Automatic Speech Recognition handout (2)

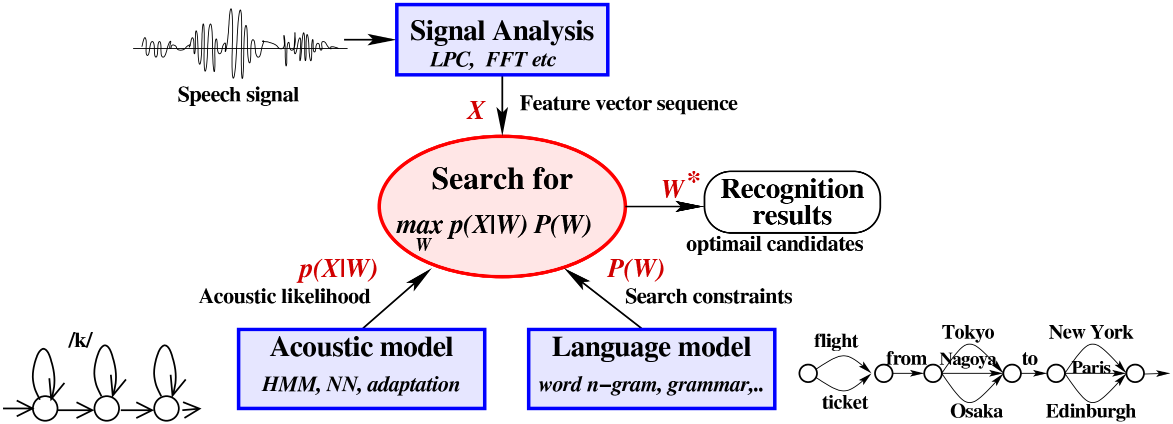
Jan - Mar 2012

Revision : 1.1

— Statistical pattern recognition —

Hiroshi Shimodaira (h.shimodaira@ed.ac.uk)

Speech Recognition recap



(after Sagayama, "Speech Translation Telephony",1994)

Bayes decision rule:

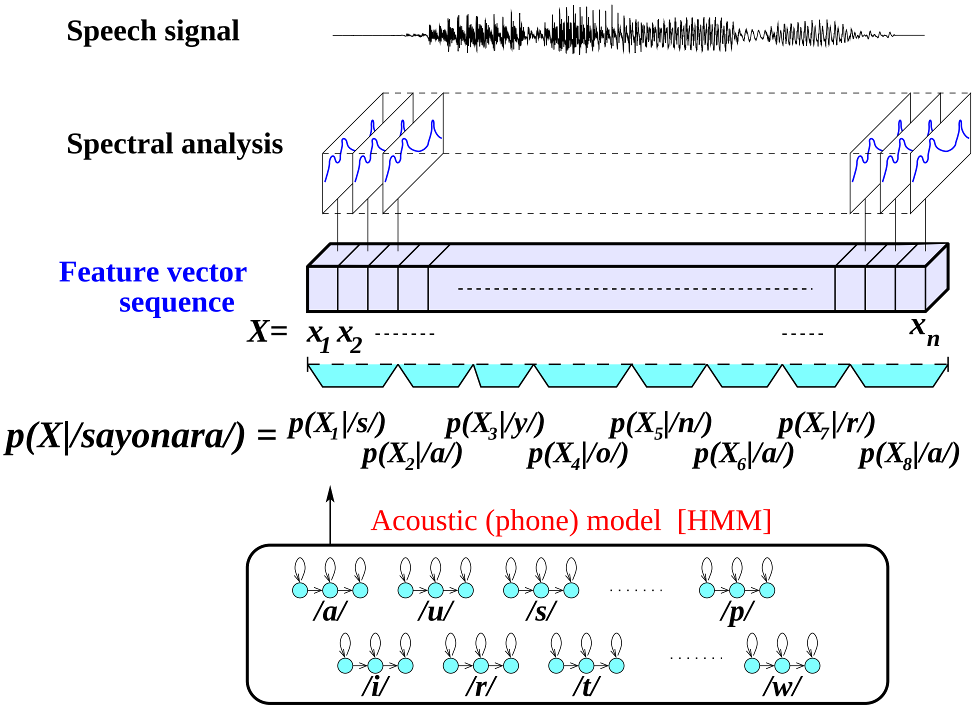
$$\boldsymbol{W}^* = \arg\max_W P(W|X) = \arg\max_W \frac{p(X|W)P(W)}{p(X)}$$

$$= \arg\max_W p(X|W)P(W)$$

$X = (\boldsymbol{x}_1, \boldsymbol{x}_2, \cdots, \boldsymbol{x}_T)$: feature vectors observed

$W = (w_1, w_2, \cdots, w_L)$: hypothesised words

Speech Recognition recap *(cont. 2)*



How to calculate $p(X|/s/)$?

The conditional probability that we observe a feature sequence X for phone /s/ :

$$p(X|/s/) = p(\boldsymbol{x}_1, \cdots, \boldsymbol{x}_{T_1}|/s/), \quad \boldsymbol{x}_i = (x_{1i}, \cdots, x_{di})^t \in \mathcal{R}^d$$

We know

- **HMM can be employed to calculate this. (Viterbi algorithm, Forward / Backward algorithm)**
- **HMM should be trained beforehand with a set of *training samples*. (Baum-Welch algorithm (EM algorithm), Viterbi training)**

To investigate these algorithms in detail, let's start with the simplest case: the length of the sequence is one ($T = 1$), and the dimensionality of x is one ($d = 1$).

$$p(X|/s/) \longrightarrow p(x|/s/)$$

How to calculate $p(x|/s/)$?

$p(x|/s/)$: **conditional probability**
(conditional probability density function (pdf) of x)

- **We learnt that a Gaussian / normal distribution function can be employed to approximate the pdf by:**

$$P(x|/s/) = \frac{1}{\sqrt{2\pi}\sigma_{/s/}} e^{-\frac{(x-\mu_{/s/})^2}{2\sigma^2_{/s/}}}$$

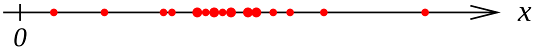
- **The function has only two parameters,** $\mu_{/s/}$ **and** $\sigma_{/s/}$

- **Given a set of training samples** $\{x_1, \cdots, x_N\}$**, we estimate** $\mu_{/s/}$ **and** $\sigma_{/s/}$ **by**

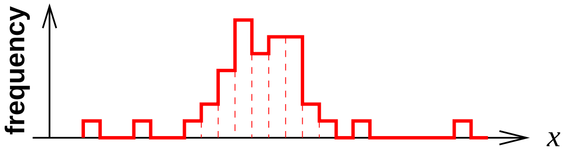
$$\hat{\mu}_{/s/} = \frac{1}{N}\sum_{i=1}^{N} x_i, \qquad \hat{\sigma}^2_{/s/} = \frac{1}{N}\sum_{i=1}^{N}(x_i - \mu_{/s/})^2$$

how reliable are these estimates?
how similar $p(x|\hat{\mu}_{/s/}, \hat{\sigma}_{/s/})$ **is to the true one?**

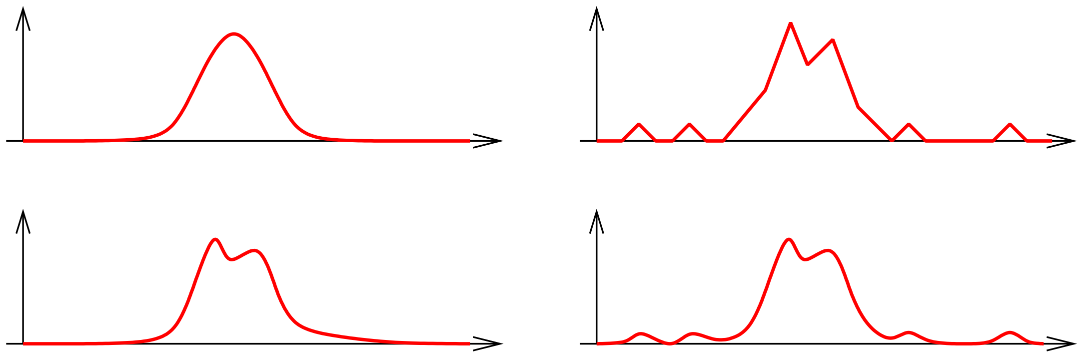
How to estimate $p(x|/s/)$?



(a) observation

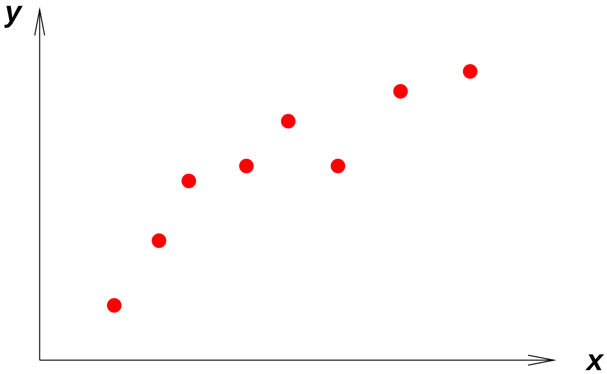


(b) histogram

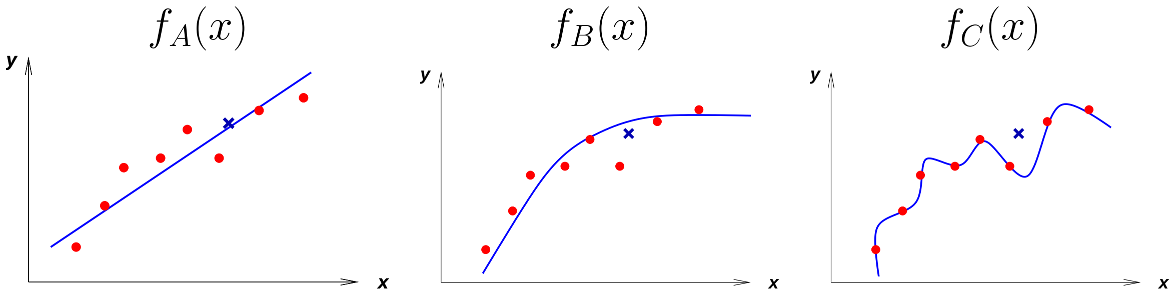


How to estimate $p(x|/s/)$?(cont. 2)

How does $y = f(x)$ look like?



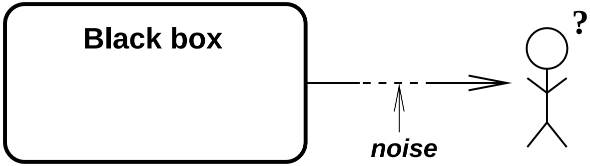
How to estimate $p(x|/s/)$? (cont. 3)



	$f_A(x)$	$f_B(x)$	$f_C(x)$
# of parameters	**small**	**medium**	**large**
Error on training samples	**moderate**	**small**	**0 (zero)**
Error on other samples	**moderate?**	**moderate/large?**	**huge?**

Technical terms: robustness, generalisation, over-fitting

How to estimate $p(x|/s/)$?(cont. 4)



- What we've observed are just samples from an unobservable system. Even worse, the samples might be distorted.
- With limited amount of samples, it is hard to estimate / identify the system completely, because more than one solution can exist and we do not know which is the best one.
- To find the best one, we need more knowledge about the system.
- Estimating the system from observation is regarded as an "inverse problem" or "ill-posed problem".
- To solve an inverse problem, we usually define an optimisation problem with some constraints (assumptions, models).

Estimating pdf

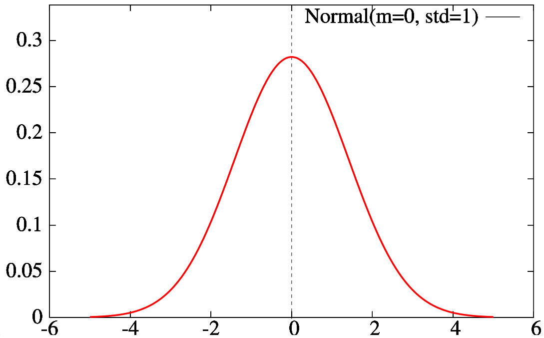
- **Non-parametric approach**
  - **histogram**
  - **kernel method, Parzen window**
  - **neural network**
- **Parametric approach**
  - **Gaussian / normal distribution**
  - **Gaussian mixture model**
  - **etc.**

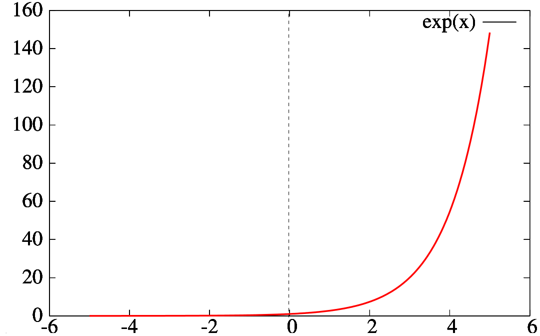
Gaussian Distribution: 1-dim case

$$p(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

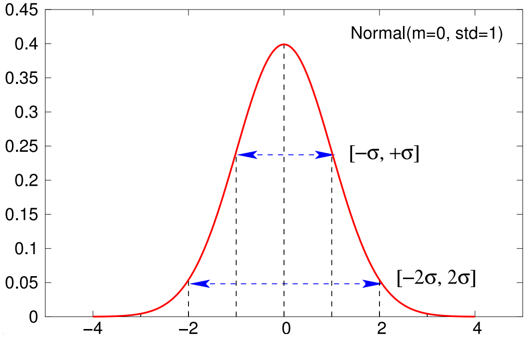
$$y = e^x$$

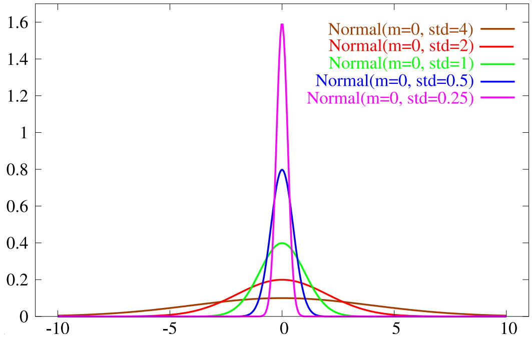
$$e = 2.7183\cdots$$





Gaussian Distribution: 1-dim case *(cont. 2)*





$$P(-\sigma < x < \sigma) \sim 0.683$$
$$P(-2\sigma < x < 2\sigma) \sim 0.954$$
$$P(-3\sigma < x < 3\sigma) \sim 0.997$$

$$p(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

Gaussian Distribution: 1-dim case *(cont. 3)*

$$P(x|\mu,\sigma) = \frac{1}{\sqrt{2\pi}\sigma}\exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right) \quad \triangleq \quad \mathcal{N}(x;\mu,\sigma)$$

Given samples $\{x_1, \cdots, x_N\} = \{x_n\}_{n=1}^{N}$,
we know how to estimate (infer) the parameters μ, σ:

$$\begin{aligned}
\hat{\mu} &= \frac{1}{N}\sum_{n=1}^{N} x_n \\
\hat{\sigma}^2 &= \frac{1}{N}\sum_{n=1}^{N} (x_n - \hat{\mu})^2 = \frac{1}{N}\sum_{n=1}^{N} x_n^2 - \left(\frac{1}{N}\sum_{n=1}^{N} x_n\right)^2
\end{aligned}$$

However,

- **Why we can say this is a right inference?**
- **Are there any other inferences?**

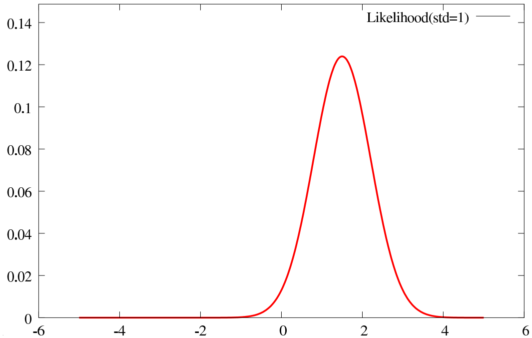
Maximum likelihood estimation

$$p(x|\mu,\sigma) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)$$

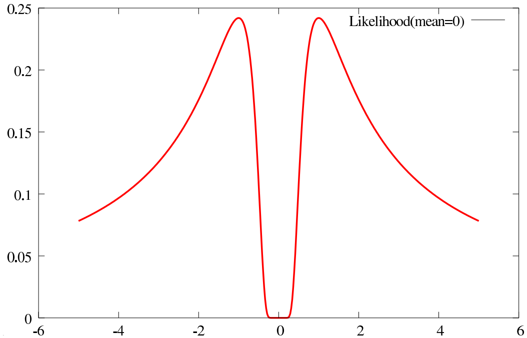
Interpretation of $p(x|\mu,\sigma)$

- **If μ, σ are fixed $\rightarrow p(x|\mu,\sigma)$ is a function of x.**
  $p(x|\mu,\sigma)$ **: a probability density function.**
- **If x is fixed $\rightarrow p(x|\mu,\sigma)$ is a function of μ, σ.**
  $p(x|\mu,\sigma)$ **: a 'likelihood' function denoted as $L(\mu,\sigma;x)$**

Maximum likelihood estimation *(cont. 2)*



likelihood as a function of μ



likelihood as a function of σ

$$L(\mu, \sigma; x) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2} \right)$$

Maximum likelihood estimation (cont. 3)

ML estimation (MLE)

Assume $X = \{x_n\}_{n=1}^{N}$ **are chosen independently (i.i.d.)** [1]**.**
The probability of observing X **is**

$$p(x_1, \cdots, x_N|\mu, \sigma) = p(x_1|\mu, \sigma) \cdots p(x_N|\mu, \sigma) = \prod_{n=1}^{N} p(x_n|\mu, \sigma)$$

$$\triangleq L(\mu, \sigma|X)$$

Find parameters $\mu, \ \sigma$ **that maximise the likelihood:**

$$\max_{\mu,\sigma} L(\mu, \sigma|X)$$

[1] i.i.d: independent identically-distributed, i.e. data points that are drawn independently from the same distribution.

Maximum likelihood estimation *(cont. 4)*

(Solution) **Take the derivative of** $\log L(\mu, \sigma | X)$ **and set it to zero.**

$$\ell(\mu, \sigma | X) = \log L(\mu, \sigma | X) = \log \prod_{n=1}^{N} p(x_n | \mu, \sigma) = \sum_{n=1}^{N} \log p(x_n | \mu, \sigma)$$

$$= -N \log(\sqrt{2\pi}\sigma) - \sum_{n=1}^{N} \frac{(x_n - \mu)^2}{2\sigma^2}$$

$$\frac{\partial \ell}{\partial \mu} = -2 \sum_{n=1}^{N} \frac{x_n - \mu}{2\sigma^2} = 0 \quad \Rightarrow \quad \mu = \frac{1}{N} \sum_{n=1}^{N} x_n$$

$$\frac{\partial \ell}{\partial \sigma} = -N \frac{1}{\sqrt{2\pi\sigma}} + \sum_{n=1}^{N} \frac{(x_n - \mu)2}{\sigma^3} = 0$$

$$\Rightarrow \quad \sigma^2 = \frac{1}{N} \sum_{n=1}^{N} (x_n - \mu)^2$$

Limitation of a Gaussian pdf

- **Gaussian distribution is assumed**
- **does not fit more complicated distributions (bimodal, etc.)**

Solutions

- **Transform features**
  (coordinate non-linear transformation)
- **Use other distribution functions**
  - **Poisson dist.**
  - **exponential dist.**
  - **gamma dist.**
  - **log-normal dist.**
- **Use a mixture of Gaussian distribution functions**
- **Use neural networks, kernel approaches**

Gaussian Mixture Model (GMM)

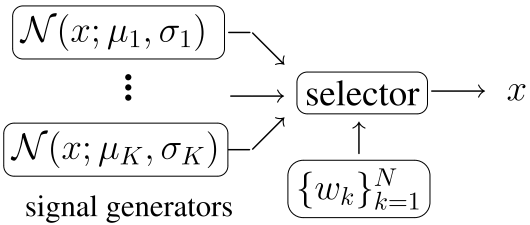
Idea **Approximate a "true" distribution by more than one Gaussian distribution.**

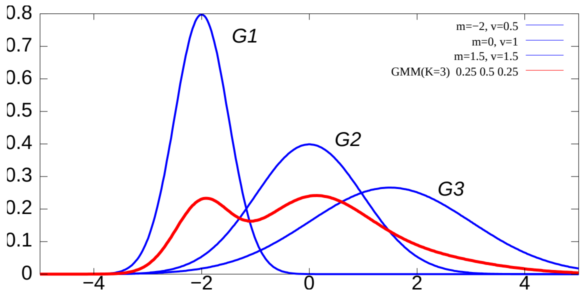
$$p(x|\Lambda) = \sum_{k=1}^{K} w_k \, p(x|\mu_k, \sigma_k) = \sum_{k=1}^{K} w_k \, \mathcal{N}(x; \mu_k, \sigma_k)$$

where $\Lambda = \{\lambda_k\}_{k=1}^{K} = \{\mu_k, \sigma_k\}_{k=1}^{K}$ $\cdots$ **a set of model parameters**

$\sum_{k=1}^{K} w_k = 1$

View as a generative-model

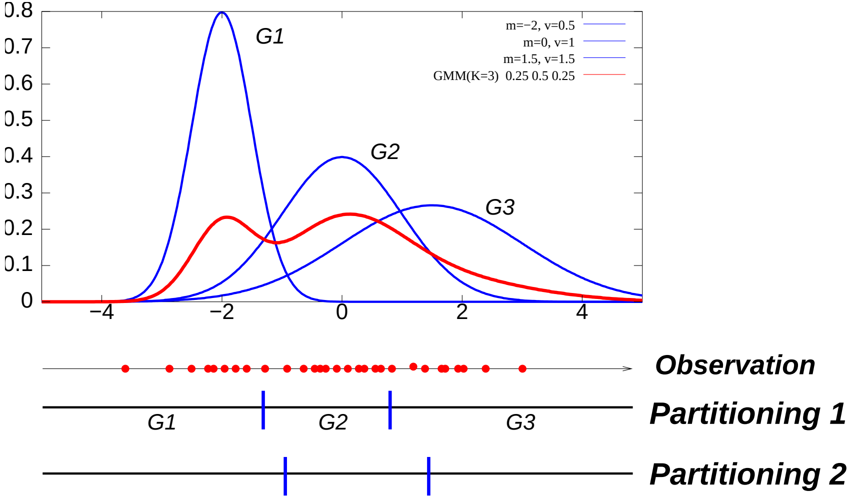




Training GMM

Given a set of training samples $x_1, x_2, \cdots, x_N$,
assuming a mixture of K Gaussians,
we don't know from which Gaussian model each x_i came.

$\Rightarrow$ ML training can not be applied directly !



Training GMM *(cont. 2)*

Solution Assume each x_n has a label y_n showing from which model the data came.
i.e. y_n takes an index of Gaussian model $1, \cdots, K$.

observations	x_1	x_2	$\cdots$	x_N
labels (unseen)	y_1	y_2	$\cdots$	y_N

Then we would be able to use ML estimation recursively to find the optimal partitioning.

Approach A (hard k-means clustering) $\cdots$ assume each x_n belongs to an exact Gaussian y_n.

Approach B (soft k-means clustering) $\cdots$ assume each x_n belongs to Gaussian k with probability $P(y_n = k)$.
(y_n is treated as a random variable)

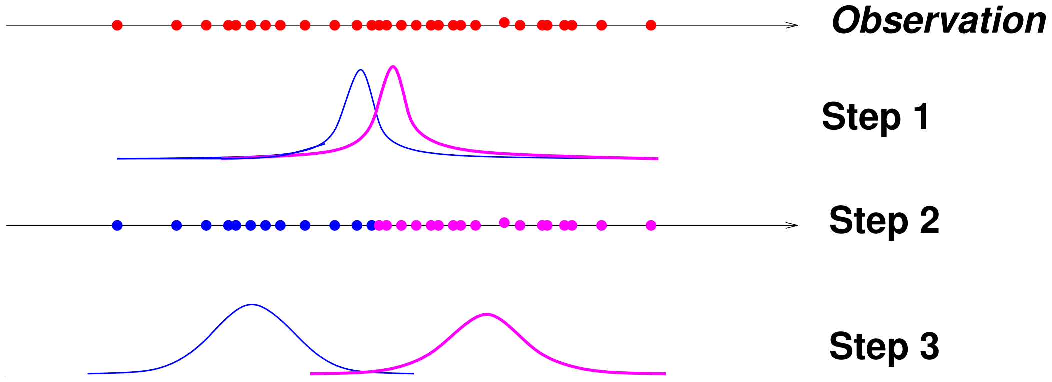
Training GMM by k-means clustering (hard version)

Step 1: **Initialisation. Set model parameters Λ arbitrarily.**

Step 2: **Clustering:** $y_n = \arg\max_k p(x_n|\lambda_k)$ for $k = 1, \ldots, N$

Step 3: **Re-estimation: estimate model parameters Λ.**

Step 4: **Iteration: Repeat steps 2 and 3 until a predefined convergence criterion is satisfied. (estimate $\{w_k\}$ when exits.)**



Training GMM by k-means clustering (soft version)

Problem of the hard k means clustering: **non guarantee of ML.**

The EM algorithm enables us to maximise the likelihood

$$L(\Lambda|X) = \prod_{n=1}^{N} p(x_n|\Lambda)$$

by iteratively maximising Q function with respect to $\Lambda^{(t+1)}$:

$$Q(\Lambda^{(t)}, \Lambda^{(t+1)}) = \sum_{Y} P(Y|X, \Lambda^{(t)}) \log p(X, Y|\Lambda^{(t+1)})$$

This is the expectation of log-likelihood from complete data (X, Y). To maximise Q, we take the derivative of Q with respect to each parameter and set it to zero.

The Expectation-Maximisation Algorithm

The EM algorithm [Dempster, Laird, and Rubin, 1977]

Step 1: **Initialisation: Set $t = 0$, and choose an initial estimate $\Lambda^{(0)}$.**

Step 2: **E-Step: Compute $Q(\Lambda^{(t)}, \Lambda)$ based on current parameter $\Lambda^{(t)}$.**

Step 3: **M-Step: Compute $\Lambda^* = \arg\max_{\Lambda} Q(\Lambda^{(t)}, \Lambda)$ to maximise Q.**

Step 4: **Iteration: Set $\Lambda^{(t+1)} = \Lambda^*$ and $t \leftarrow t + 1$, repeat from Step 2 until convergence.**

(Note)

- **The algorithm is a generalisation of MLE, when we have incomplete data (i.e. some data are unobservable (hidden)).**
- **The algorithm gives just a general idea, and actual implementation differs depending on the problems.**

EM algorithm for GMM

$$
\begin{aligned}
Q(\Lambda^{(t)}, \Lambda^{(t+1)}) &= \sum_{n=1}^{N} Q_i(\Lambda^{(t)}, \Lambda^{(t+1)}) = \sum_{n=1}^{N} \sum_{y_n} P(y_n|x_n, \Lambda^{(t)}) \log p(x_n, y_n|\Lambda^{(t+1)}) \\
&= \sum_{n=1}^{N} \sum_{y_n} \frac{p(x_n, y_n|\Lambda^{(t)})}{p(x_n|\Lambda^{(t)})} \log p(x_n, y_n|\Lambda^{(t+1)}) \\
&= \sum_{k=1}^{K} \gamma_k \log w_k^{(t+1)} + \sum_{k=1}^{K} Q_\pi(\Lambda^{(t)}, \lambda_k^{(t+1)})
\end{aligned}
$$

where

$$
\gamma_k = \sum_{n=1}^{N} \gamma_k^n, \qquad \gamma_k^n = \frac{w_k^{(t)} p(x_n|\lambda_k^{(t)})}{p(x_n|\Lambda^{(t)})}
$$

$$
Q_\pi(\Lambda^{(t)}, \lambda_k^{(t+1)}) = \sum_{n=1}^{N} \gamma_k^n \log p(x_n|\lambda_k^{(t+1)})
$$

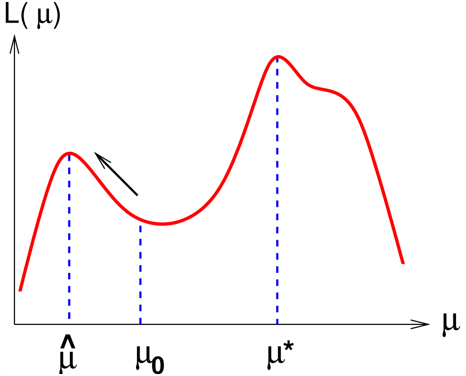
$$
\begin{aligned}
w_k^{(t+1)} &= \gamma_k / \sum_{k=1}^{K} \gamma_k = \gamma_k / N \\
m_k^{(t+1)} &= \sum_{n=1}^{N} \gamma_k^n x_n / \sum_{n=1}^{N} \gamma_k^n, \quad \sigma_k^{(t+1)} = \sum_{n=1}^{N} \gamma_k^n (x - m_k^{(t)})^2 / \sum_{n=1}^{N} \gamma_k^n
\end{aligned}
$$

Limitation of ML-trained GMM

- **Local optimum problem (the EM algorithm does not guarantee the global optimum)**
- **No mechanism for determining K (# of mixtures).**

Initial value problem of GMM

- **The EM iteration converges at a local maximum, which mighty be different from the global maximum.**
- **The performance of the algorithm highly relies on how appropriate initial value the algorithm starts from.**
- **As we increase the number of Gaussian mixture components, the chance we suffer the problem increases.**



Initial value problem of GMM *(cont. 2)*

Techniques to avoid the local maximum problem

Method-A: **Use k-means clustering (hard) to find initial partitions of data, and calculate initial parameters of GMM.**

Method-B: **Increase the mixture number k of GMM gradually. (start from $k = 1$, and then $k \leftarrow k * 2$.)**

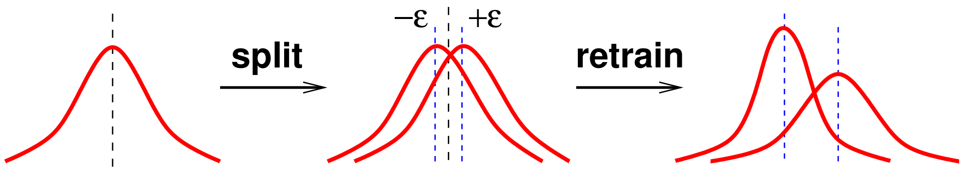


Fig. Mixture splitting

Remaining problems with GMM

- Avoiding the local maximum problem is not enough
- There is another problem: “over-fitting problem”
  - the likelihood on training data increases, as k (# of mixtures) increases.
  - The ML training does not tell us what k is optimal.

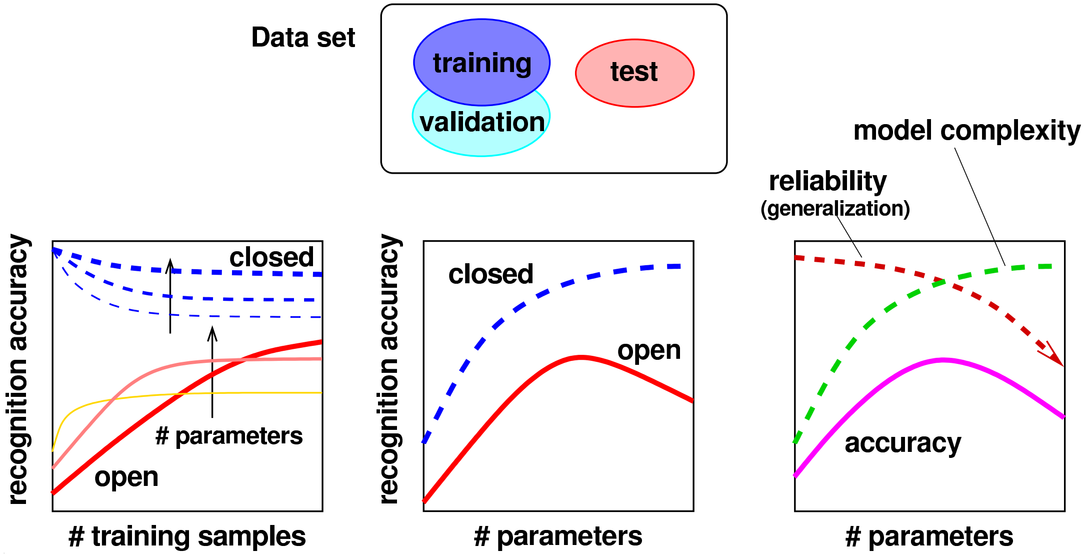
Solutions

- use model selection technique such as
  - AIC (Akaike’s information criterion)
  - MDL (minimum description length)

  $\longrightarrow$ “Occam’s razor”:
  “Entia non sunt multiplicanda praeter necessitatem”
  (accepts the simplest explanation that fits the data)

- use validation data

Remaining problems with GMM*(cont. 2)*



Maximum a posterior (MAP) estimation

X : a set of training data $\{x_n\}_{n=1}^{N}$
Λ : a set of parameters to estimate

ML estim. $\Lambda^* = \arg\max_{\Lambda} p(X|\Lambda)$

MAP estim. $\Lambda^* = \arg\max_{\Lambda} p(\Lambda|X) = \arg\max_{\Lambda} p(X|\Lambda)p(\Lambda)$

$p(\Lambda)$: a priori distribution over Λ
$p(\Lambda|X)$: a posteriors distribution after observing X

$$p(\Lambda|X) = \frac{p(X|\Lambda)p(\Lambda)}{p(X)} = \frac{p(X|\Lambda)p(\Lambda)}{\int p(X|\Lambda')p(\Lambda)'d\Lambda'}$$

- **The object function takes the reliability of parameters, i.e. the a priori distribution, into account**
- **Regarded as a penalised (regularised) version of MLE**
- **How to define/calculate the prior $p(\Lambda)$?**
  $\Rightarrow$ several approaches have been proposed

Bayesian estimation

p.d.f. estimation based on

- **point estimation**

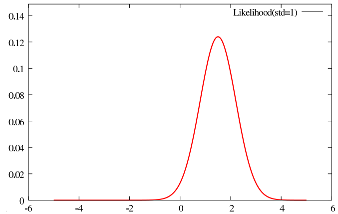
$$\hat{p}(\boldsymbol{x}|X) = p(\boldsymbol{x}|X, \Lambda^*), \quad \Lambda^* = \begin{cases} \max\limits_{\Lambda} p(X|\Lambda), & \textbf{MLE} \\ \max\limits_{\Lambda} p(X|\Lambda)p(\Lambda), & \textbf{MAP} \end{cases}$$

assuming parameter Λ is fixed but unknown.

- **interval estimation (Bayesian estimation)**

$$\hat{p}(\boldsymbol{x}|X) = \int p(\boldsymbol{x}|X, \Lambda)p(\Lambda|X)d\Lambda$$

$$\text{where} \quad p(\Lambda|X) = \frac{p(X|\Lambda)p(\Lambda)}{\int p(X|\Lambda)p(\Lambda)d\Lambda}$$



assuming the parameters Λ are random variables with a prior distribution $p(\Lambda)$

Multivariate Gaussian pdf

One variable

$$\textbf{data:}\quad \boldsymbol{x}_n = (x_n)$$

$$\mu = \frac{1}{N}\sum_{i=1}^{N} x_i, \qquad \sigma^2 = \frac{1}{N}\sum_{i=1}^{N}(x_i - \mu)^2$$

Two variables

$$\boldsymbol{x}_n = (x_{1n}, x_{2n})^t = \begin{pmatrix} x_{1n} \\ x_{2n} \end{pmatrix}, \quad \boldsymbol{\mu} = \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix}, \quad \Sigma = \begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{pmatrix}$$

$$\mu_1 = \frac{1}{N}\sum_{i=1}^{N} x_{1n}, \qquad \mu_2 = \frac{1}{N}\sum_{i=1}^{N} x_{2n}$$

$$\sigma_{11} = \frac{1}{N}\sum_{i=1}^{N}(x_{1i} - \mu_1)^2, \qquad \sigma_{22} = \frac{1}{N}\sum_{i=1}^{N}(x_{2i} - \mu_2)^2$$

$$\sigma_{12} = \frac{1}{N}\sum_{i=1}^{N}(x_{1i} - \mu_1)(x_{2i} - \mu_2) = \sigma_{21}$$

Multivariate Gaussian pdf *(cont. 2)*

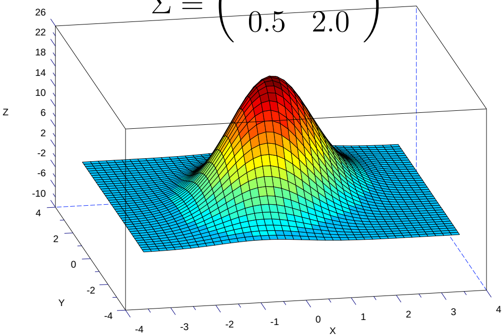
mean vector: $\boldsymbol{\mu} = (\mu_1, \mu_2)^t$

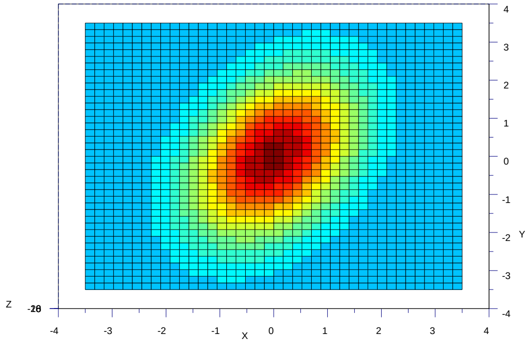
covariance matrix: $\Sigma = \begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{pmatrix}$

$$p(\boldsymbol{x}|\boldsymbol{\mu}, \Sigma) = \frac{1}{(2\pi)^{d/2}|\Sigma|^{1/2}} \exp\left(-\frac{1}{2}(\boldsymbol{x}-\boldsymbol{\mu})^t \Sigma^{-1}(\boldsymbol{x}-\boldsymbol{\mu})\right)$$

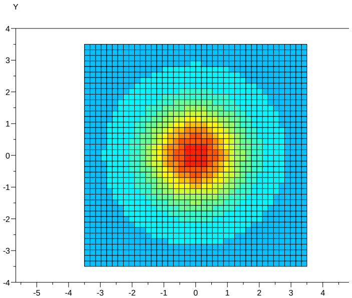
where d denotes dimensions (here $d = 2$).

$\Sigma = \begin{pmatrix} 1.0 & 0.5 \\ 0.5 & 2.0 \end{pmatrix}$

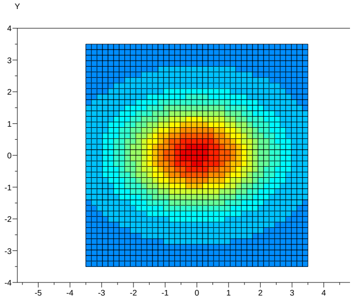




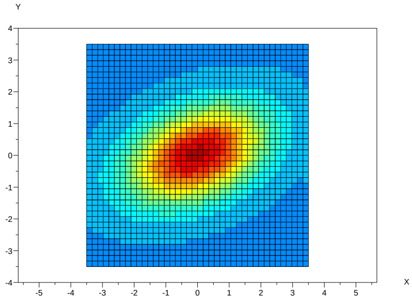
$$\Sigma = \begin{pmatrix} 1.0 & 0.0 \\ 0.0 & 1.0 \end{pmatrix}$$



$$\Sigma = \begin{pmatrix} 2.0 & 0.0 \\ 0.0 & 1.0 \end{pmatrix}$$



$$\Sigma = \begin{pmatrix} 2.0 & 0.5 \\ 0.5 & 1.0 \end{pmatrix}$$

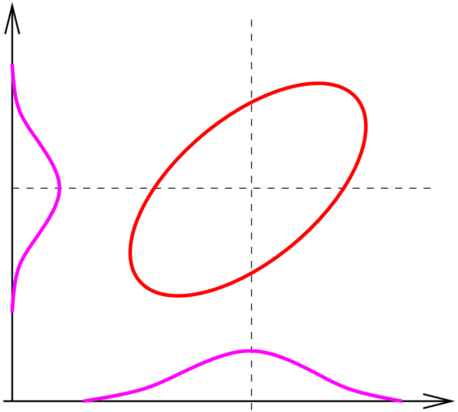


x_1 **and** x_2 **are called uncorrelated if** $\sigma_{12} = 0$**.**

Gaussian pdf with diagonal covariances

Gaussian pdf with a full-covariance

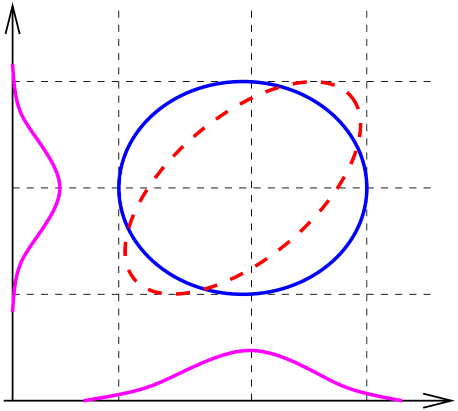
$$\Sigma = \begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{pmatrix}$$



Gaussian pdf with diagonal-covariances

$$\Sigma = \begin{pmatrix} \sigma_{11} & 0 \\ 0 & \sigma_{22} \end{pmatrix}$$

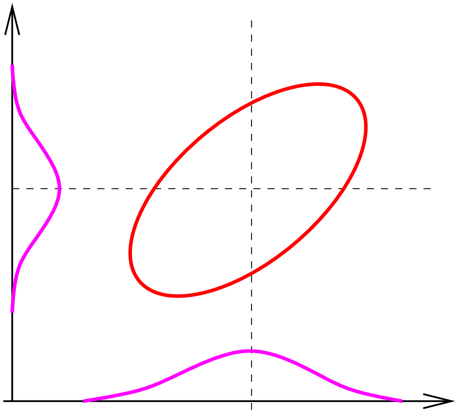
$$p(\boldsymbol{x}|\boldsymbol{\mu}, \Sigma) = p(x_1|\mu_1, \sigma_{11})\, p(x_2|\mu_2, \sigma_{22})$$



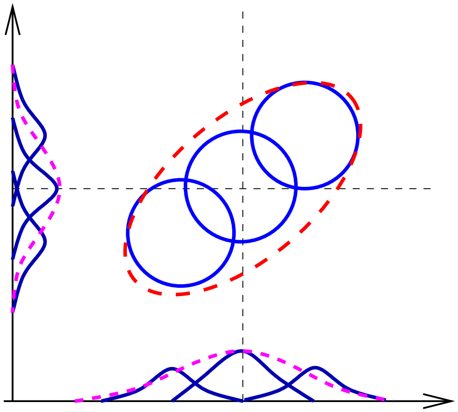
Gaussian pdf with diagonal covariances*(cont. 2)*

GMM with diag. covs. can model dependencies (correlations) !

Gaussian pdf
with full-covariance



GMM pdf
with diagonal-covariance



Gaussian pdf with diagonal covariances*(cont. 3)*

Question **why we normally use GMM with diagonal covariances rather than the one with full covariances?**

What has not been discussed yet

$$\boldsymbol{W}^* = \arg\max_W P(W|X) = \arg\max_W p(X|W)P(W)$$

$$X = (\boldsymbol{x}_1, \boldsymbol{x}_2, \cdots, \boldsymbol{x}_T) \; : \quad \text{feature vectors observed}$$

$$W = (w_1, w_2, \cdots, w_L) \; : \quad \text{hypothesised words}$$

- **Assumptions so far**
  - $T = 1$ **:** X **consists of s single frame**
  - $L = 1$ **and** $W = /s/$ **(single phoneme** $/s/$**)**
- **Calculation of** $P(X|W)$ **and** $P(W)$ **under those conditions:**
  - X **is a sequence of feature vectors**
  - W **is a sequence of words/phonemes (**L **is unknown)**
  - **boundaries of words/phonemes (+ silence) are unknown**

Basic idea of estimating error rate

1. **Prepare two subsets of the dataset:**

   training set — to train the recogniser

   test(validation) set — to evaluate recognition performance

   * these two subsets should be mutually exclusive (no overlap between them)

2. **Train the recogniser using the training set.**

3. **Carry out a recognition experiment of the recogniser on the test set, and calculate the recognition error:**

$$R_{\mathbf{emp}} = \frac{\#\text{ of misrecognised data}}{\#\text{ of input data}} \quad \cdots \left(\begin{array}{l} \text{test set error, or} \\ \text{empirical error} \end{array} \right)$$

4. **Repeat the recognition experiment with changing training and test sets to estimate the expectation of error:**

   generalisation error: $R = E[R_{\mathbf{emp}}]$

http://en.wikipedia.org/wiki/Generalization_error

k-fold Cross-Validation (CV)

Step 1 **split the dataset randomly into k disjoint sets of equal size.**

Step 2 **train and test the recogniser k times: each time with choosing one set for validation and the remaining $k-1$ sets for training.**

Step 3 **Estimate the performance by taking the mean of the k errors.**

k-fold CV	subsets
2-fold CV	Set2-0, Set2-1
5-fold CV	Set5-0, Set5-1, Set5-2, Set5-3, Set5-4

Similar techniques: leave-one-out, held-out, jackknife, bootstrap

References:

`http://en.wikipedia.org/wiki/Cross_validation`
`http://www.faqs.org/faqs/ai-faq/neural-nets/part3/section-12.html`

References

- Books:
  - Christopher M. Bishop, “Pattern Recognition and Machine Learning”, ISBN 0387310738, Springer-Verlag, 2006.
- Web pages:

  Normal (Gaussian) Distribution
  - http://en.wikipedia.org/wiki/Normal_distribution

  Maximum Likelihood Estimation (MLE)
  - http://en.wikipedia.org/wiki/Maximum_likelihood

  GMM
  - http://en.wikipedia.org/wiki/Mixture_model

  EM algorithm
  - http://en.wikipedia.org/wiki/Expectation-maximization_algorithm