Computational Foundations of Cognitive Science

Lecture 13: Determinants and Eigenvectors

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Reading: Anton and Busby, Chs. 4.1, 4.4

Determinants for 2×2 and 3×3 Matrices

Determinants determine if a matrix is invertible (Lecture 12). They are also important for eigenvectors. Recall:

Definition: Determinant of a 2×2 Matrix

The determinant of a 2×2 matrix A is given by:

$$\det(A) = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}$$

We can extend this to 3×3 matrices:

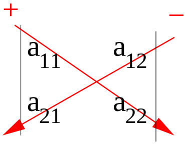
Definition: Determinant of a 3×3 Matrix

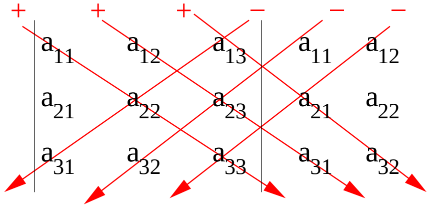
The determinant of a 3×3 matrix A is given by:

$$\det(A) = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32}$$
$$-a_{13}a_{22}a_{31} - a_{12}a_{21}a_{33} - a_{11}a_{23}a_{32}$$

Determinants for 2 × 2 and 3 × 3 Matrices

The following diagrams make these formulae easier to remember: the determinant consists of the products obtained by following the arrows, with the appropriate sign:





Example

$$\begin{vmatrix} 1 & 2 & 3 \\ -4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix} = 45 + 84 - 96 - 105 - 48 - (-72) = -48$$

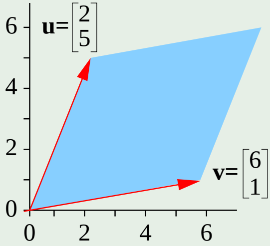
Geometric Interpretation of Determinants

Theorem: Geometric Interpretation of Determinants

If A is a 2×2 matrix, then abs(det(A)) represents the area of the parallelogram determined by the two column vectors of A when they are positioned so their initial points coincide.

Note: abs(x) is the absolute value of x.

Example



$A = \begin{bmatrix} 2 & 6 \\ 5 & 1 \end{bmatrix} = \begin{bmatrix} \mathbf{u} & \mathbf{v} \end{bmatrix}$. The area of the parallelogram determined by **u** and **v** is:

$\text{abs}(\det(A)) = \text{abs}(\begin{vmatrix} 2 & 6 \\ 5 & 1 \end{vmatrix}) =$

$\text{abs}(-28) = 28.$

General Determinants

This method only works for 2×2 and 3×3 matrices. For larger matrices, we need a more general definition.

An *elementary product* from an $n \times n$ matrix A is a product of n entries from A, where no two come from the same row or column:

$$\pm a_{1j_1} a_{2j_2} \cdots a_{nj_n}$$

Here, the column indices $\{j_1, j_2, \ldots, j_n\}$ form a permutation of the integers from 1 to n.

The sign of the elementary product is based on the minimum number of interchanges required to put the permutation in natural order. If it is even, then the sign is $+$, if it is odd, then $-$.

General Determinants

Definition: Determinant

The determinant of a square matrix A is denoted by $\det(A)$ and is defined to be sum of all elementary products from A:
$\det(A) = \sum \pm a_{1j_1} a_{2j_2} \cdots a_{nj_n}$

Computing determinants based on this definition is computationally intractable for large n, as the number of permutations is:

$$n! = n(n-1)(n-2)\cdots 1$$

This figure grows very quickly: $3! = 6$, $4! = 24$, $5! = 120$, $10! = 3,628,800$. More efficient algorithms are available, and are used in computational tools such as Matlab.

Determinants of Matrices with Special Form

Theorem: Determinants for Matrices with Zero Rows or Columns

If A is a square matrix with a row or a column of zeros, then $\det(A) = 0$.

Each elementary product has an entry from each row, therefore each product has a factor that is zero, hence the product is zero. So the sum of all elementary products is zero. Same for columns.

Theorem: Determinants for Triangular Matrices

If A is a triangular matrix, then $\det(A)$ is the product of the entries on the main diagonal.

The only elementary product that is not zero is $\pm a_{11}a_{22}\cdots a_{nn}$, which is the product of the entries on the main diagonal.

Determinants of Matrices with Special Form

Example

$$\begin{vmatrix} -2 & 5 & 7 \\ 0 & 3 & 8 \\ 0 & 0 & 5 \end{vmatrix} = (-2)\cdot 3 \cdot 5 = -30 \qquad \begin{vmatrix} 1 & 0 & 0 & 0 \\ 4 & 9 & 0 & 0 \\ -7 & 6 & -1 & 0 \\ 3 & 8 & -5 & -2 \end{vmatrix} =$$

$$1 \cdot 9(-1)(-2) = 18$$

A computationally more efficient way of computing an $n \times n$ determinant is *cofactor expansion*, which expresses this determinant in terms of $(n-1) \times (n-1)$ determinants.

Cofactor expansion can be applied recursively until we have a set of 2×2 determinants to compute. See Anton and Busby, Ch. 4.1.

Eigenvalues

Definition: Eigenvalues and Eigenvectors

If A is an $n \times n$ matrix, then a scalar λ is called an eigenvalue of A if there is a nonzero vector $\mathbf{x}$ (called eigenvector) such that $A\mathbf{x} = \lambda\mathbf{x}$.

We compute the eigenvalues of A by solving the *characteristic equation* of A: $\det(\lambda I - A) = 0$

Example

To find the eigenvalues of $A = \begin{bmatrix} 1 & 3 \\ 4 & 2 \end{bmatrix}$, first compute:

$\lambda I - A = \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 3 \\ 4 & 2 \end{bmatrix} = \begin{bmatrix} \lambda - 1 & -3 \\ -4 & \lambda - 2 \end{bmatrix}$. Then the characteristic equation is: $\begin{vmatrix} \lambda - 1 & -3 \\ -4 & \lambda - 2 \end{vmatrix} = 0$, which yields: $\lambda^2 - 3\lambda - 10 = 0$, which has the solutions $\lambda = -2$ and $\lambda = 5$.

Eigenvectors

Once we have the eigenvalues, we can compute the eigenvectors by solving the equation $A\mathbf{x} = \lambda\mathbf{x}$, equivalent to: $(\lambda I - A)\mathbf{x} = \mathbf{0}$.

Example

For the previous example, the equation $(\lambda I - A)\mathbf{x} = \mathbf{0}$ is:
$\begin{bmatrix} \lambda - 1 & -3 \\ -4 & \lambda - 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. We substitute $\lambda = -2$ and obtain:
$\begin{bmatrix} -3 & -3 \\ -4 & -4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. Hence we need to solve the following system of linear equations: $\begin{matrix} -3x & + & -3y & = & 0 \\ -4x & + & -4y & = & 0 \end{matrix}$, which yields the solutions $x = -t$ and $y = t$, hence the eigenvector is $\begin{bmatrix} -t \\ t \end{bmatrix}$. The eigenvector for $\lambda = 5$ is $\begin{bmatrix} \frac{3}{4}t \\ t \end{bmatrix}$ (details omitted).

Eigenvectors

By definition, the eigenvalues and eigenvectors of A make the following equation true: $A\mathbf{x} = \lambda\mathbf{x}$.

We can verify this by substituting $\mathbf{x}$ into this equation.

Example

First eigenvector of A:

$$A\mathbf{x} = \begin{bmatrix} 1 & 3 \\ 4 & 2 \end{bmatrix} \begin{bmatrix} -t \\ t \end{bmatrix} = \begin{bmatrix} 2t \\ -2t \end{bmatrix} = -2 \begin{bmatrix} -t \\ t \end{bmatrix} = -2\mathbf{x}$$

Second eigenvector of A:

$$A\mathbf{x} = \begin{bmatrix} 1 & 3 \\ 4 & 2 \end{bmatrix} \begin{bmatrix} \frac{3}{4}t \\ t \end{bmatrix} = \begin{bmatrix} \frac{15}{4}t \\ 5t \end{bmatrix} = 5 \begin{bmatrix} \frac{3}{4}t \\ t \end{bmatrix} = 5\mathbf{x}$$

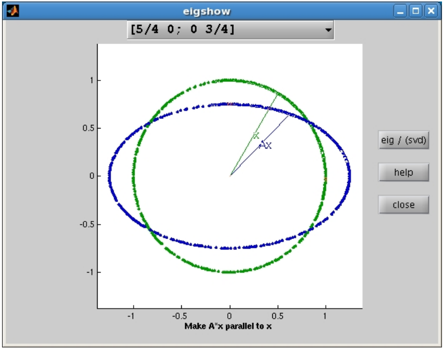
Mid-lecture Problem

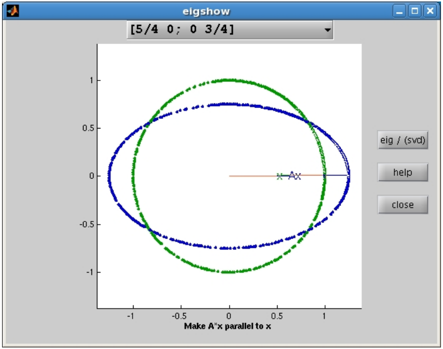
Suppose $A = \begin{bmatrix} 2 & 2 \\ -1 & 5 \end{bmatrix}$ is a matrix with the eigenvector $\mathbf{x} = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$ and the corresponding eigenvalue $\lambda = 4$. Verify that $\lambda^2 = 16$ is an eigenvalue of $A^2 = AA$, for the same eigenvector.

Is this a general property? In other words, if λ is an eigenvalue of A, then is λ^2 always an eigenvalue of A^2?

Eigenvectors

Let's use Matlab's `eigshow` to find the eigenvectors of $A = \begin{bmatrix} \frac{5}{4} & 0 \\ 0 & \frac{3}{4} \end{bmatrix}$:





If $\mathbf{x}$ is parallel to $A\mathbf{x}$, then $\mathbf{x}$ is an eigenvector. Here: $\mathbf{x} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\lambda = \frac{5}{4}$.

Eigenvalues of Triangular Matrices

If A is a triangular matrix with the diagonal $a_{11}, a_{22}, \ldots, a_{nn}$, then $\lambda I - A$ is triangular with diagonal $\lambda - a_{11}, \lambda - a_{22}, \ldots, \lambda - a_{nn}$. The characteristic polynomial of A is:

$$\det(\lambda I - A) = (\lambda - a_{11})(\lambda - a_{22}) \cdots (\lambda - a_{nn})$$

which implies that the eigenvalues of A are:

$$\lambda_1 = a_{11}, \lambda_2 = a_{22}, \ldots, \lambda_n = a_{nn}$$

Theorem: Eigenvalues of Triangular Matrices

If A is a triangular matrix (upper triangular, lower triangular, or diagonal), then the eigenvalues of A are the entries on the main diagonal of A.

Example: on previous slide.

Eigenvalues of 2 × 2 Matrices

The characteristic equation of a 2 × 2 matrix is:

$$\det(\lambda I - \begin{bmatrix} a & b \\ c & d \end{bmatrix}) = 0$$

This can be written as:

$$\lambda^2 - \text{tr}(A)\lambda + \det(A) = 0$$

because we can make the following simplification:

$$\det(\lambda I - \begin{bmatrix} a & b \\ c & d \end{bmatrix}) = \begin{vmatrix} \lambda - a & -b \\ -c & \lambda - d \end{vmatrix} = (\lambda - a)(\lambda - d) - bc =$$
$$\lambda^2 - (a + d)\lambda + (ad - bc) = \lambda^2 - \text{tr}(A)\lambda + \det(A)$$

Eigenvalues of 2×2 Matrices

From algebra we know that a quadratic equation $ax^2 + bx + c = 0$ has two solutions if $b^2 - 4ac > 0$, one solution if $b^2 - 4ac = 0$, and no solutions if $b^2 - 4ac < 0$.

We set $a = 1$, $b = -\text{tr}(A)$, and $c = \det(A)$ and obtain:

Theorem: Eigenvalues of 2×2 Matrices

If A is a 2×2 matrix, then the characteristic equation of A is $\lambda^2 - \text{tr}(A)\lambda + \det(A) = 0$ and

- A has two eigenvalues if $\text{tr}(A)^2 - 4\det(A) > 0$
- A has one eigenvalue if $\text{tr}(A)^2 - 4\det(A) = 0$
- A has no eigenvalues if $\text{tr}(A)^2 - 4\det(A) < 0$

Note: in the case of $\text{tr}(A)^2 - 4\det(A) < 0$, A has imaginary eigenvalues. We will ignore this.

Eigenvalues of 2×2 Matrices

Example

Find the eigenvalues of $A = \begin{bmatrix} 2 & 2 \\ -1 & 5 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & -1 \\ 1 & 2 \end{bmatrix}$.

We have $\text{tr}(A) = 7$ and $\det(A) = 12$, so the characteristic equation of A is $\lambda^2 - 7\lambda + 12 = 0$. We solve this using the quadratic formula and get: $\lambda = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{7 \pm \sqrt{(-7)^2 - 4 \cdot 12}}{2}$, so $\lambda = 4$ and $\lambda = 3$.

We have $\text{tr}(B) = 2$ and $\det(B) = 1$ and a characteristic equation $\lambda^2 - 2\lambda + 1 = 0$, which yields $\lambda = \frac{2 \pm \sqrt{(-2)^2 - 4}}{2} = 1$.

Summary

- There are formulae for computing determinants for 2×2 and 3×3 matrices;
- elementary products can be used to compute general determinants (but: very inefficient);
- $\det(A) = 0$ if A has a row or column of zeros;
- $\det(A) = a_{11}a_{22} \cdots a_{nn}$ if A is triangular;
- If $A\mathbf{x} = \lambda\mathbf{x}$, then λ is an eigenvalue and $\mathbf{x}$ an eigenvector;
- they can be computed by solving the characteristic equation of A, $\det(\lambda I - A) = 0$;
- the eigenvalues of a triangular matrix are the entries on its diagonal;
- the characteristic equation of a 2×2 matrix is $\lambda^2 - \text{tr}(A)\lambda + \det(A) = 0$.