Computational Foundations of Cognitive Science

Lecture 17: Introduction to Probability Theory; Combinatorial Methods

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Reading: Freund, Chs. 1.1–1.3.

What is Probability Theory?

Probability theory deals with *combinatorics:*

- given a set of items, how many different orders are there?

Examples

How many possible three letter words are there in English?

A sentence can have a subject, a verb, and an object. In English, these occur in the order SVO. How many other orders are theoretically possible in other languages?

What is Probability Theory?

Probability theory deals with *prediction:*

- given an event has occurred, how likely is it that another event will occur?

Examples

Given that the first letter of a word is *k*, how likely is it that the next letter will be *s*?

Given that you've just heard the word *amok*, how likely is it that the previous word was *run*?

What is Probability Theory?

Probability theory deals with *inference:*

- given some prior knowledge about an event and some new evidence regarding the event, what can we infer?

Example

If a test to detect a disease whose prevalence is 1/1000 has a false-positive rate of 5%, what is the chance that a person found to have a positive result actually has the disease, assuming you know nothing about the person's symptoms or signs?

Example: Probability and Language

Probabilities in language processing:

- more probable words are recognized faster, produced more quickly;
- for ambiguous words, the more probable meaning is retrieved more quickly;
- for ambiguous sentences, the more probable reading is preferred over the less probable one;
- when speakers know the beginning of a sentence, they can predict the next word.

Example: Probability and Language

Probabilities in language acquisition:

- learners segment words into sounds by using probable sound combinations;
- learners acquire the meaning of a word by figuring out which other words it is likely to occur with;
- learners acquire the structure of sentences based on probable combination word categories.

Example: Probability and Reasoning

Probabilities in human reasoning and decision making:

- reasoning can be formalized using logic (e.g., $a \rightarrow b$ means a implies b);
- however, it turns out that this is not a very good model human reasoning, which often involves uncertain information;
- alternative: formalization in probabilistic terms (e.g., $P(a \rightarrow b)$ means a implies b with a certain probability);
- the probability of a rule can change with experience (i.e., depending on how often it has been applied);
- in general, human decision making can be viewed as a form of probabilistic inference (Bayesian inference).

Example: Probability and Memory

Probabilities in human memory:

- the probability of correctly recalling an item depends on amount of practice;
- the probability of forgetting an item depends on amount of time elapsed;
- items that occur more frequently are recalled more accurately and more quickly;
- items that stay in short term memory longer are more likely to be transfered to long term memory.

What is Combinatorics?

Before we move to probability theory, we need to introduce basic *combinatorics.*

Combinatorics is the *science of counting.* For a given set of elements, determine what arrangements of the elements are possible, and how many there are.

Useful for *probability theory:* the probability of a set often depends on how many different possibilities (combinations) of elements there are in the set.

Multiplications of Choices

Theorem: multiplication of choices

If an operation consists of k steps, of which the first step can be done in n_1 ways, the second step can be done in n_2 ways, etc., then the whole operation can be done in $n_1 \cdot n_2 \dots n_k$ ways.

Here, an *operation* can be any procedure, process, or method of selection.

Example

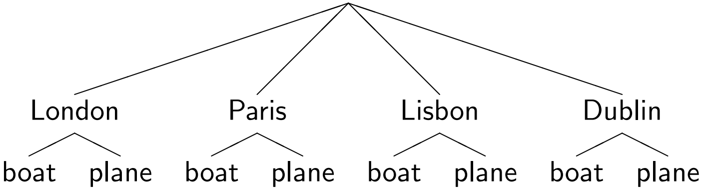
How many possible three letter words are there in English? There are 26 choices for the first letter, 26 choices for the second letter, and 26 choices for the third letter. The overall number of combinations is therefore $26 \cdot 26 \cdot 26 = 26^3 = 17,576$.

Multiplications of Choices

Example

Assume you want to travel to either London, Paris, Lisbon, or Dublin, by either boat or plane. Then there are $n_1 \cdot n_2 = 4 \cdot 2 = 8$ ways in which this can be done.

This can be visualized using a *tree diagram:*



Permutations

Example

A sentence can have a subject, a verb, and an object. In English, these occur in the order SVO. How many other orders are theoretically possible in other languages?

We assume that each of S, V, and O occur only once. For the first position in the sentence, we have three choices, for the second position, two choices, and for the third position, one choice. The total number of combinations is therefore $3 \cdot 2 \cdot 1 = 6$.

Permutations

This argument can be generalized. Assume a set of n objects. Then the number of possible orders is $n(n-1)(n-2)\dots 3 \cdot 2 \cdot 1 = n!$.

Theorem: permutations of distinct objects

The number of permutations of n distinct objects is $n!$.

Example

Assume a text consists of 10 sentences. A copy editor wants to re-order the text to improve its readability. He can choose from $10! = 3,628,800$ different orders.

Permutations

Theorem: permutations of distinct objects with grouping

The number of permutations of n distinct objects taken r at a time is (for $r = 0, 1, 2, \ldots, n$):

$$_nP_r = \frac{n!}{(n-r)!}$$

Example

In a game of cards, assume you have five cards, of which you select two. The number of ways this can be done is:

$$_5P_2 = \frac{5!}{(5-2)!} = \frac{5!}{3!} = 5 \cdot 4 = 20$$

Mid-lecture Problem

Assume four people play a board game, and they sit in a circle. In how many different ways can you arrange the players?

In general, how many permutations of n objects are there if the objects are arranged in a circle? We are looking for the number of *circular permutations.*

Permutations

So far we have assumed that the n objects from which we select r objects are all distinct. What happens, however, if we are dealing with identical objects?

Example

How many different permutations are there of the letters in the word *book*?

Naively, there are $4! = 24$ different permutations of b, o_1, o_2, and k. However, bo_1ko_2 and bo_2ko_1 are in fact the same word *boko*. Hence the total number of permutations of the letters is $\frac{24}{2} = 12$.

Permutations

When we generalize this reasoning, we arrive at the following theorem:

Theorem: permutations of identical objects

The number of permutations of n objects of which n_1 are of one kind, n_2 are of a second kind, ..., n_k are of a kth kind, and $n_1 + n_2 + \cdots + n_k = n$ is:

$$\frac{n!}{n_1! \cdot n_2! \dots n_k!}$$

Combinations

Often, we want to determine the number of ways in which r objects can be selected from among n distinct objects *without regard to order.* Such selections (arrangements) or called *combinations.*

Example

To run an experiment, we select 10 subjects from an undergraduate class of 25. If we care about the order in which the subjects are selected, then the number of possible selections is:

$$_{25}P_{10} = \frac{25!}{15!} = 1.186 \cdot 10^{13}$$

If we don't care about the order, then we have to divide this by 10!, i.e., number of different orders for 10 subjects:

$$\frac{_{25}P_{10}}{10!} = 3,268,760$$

Binomial Coefficients

Again, we can generalize this: we have ${}_nP_r$ permutations when we select r out of n objects, and $r!$ ways of ordering the r objects.

Theorem: combinations of distinct objects

The number of combinations of n distinct objects taken r at a time is (for $r = 0, 1, 2, \ldots, n$):

$$\binom{n}{r} = \frac{{}_nP_r}{r!} = \frac{n!}{r!(n-r)!}$$

Note that combinations are the same as *subsets:* we compute the total number of subsets of r objects that can be selected from a set of n distinct objects (sets are always *unordered*).

Binomial Coefficients

Example

We are tossing a coin six times. In how many different ways can this yield two heads and four tails?

We use the binomial coefficient to compute the number of ways in which we can select the two tosses that yield heads:

$$\binom{6}{2} = \frac{6!}{2!(6-2)!} = \frac{6!}{2! \cdot 4!} = 15$$

Binomial Coefficients

We can do arithmetic on binomial coefficients. Here are a few operations.

Theorem: rules for binomials

For any positive integers n and $r = 0, 1, 2, \ldots, n$:

$$\binom{n}{r} = \binom{n}{n-r}$$

For any positive integers n and $r = 1, 2, \ldots, (n-1)$:

$$\binom{n}{r} = \binom{n-1}{r} + \binom{n-1}{r-1}$$

Summary

- probability theory deals with prediction and inference;
- in cognitive science, probabilistic processes occur, e.g., in language, reasoning, and memory;
- combinatorics answers the question: given a set of items, how many different orders are there?
- the number of permutations of n objects is $n!$;
- if r objects are selected at a time, the number of permutations is ${}_nP_r$;
- the number of combinations (subsets) of r objects selected out of n is $\binom{n}{r}$.