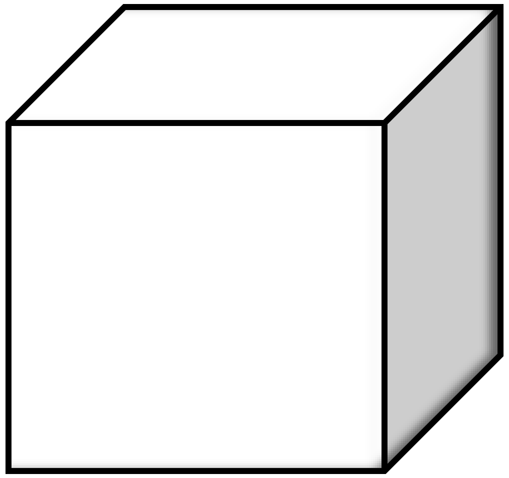
- Parametric surfaces / curves
    - Polynomials
    - Bezier curves, B-splines



Motivation

Smoothness

Many applications require smooth surfaces



[scene360.com]

Can produce smooth surfaces with less parameters

- Easier to design
- Can efficiently preserve complex structures

Original Spline

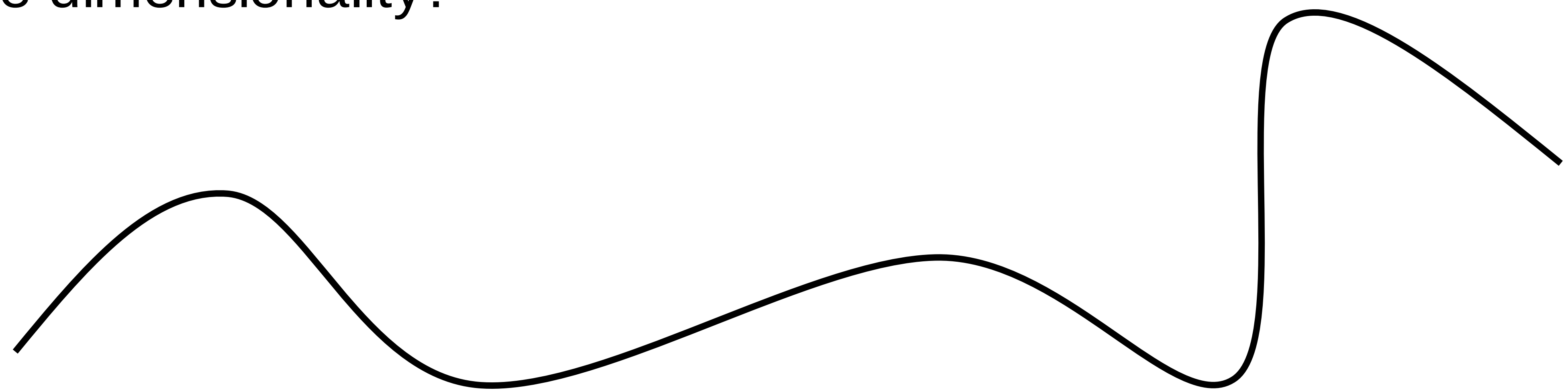
From draftsmanship to CG

- Control
  - user specified control points
  - analogy: ducks
- Smoothness
  - smooth functions
  - usually low order polynomials
  - analogy: physical constraints, optimization

What is a curve?

A set of points that the pen traces over an interval of time

What is the dimensionality?



Implicit form: $\mathrm{f}(\mathrm{x},\mathrm{y}) = x^2 + y^2 - 1 = 0$

- Find the points that satisfy the equation

Parametric form: $(\mathrm{x},\mathrm{y}) = \mathrm{f}(\mathrm{t}) = (cost, sint),\ \ t \in [0,2\pi)$

- Easier to draw

What is a spline curve?

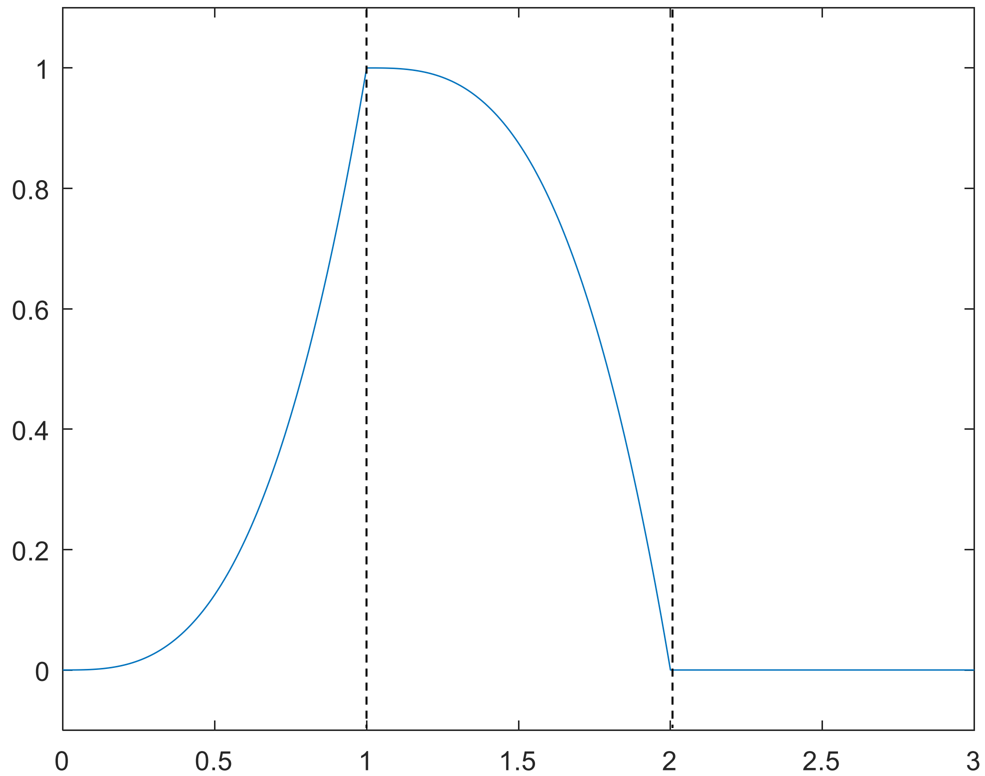
in this context

$f(t)$ is a

- parametric curve
- piecewise polynomial function that switches between different functions for different t intervals

Example:

$$f(t) = \begin{cases} t^3, & \text{if } 0 \leq t < 1 \\ 1-(t-1)^3, & \text{if } 1 \leq t < 2 \\ 0, & \text{otherwise} \end{cases}$$

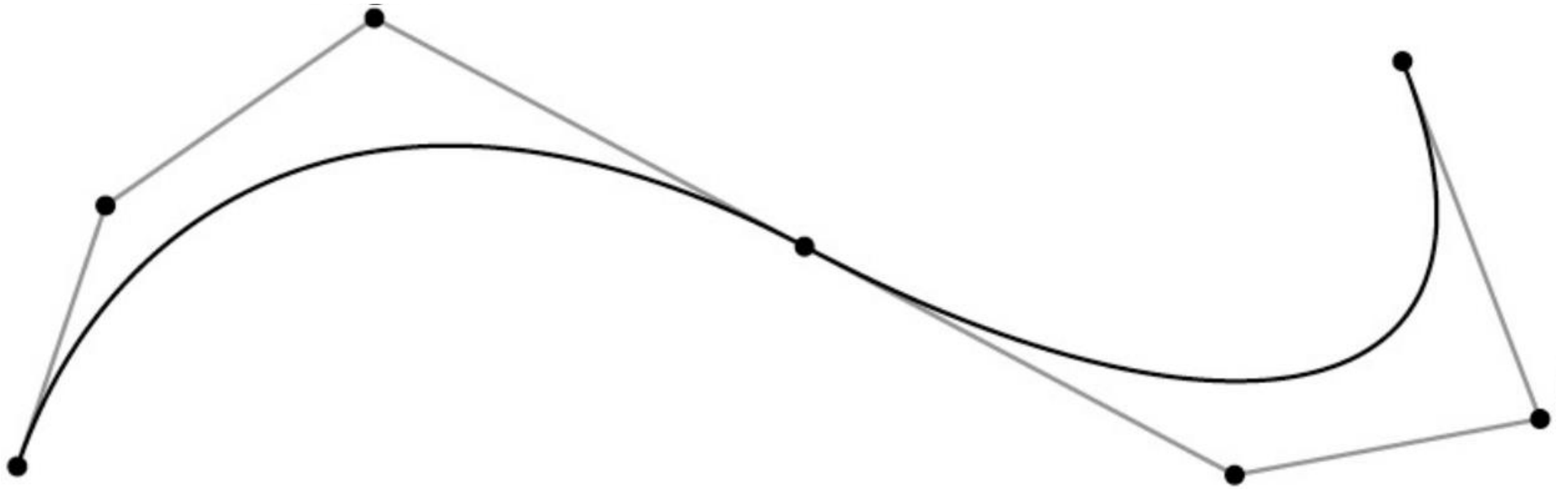


Defining spline curves

- Discontinuities at the integers [t=k]
- Each spline piece is defined over [k,k+1] (e.g. a cubic spline)

$$f(t) = at^3 + bt^2 + ct + d$$

- Different coefficients for every interval
- Control of spline curves
  - Interpolate
  - Approximate



Today

- Spline segments
  - Linear
  - Quadratic
  - Hermite
  - Bezier
- Chaining splines

- Notation
  - vectors bold and lowercase $\boldsymbol{v}$
  - points as column vector $\boldsymbol{p} = \begin{pmatrix} p_x & p_y \end{pmatrix}$
  - matrices bold and uppercase $\boldsymbol{M}$

Spline segments

Linear Segment

A line segment connecting point $\boldsymbol{p_o}$ to $\boldsymbol{p_1}$

Such that $f(0) = \boldsymbol{p_0}$ and $f(1) = \boldsymbol{p_1}$

$$f_x(t) = (1-t)\boldsymbol{x_o} + t\boldsymbol{x_1}$$

$$f_y(t) = (1-t)\boldsymbol{y_o} + t\boldsymbol{y_1}$$

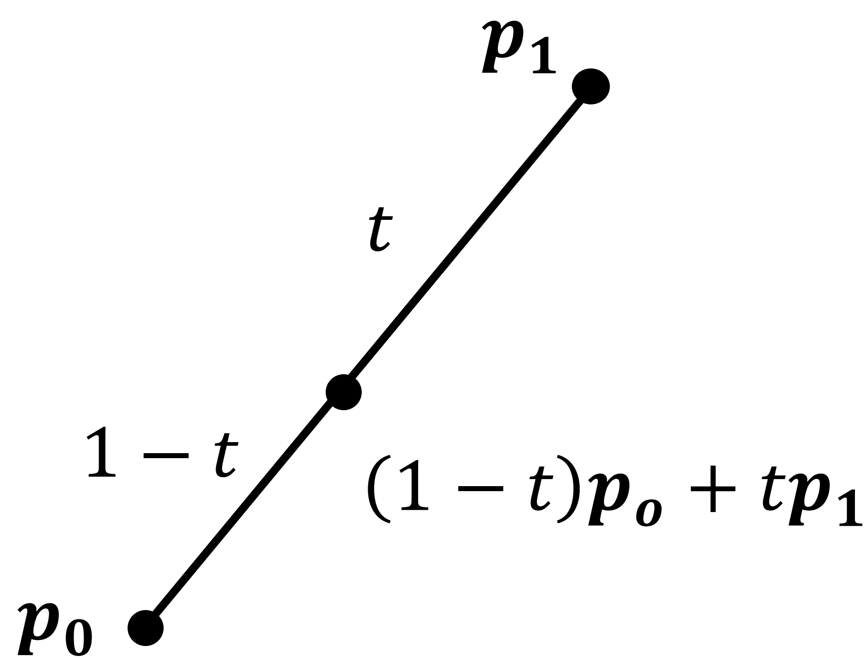
Vector formulation

$$f(t) = (1-t)\boldsymbol{p_o} + t\boldsymbol{p_1}$$

Matrix formulation

$$f(t) = \begin{pmatrix} t & 1 \end{pmatrix} \begin{pmatrix} ? & ? \\ ? & ? \end{pmatrix} \begin{pmatrix} \boldsymbol{p_0} \\ \boldsymbol{p_1} \end{pmatrix}$$

$$f(t) = \begin{pmatrix} t & 1 \end{pmatrix} \begin{pmatrix} -1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \boldsymbol{p_0} \\ \boldsymbol{p_1} \end{pmatrix}$$



Matrix form of spline

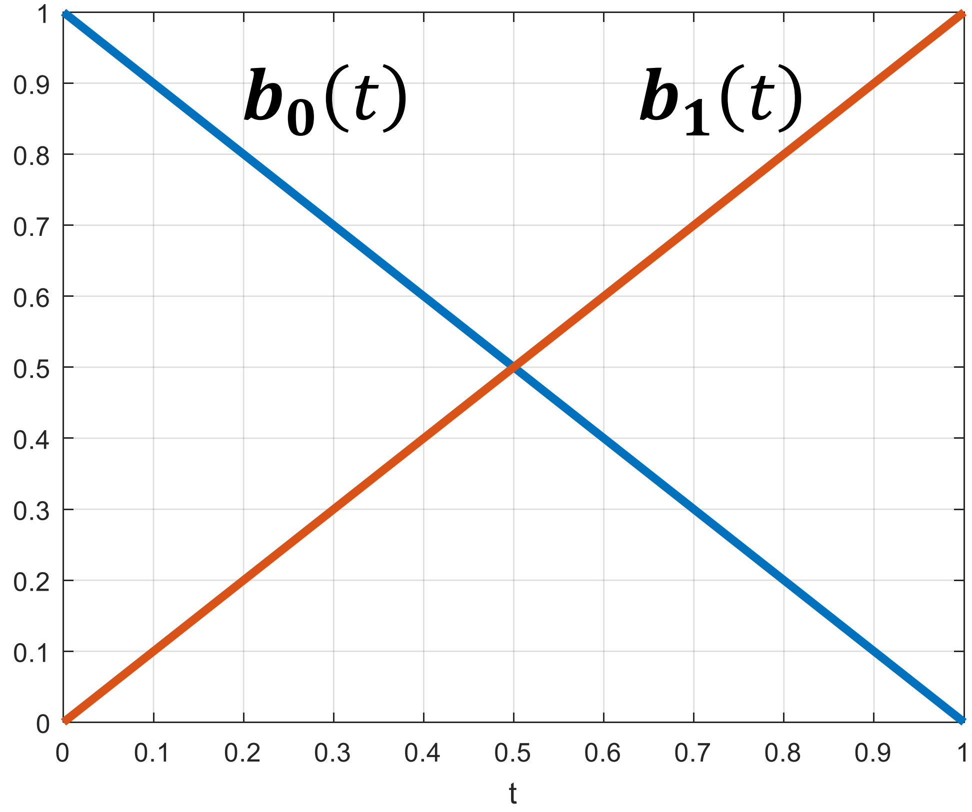
$$f(t) = \begin{pmatrix} t & 1 \end{pmatrix} \begin{pmatrix} -1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \boldsymbol{p_0} \\ \boldsymbol{p_1} \end{pmatrix}$$

Blending functions $\boldsymbol{b}(t)$ specify how to blend the values of the control point vector

$$\boldsymbol{f}(t) = \boldsymbol{b_0}(t)\boldsymbol{p_0} + \boldsymbol{b_1}(t)\boldsymbol{p_1}$$

$$\boldsymbol{b_0}(t) = 1 - t$$

$$\boldsymbol{b_1}(t) = t$$



Beyond line segment

Quadratic

A quadratic ($\boldsymbol{f}(t) = a_0 + a_1 t + a_2 t^2$) passes through $\boldsymbol{p_0}$, $\boldsymbol{p_1}$, $\boldsymbol{p_2}$ s.t.

$$\boldsymbol{p_0} = f(0) \quad = a_0 + 0 \quad a_1 + 0^2 \quad a_2$$

$$\boldsymbol{p_1} = f(0.5) = a_0 + 0.5\, a_1 + 0.5^2\, a_2$$

$$\boldsymbol{p_2} = f(1) \quad = a_0 + 1 \quad a_1 + 1^2 \quad a_2$$

Points can be written in terms of constraint matrix $\boldsymbol{C}$

$$\begin{pmatrix} \boldsymbol{p_0} \\ \boldsymbol{p_1} \\ \boldsymbol{p_2} \end{pmatrix} = \boldsymbol{Ca} \Rightarrow \begin{pmatrix} \boldsymbol{p_0} \\ \boldsymbol{p_1} \\ \boldsymbol{p_2} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0.5 & 0.25 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \end{pmatrix} \Rightarrow \begin{pmatrix} a_0 \\ a_1 \\ a_2 \end{pmatrix} = \boldsymbol{C^{-1}} \begin{pmatrix} \boldsymbol{p_0} \\ \boldsymbol{p_1} \\ \boldsymbol{p_2} \end{pmatrix}$$

$\boldsymbol{f}(t)$ can be written in terms of basis matrix $\boldsymbol{B} = \boldsymbol{C^{-1}}$ and points $\boldsymbol{p}$

$$\boldsymbol{f}(t) = \boldsymbol{tBp} = \boldsymbol{tC}^{-1}\boldsymbol{p} = (t^2 \;\; t \;\; 1) \begin{pmatrix} 2 & -4 & 2 \\ -3 & 4 & -1 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} \boldsymbol{p_0} \\ \boldsymbol{p_1} \\ \boldsymbol{p_2} \end{pmatrix}$$

Matrix form of spline

Blending functions

$$\boldsymbol{f}(t) = \begin{pmatrix} t^2 & t & 1 \end{pmatrix} \begin{pmatrix} 2 & -4 & 2 \\ -3 & 4 & -1 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} \boldsymbol{p_0} \\ \boldsymbol{p_1} \\ \boldsymbol{p_2} \end{pmatrix}$$

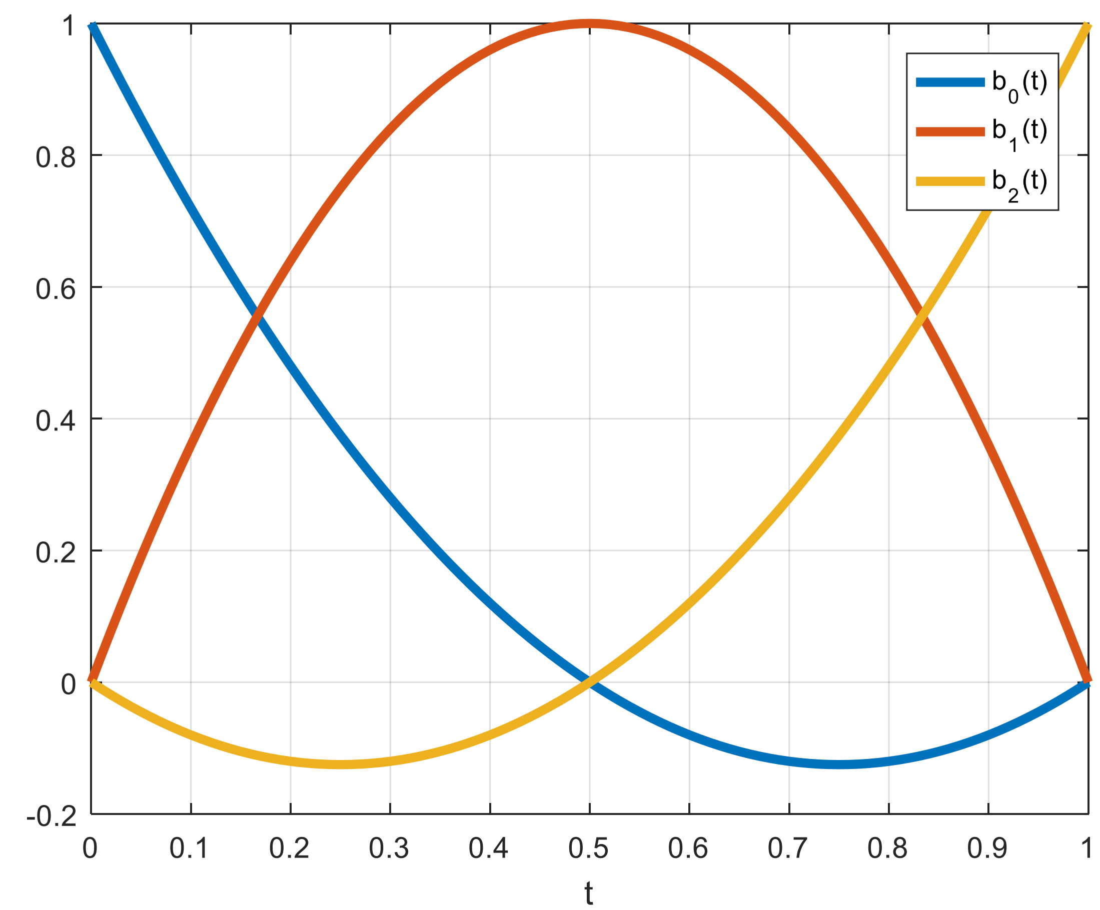
Blending functions $\boldsymbol{b}(t)$ specify how to blend the values of the control point vector

$$\boldsymbol{f}(t) = \boldsymbol{b_0}(t)\boldsymbol{p_0} + \boldsymbol{b_1}(t)\boldsymbol{p_1} + \boldsymbol{b_2}(t)\boldsymbol{p_2}$$

$$\boldsymbol{b_0}(t) = 2t^2 - 3t + 1$$

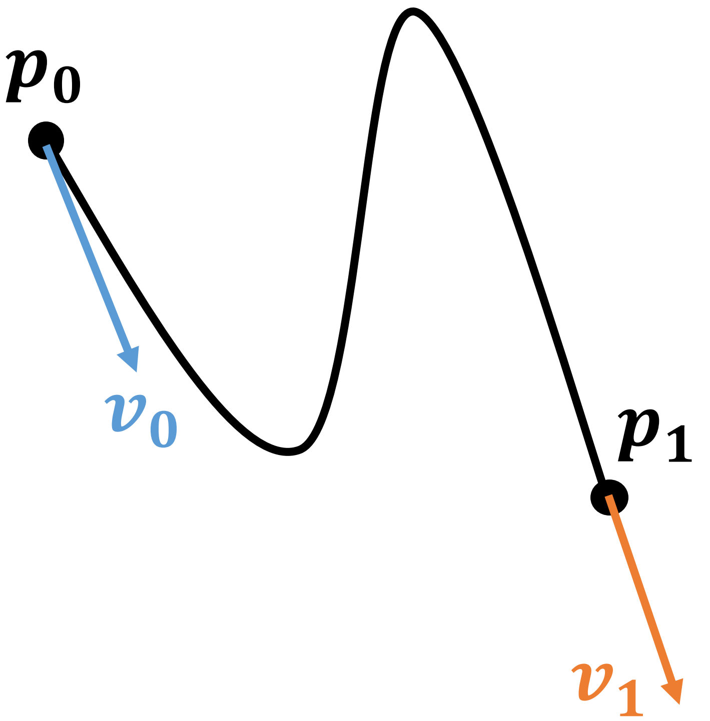
$$\boldsymbol{b_1}(t) = -4t^2 - 4t$$

$$\boldsymbol{b_2}(t) = 2t^2 - 1$$



Hermite spline

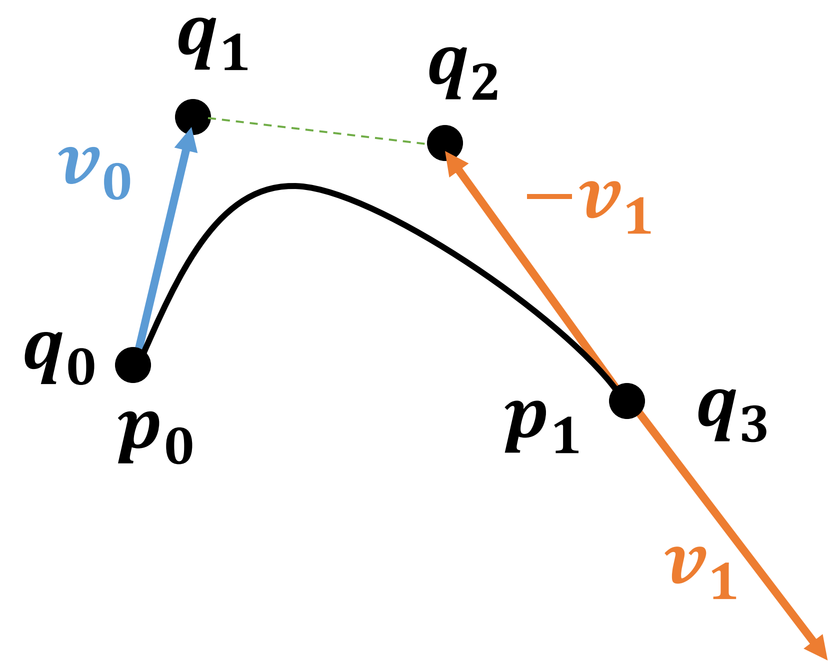
- Piecewise cubic ($\boldsymbol{f}(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3$)
- Additional constraint on tangents (derivatives)
  - $\boldsymbol{f}(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3$
  - $\boldsymbol{f}'^{(t)} = a_1 + 2a_2 t + 3a_3 t^2$
  - $\boldsymbol{p_0} = f(0) \quad = a_0$
  - $\boldsymbol{p_1} = f(1) \quad = a_0 + a_1 + a_2 + a_3$
  - $\boldsymbol{v}_1 = f'(0) \quad = a_1$
  - $\boldsymbol{v}_2 = f'(1) \quad = a_1 + 2a_2 + \ 3a_3$



- Simpler matrix form

$$\boldsymbol{f}(t) = (t^3 \ t^2 \ \ t \ \ 1)\begin{pmatrix} 2 & -2 & 1 & 1 \\ -3 & 3 & -2 & -1 \\ 0 & 0 & 1 & 0 \\ 2 & 0 & 0 & 2 \end{pmatrix}\begin{pmatrix} \boldsymbol{p_0} \\ \boldsymbol{p_1} \\ \boldsymbol{v_1} \\ \boldsymbol{v_2} \end{pmatrix}$$

Hermite to Bézier



Specify tangents as points

- $p_0 = q_0, p_1 = q_3, v_0 = 3(q_1 - q_0), v_1 = 3(q_3 - q_2)$
- $\begin{pmatrix} \boldsymbol{p_0} \\ \boldsymbol{p_1} \\ \boldsymbol{v_1} \\ \boldsymbol{v_2} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ -3 & 3 & 1 & 0 \\ 0 & 0 & -3 & 3 \end{pmatrix} \begin{pmatrix} \boldsymbol{q_0} \\ \boldsymbol{q_1} \\ \boldsymbol{q_2} \\ \boldsymbol{q_3} \end{pmatrix}$
- Update Hermite eq. (from previous slide)

$$\boldsymbol{f}(t) = (t^3 \; t^2 \; t \; 1) \begin{pmatrix} 2 & -2 & 1 & 1 \\ -3 & 3 & -2 & -1 \\ 0 & 0 & 1 & 0 \\ 2 & 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ -3 & 3 & 1 & 0 \\ 0 & 0 & -3 & 3 \end{pmatrix} \begin{pmatrix} \boldsymbol{q_0} \\ \boldsymbol{q_1} \\ \boldsymbol{q_2} \\ \boldsymbol{q_3} \end{pmatrix}$$

Bézier matrix

$$\boldsymbol{f}(t) = (t^3\ t^2\ \ t\ \ 1)\begin{pmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 3 & 0 & 0 \\ 2 & 0 & -6 & 6 \end{pmatrix}\begin{pmatrix} \boldsymbol{q_0} \\ \boldsymbol{q_1} \\ \boldsymbol{q_2} \\ \boldsymbol{q_3} \end{pmatrix}$$

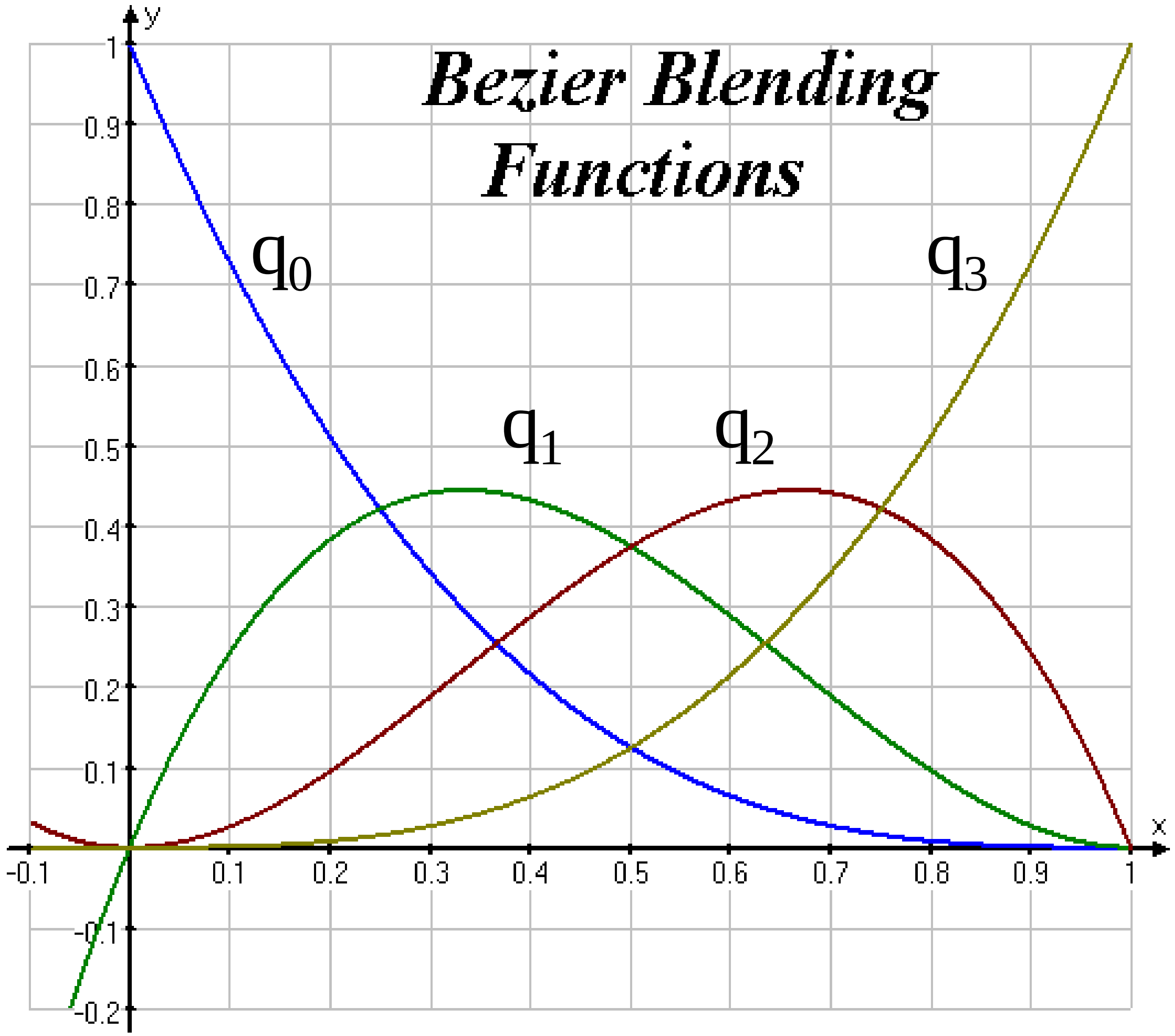
- $\boldsymbol{f}(t) = \sum_{n=0}^{d} \boldsymbol{b_{n,3}} \boldsymbol{q_n}$
- Blending functions $\boldsymbol{b}(t)$ has a special name in this case:
- Bernstein polynomials

$$b_{n,k} = \binom{n}{k} t^k (1-t)^{n-k}$$

and that defines Bézier curves for any degree

Bézier blending functions

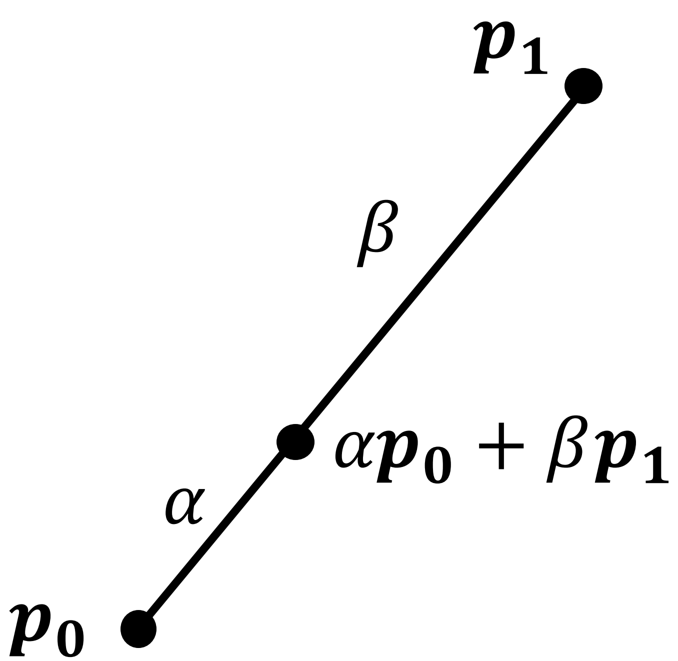
- The functions sum to 1 at any point along the curve.
- Endpoints have full weight

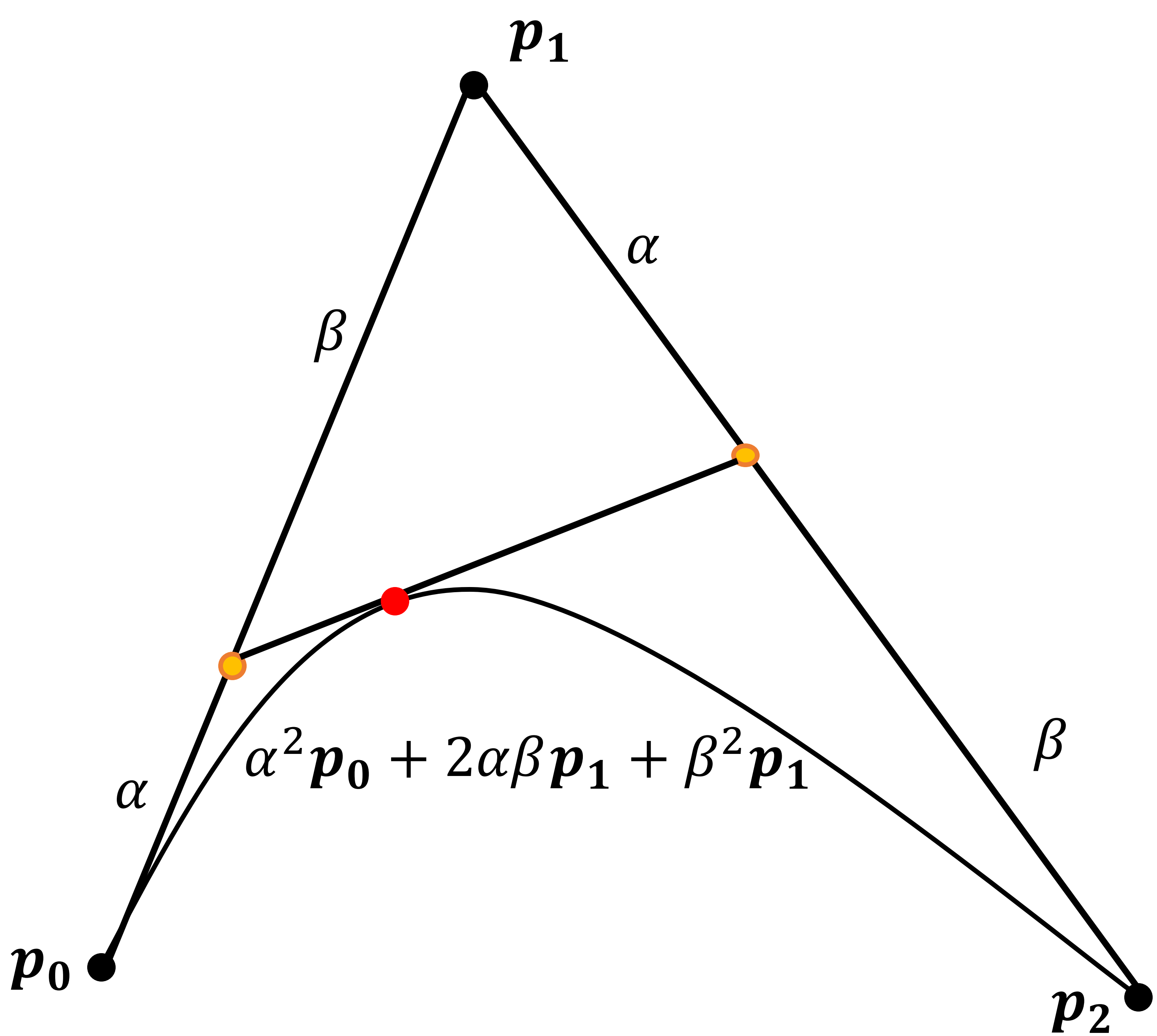


Another view to Bézier segments

de Casteljau algorithm

Blend each linear spline with α and $\beta = 1 - \alpha$





Review

http://www.inf.ed.ac.uk/teaching/courses/cg/d3/hermite.html

http://www.inf.ed.ac.uk/teaching/courses/cg/d3/bezier.html

http://www.inf.ed.ac.uk/teaching/courses/cg/d3/Casteljau.html