HMM Algorithms

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Automatic Speech Recognition— ASR Lecture 5
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Overview

HMM algorithms

- HMM recap
- HMM algorithms (2)
  - Likelihood computation (forward algorithm)
  - Finding the most probable state sequence (Viterbi algorithm)
  - Estimating the parameters (forward-backward and EM algorithms)

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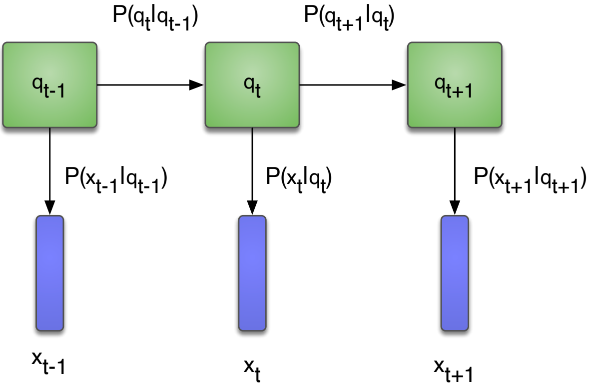
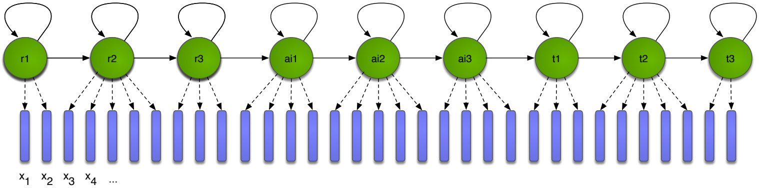
Recap: the HMM



- A *generative model* for the sequence $X = (x_1, \ldots, x_T)$
- Discrete states q_t are unobserved
- q_{t+1} is conditionally independent of $q_1, \ldots, q_{t-1}$, given q_t
- Observations x_t are conditionally independent of each other, given q_t.

HMMs for ASR

The three-state left-to-right topology for phones:



Computing likelihoods with the HMM

Joint likelihood of X and $Q = (q_1, \ldots, q_T)$:

$$P(X, Q|\lambda) = P(q_1)P(\mathrm{x}_1|q_1)P(q_2|q_1)P(\mathrm{x}_2|q_2)\ldots \quad (1)$$

$$= P(q_1)P(\mathrm{x}_1|q_1)\prod_{t=2}^{T} P(q_t|q_{t-1})P(\mathrm{x}_t|q_t) \quad (2)$$

$P(q_t)$ denotes the initial occupancy probability of each state

HMM parameters

The parameters of the model, λ, are given by:

- Transition probabilities $a_{kj} = P(q_{t+1} = j | q_t = k)$
- Observation probabilities $b_j(\mathrm{x}) = P(\mathrm{x} | q = j)$

The three problems of HMMs

Working with HMMs requires the solution of three problems:

1. **Likelihood** Determine the overall likelihood of an observation sequence $X = (x_1, \ldots, x_t, \ldots, x_T)$ being generated by a known HMM topology, $\mathcal{M}$.
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2. **Decoding and alignment** Given an observation sequence and an HMM, determine the most probable hidden state sequence
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   $\rightarrow$ the *forward algorithm*
2. **Decoding and alignment** Given an observation sequence and an HMM, determine the most probable hidden state sequence
   $\rightarrow$ the *Viterbi algorithm*
3. **Training** Given an observation sequence and an HMM, find the state occupation probabilities, in order to find the best HMM parameters λ
   $\rightarrow$ the *forward-backward* and *EM* algorithms

Viterbi algorithm

- Instead of finding the likelihood over all possible state sequences, as we do in the Forward algorithm, just consider just the most probable path:

$$P^*(X|\mathcal{M}) = \max P(X, Q|\mathcal{M})$$

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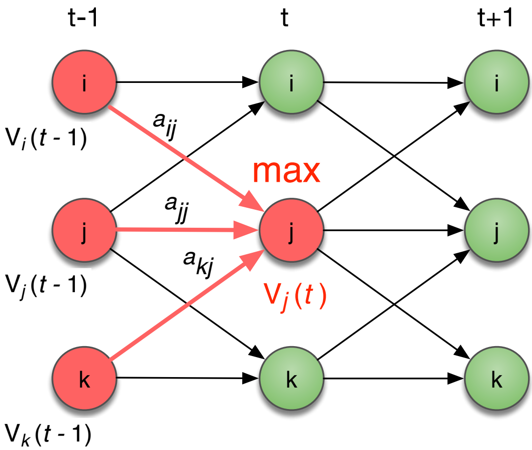
- We need to keep track of the states that make up this path by keeping a sequence of *backpointers* to enable a Viterbi *backtrace*: the backpointer for each state at each time indicates the previous state on the most probable path

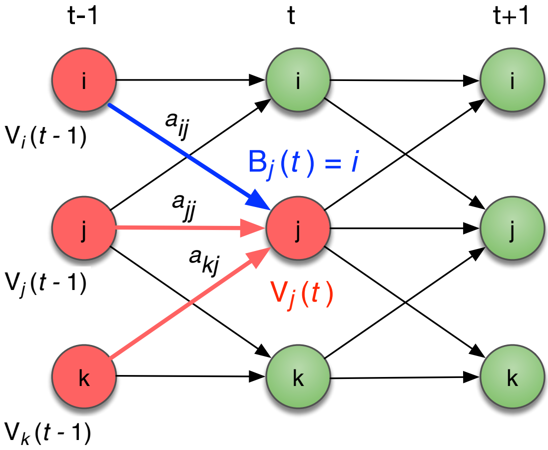
Viterbi Recursion

$$V_j(t) = \max_i V_i(t-1) a_{ij} b_j(x_t)$$



Viterbi Recursion

Backpointers to the previous state on the most probable path



2. Decoding: The Viterbi algorithm

- Initialisation

$$
\begin{aligned}
V_0(0) &= 1 \\
V_j(0) &= 0 \qquad \text{if } j \neq 0 \\
B_j(0) &= 0
\end{aligned}
$$

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- Recursion

$$V_j(t) = \max_{i=0}^{J} V_i(t-1) a_{ij} b_j(x_t)$$
$$B_j(t) = \arg\max_{i=0}^{J} V_i(t-1) a_{ij} b_j(x_t)$$

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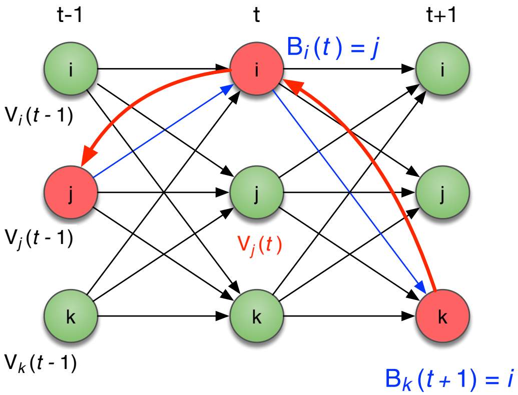
- Termination

$$P^* = V_E = \max_{i=1}^{J} V_T(\, i \,) a_{iE}$$
$$s_T^* = B_E = \arg\max_{i=1}^{J} V_i(T) a_{iE}$$

Viterbi Backtrace

Backtrace to find the state sequence of the most probable path



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- Maximum likelihood training: find the parameters that maximise

$$
\begin{aligned}
F_{\mathrm{ML}}(\lambda) &= \log P(X | \mathcal{M}, \lambda) \\
&= \log \sum_{Q \in \mathcal{Q}} P(X, Q | \mathcal{M}, \lambda)
\end{aligned}
$$

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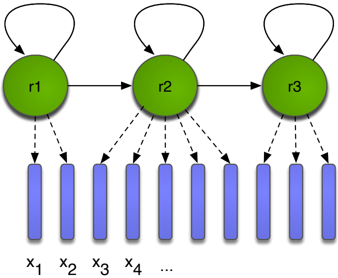
- Define indicator variable $z_{jt} = 1$ if the HMM is in state j at time t, and $z_{jt} = 0$ otherwise If we knew the state-time alignment, this variable would be observed, and we could use it to obtain the standard maximum likelihood estimates for the mean of the observation probability distribution:

$$\hat{\mu}_j = \frac{\sum_t z_{jt} \mathbf{x}_t}{\sum_t z_{jt}}$$

Example



$$a_{11} = \frac{1}{2} \qquad a_{22} = \frac{4}{5} \qquad a_{33} = \frac{2}{3}$$

$$a_{12} = \frac{1}{2} \qquad a_{23} = \frac{1}{5} \qquad a_{3E} = \frac{1}{3}$$

EM Algorithm

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- *State occupation probability*: The probability $\gamma_j(t)$ of occupying state j at time t given the sequence of observations.
- We can use this for an iterative algorithm for HMM training: the EM algorithm
- Application of EM algorithm to HMMs is called 'Baum-Welch algorithm

EM Algorithm

If we have some initial parameters λ_0 and we want to find new parameters to maximise the likelihood $F_{\mathrm{ML}}(\lambda)$, then we can instead maximise

$$\sum_{Q \in \mathcal{Q}} P(Q|X, \mathcal{M}, \lambda_0) \log P(X, Q|\mathcal{M}, \lambda)$$

E-step estimate the state occupation probabilities given the current parameters (Expectation)

M-step re-estimate the HMM parameters based on the estimated state occupation probabilities (Maximisation)

Why does this work? See next lecture.

Backward probabilities

- To estimate the state occupation probabilities we need to define (recursively) another set of probabilities—the *Backward probabilities*

$$\beta_j(t) = p(x_{t+1}, \ldots, x_T \mid q_t = j, \mathcal{M})$$

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  - Recursion

$$\beta_i(t) = \sum_{j=1}^{J} a_{ij} b_j(x_{t+1}) \beta_j(t+1) \quad \text{for } t = T-1, \ldots, 1$$

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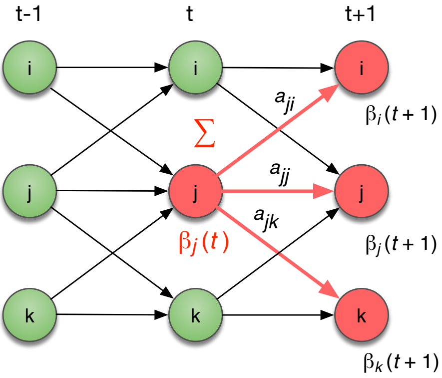
  - Termination

    $$p(\mathrm{X} \,|\, \mathcal{M}) = \beta_0(0) = \sum_{j=1}^{J} a_{0j} b_j(\mathrm{x}_1) \beta_j(1) = \alpha_E$$

Backward Recursion

$$\beta_j(t) = p(x_{t+1}, \ldots, x_T \mid q_t = j, \mathcal{M}) = \sum_{j=1}^{J} a_{ij} b_j(x_{t+1}) \beta_j(t+1)$$



State Occupation Probability

- The **state occupation probability** $\gamma_j(t)$ is the probability of occupying state j at time t given the sequence of observations
- Express in terms of the forward and backward probabilities:

$$\gamma_j(t) = P(q_t = j \,|\, \mathrm{X}, \mathcal{M}) = \frac{1}{\alpha_E} \alpha_j(\,t\,) \beta_j(\,t\,)$$

recalling that $p(\mathrm{X} \,|\, \mathcal{M}) = \alpha_E$

- Since

$$\begin{aligned} \alpha_j(t)\beta_j(t) &= p(\mathrm{x}_1, \ldots, \mathrm{x}_t, q_t = j \,|\, \mathcal{M}) \\ &\qquad p(\mathrm{x}_{t+1}, \ldots, \mathrm{x}_T \,|\, q_t = j, \mathcal{M}) \\ &= p(\mathrm{x}_1, \ldots, \mathrm{x}_t, \mathrm{x}_{t+1}, \ldots, \mathrm{x}_T, q_t = j \,|\, \mathcal{M}) \\ &= p(\mathrm{X}, q_t = j \,|\, \mathcal{M}) \end{aligned}$$

$$P(q_t = j \,|\, \mathrm{X}, \mathcal{M}) = \frac{p(\mathrm{X}, q_t = j \,|\, \mathcal{M})}{p(\mathrm{X} \,|\, \mathcal{M})}$$

Re-estimation of transition probabilities

- Similarly to the state occupation probability, we can estimate $\xi_{i,j}(t)$, the probability of being in i at time t and j at $t+1$, given the observations:

$$\xi_t(\,i\,,\,j\,) = P(q_t = i, q_{t+1} = j \,|\, \mathrm{X}, \mathcal{M})$$
$$= \frac{p(q_t = i, q_{t+1} = j, \mathrm{X} \,|\, \mathcal{M})}{p(\mathrm{X} \,|\, \mathcal{M})}$$
$$= \frac{\alpha_i(t) a_{ij} b_j(\mathrm{x}_{t+1}) \beta_j(t+1)}{\alpha_E}$$

- We can use this to re-estimate the transition probabilities

$$\hat{a}_{ij} = \frac{\sum_{t=1}^{T} \xi_{i,j}(t)}{\sum_{k=1}^{J} \sum_{t=1}^{T} \xi_{i,k}(t)}$$

- See next lecture for re-estimation of obervation probabilities $b_j(x)$

Pulling it all together

- Iterative estimation of HMM parameters using the EM algorithm. At each iteration
  - E step For all time-state pairs
    1. Recursively compute the forward probabilities $\alpha_j(t)$ and backward probabilities $\beta_j(t)$
    2. Compute the state occupation probabilities $\gamma_j(t)$ and $\xi_{i,j}(t)$
  - M step Based on the estimated state occupation probabilities re-estimate the HMM parameters: transition probabilities a_{ij} and parameters of the obervation probabilities, $b_j(x)$
- The application of the EM algorithm to HMM training is sometimes called the Forward-Backward algorithm or Baum-Welch algorithm

Extension to a corpus of utterances

- We usually train from a large corpus of R utterances
- If x_t^r is the t th frame of the r th utterance X^r then we can compute the probabilities $\alpha_j^r(t)$, $\beta_j^r(t)$, $\gamma_j^r(t)$ and $\xi_{i,j}^r(t)$ as before
- The re-estimates are as before, except we must sum over the R utterances, ie:

$$\hat{a}_{ij} = \frac{\sum_{r=1}^{R} \sum_{t=1}^{T} \xi_{i,j}(t)}{\sum_{r=1}^{R} \sum_{k=1}^{J} \sum_{t=1}^{T} \xi_{i,k}(t)}$$

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- In addition, we usually employ "*embedded training*", in which fine tuning of phone labelling with "*forced Viterbi alignment*" or forced alignment is involved. (For details see Section 9.7 in Jurafsky and Martin's SLP)

Summary: HMMs

- HMMs provide a generative model for statistical speech recognition
- Three key problems
  1. Computing the overall likelihood: the Forward algorithm
  2. Decoding the most likely state sequence: the Viterbi algorithm
  3. Estimating the most likely parameters: the EM (Forward-Backward) algorithm
- Solutions to these problems are tractable due to the two key HMM assumptions
  1. Conditional independence of observations given the current state
  2. Markov assumption on the states

References: HMMs

- Gales and Young (2007). "The Application of Hidden Markov Models in Speech Recognition", *Foundations and Trends in Signal Processing*, **1** (3), 195–304: section 2.2.
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- Renals and Hain (2010). "Speech Recognition", *Computational Linguistics and Natural Language Processing Handbook*, Clark, Fox and Lappin (eds.), Blackwells.