Automated Reasoning

Lecture 13: Rewriting II

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Recap

- Previously: Rewriting
  - Definition of Rewrite Rule of Inference
  - Termination
  - Rewriting in Isabelle
- This time: More of the same!
  - Canonical normal forms
  - Confluence
  - Critical Pairs
  - Knuth-Bendix Completion

Canonical Normal Form

For some rewrite rule sets, order of application might affect result.

We might have:

$$\begin{tikzcd}
 & & s \arrow[dll, "*"'] \arrow[dl, "*"'] \arrow[d, "*"] \arrow[dr, "*"] \arrow[drr, "*"] & & \\
t_1 & t_2 & t_3 & t_4 & t_5
\end{tikzcd}$$

where all of t_1, t_2, t_3, t_4, t_5 are in normal form after multiple (zero or more) rewrite rule applications.

If all the normal forms are identical we can say we have a **canonical** normal form for s.

This is a very nice property!

- Means that order of rewrite rule application doesn't matter
- In general, means our rewrites are simplifying the expression in a canonical (safe) way.

Confluence and Church-Rosser

How do we know when a set of rules yields canonical normal forms?

A set of rewrite rules is **confluent** if for all terms r, s_1, s_2 such that $r \longrightarrow^* s_1$ and $r \longrightarrow^* s_2$ there exists a term t such that $s_1 \longrightarrow^* t$ and $s_2 \longrightarrow^* t$.

A set of rewrite rules is **Church-Rosser** if for all terms s_1 and s_2 such that $s_1 \leftrightarrow^* s_2$, there exists a term t such that $s_1 \longrightarrow^* t$ and $s_2 \longrightarrow^* t$.

$$\begin{tikzcd} & r \arrow[dl, "*"'] \arrow[dr, "*"] & \\ s_1 \arrow[dr, "*"'] & & s_2 \arrow[dl, "*"] \\ & t & \end{tikzcd}$$

Theorem

Church-Rosser is equivalent to confluence.

Theorem

*For **terminating** rewrite sets, these properties mean that any expression will rewrite to a canonical normal form.*

Local Confluence

The properties of Church-Rosser and confluence can be difficult to prove. A weaker definition is useful:

A set of rewrite rules is **locally confluent** if for all terms r, s_1, s_2 such that $r \longrightarrow s_1$ and $r \longrightarrow s_2$ there exists a term t such that $s_1 \longrightarrow^* t$ and $s_2 \longrightarrow^* t$.

$$\begin{tikzcd}
& r \arrow[dl] \arrow[dr] & \\
s_1 \arrow[dr, "*"'] & & s_2 \arrow[dl, "*"] \\
& t &
\end{tikzcd}$$

Theorem (Newman's Lemma)

local confluence + termination = confluence

Also: local confluence is decidable (due to Knuth and Bendix)

Both theorem and the decision procedure use idea of **critical pairs**

Choices in Rewriting

How can choices arise in rewriting?

- Multiple rules apply to a single redex: order might matter
- Rules apply to multiple redexes:
  - if they are separate: order does not matter
  - if one contains the other: order might matter

We are interested in cases where the order matters:

Rules	Rewrites	Critical Pair
$X^0 \Rightarrow 1$ $0^Y \Rightarrow 0$	0^0 rewrites to 0 and to 1	$\langle 0, 1 \rangle$
$X \cdot e \Rightarrow X$ $(X \cdot Y) \cdot Z \Rightarrow X \cdot (Y \cdot Z)$	$(x \cdot e) \cdot z$ rewrites to $x \cdot z$ and $x \cdot (e \cdot z)$	$\langle x \cdot z, x \cdot (e \cdot z) \rangle$

Critical Pairs

Given two rules $L_1 \Rightarrow R_1$ and $L_2 \Rightarrow R_2$, we are concerned with the case when there exists a *non-variable* sub-term s of L_1 such that $s[\theta] = L_2[\theta]$, with most general unifier θ.

Applying these rules in different orders gives rise to a **critical pair**, where $L_1[\theta]\{R_2[\theta]/s[\theta]\}$ denotes replacing $s[\theta]$ by $R_2[\theta]$ in $L_1[\theta]$.

$$\begin{tikzcd}
 & L_1[\theta] \arrow[dl] \arrow[dr] & \\
R_1[\theta] & & L_1[\theta]\{R_2[\theta]/s[\theta]\}
\end{tikzcd}$$

$\langle R_1[\theta], L_1[\theta]/\{R_2[\theta]/s[\theta]\}\rangle$ is the critical pair.

Note: the variables in the two rules should be *renamed* so they do **not** share any variable names.
Note: A rewrite rule may have critical pairs with itself e.g. consider the rule $f(f(x)) \Rightarrow g(x)$.

With $W \cdot e \Rightarrow W$ and $(X \cdot Y) \cdot Z \Rightarrow X \cdot (Y \cdot Z)$, where X, Y and Z are variables, we can have $\theta = [W/X, e/Y]$, any other?

Critical Pairs: Example

Consider the rewrite rules:

$$\overbrace{f(\underbrace{f(x, y)}_{s}, z)}^{L_1} \quad \Rightarrow \quad \overbrace{f(x, f(y, z))}^{R_1}$$

$$\overbrace{f(i(x_1), x_1)}^{L_2} \quad \Rightarrow \quad \overbrace{e}^{R_2}$$

The mgu θ, given our choice of non-variable subterm s of L_1, is given by $\theta = \{i(x_1)/x, x_1/y\}$ and by considering:

$$\begin{tikzcd}
 & f(f(i(x_1), x_1), z) \arrow[dl] \arrow[dr] & \\
f(i(x_1), f(x_1, z)) & & f(e, z)
\end{tikzcd}$$

We get the critical pair $\langle f(i(x_1), f(x_1, z)), f(e, z)\rangle$.

Testing for Local Confluence

If we can **conflate** (join) all the critical pairs, then have **local confluence**.

Conflation for a critical pair $\langle s_1, s_2 \rangle$ is when there is a t such that $s_1 \longrightarrow^* t$ and $s_2 \longrightarrow^* t$.

An algorithm to test for local confluence (assuming termination):

1. Find all the critical pairs in set of rewrite rules R
2. For each critical pair $\langle s_1, s_2 \rangle$:
   2.1 Find a normal form s_1' of s_1;
   2.2 Find a normal form s_2' of s_2;
   2.3 Check $s_1' = s_2'$, if not then fail.

Establishing Local Confluence

Sometimes a set of rules is not locally confluent

$$\begin{array}{l} X \cdot e \Rightarrow X \\ f \cdot X \Rightarrow X \end{array} \quad \text{is not locally confluent: } \langle f, e \rangle \text{ does not conflate.}$$

We can add the rule $f \Rightarrow e$ to make this critical pair joinable.

However, adding new rules requires care:

- Must preserve termination
- Might give rise to *new* critical pairs and so we may need to check local confluence again.

Establishing Local Confluence: Example

Consider the set R consisting of just one rewrite rule, with x a variable:

$$f(f(x)) \Rightarrow g(x)$$

which has exactly one critical pair (CP) when it is overlapped with a *renamed* copy of itself $f(f(y)) \Rightarrow g(y)$. The lhs $f(f(x))$ unifies with the subterm $f(y)$ of the renamed lhs to produce the mgu $\{f(x)/y\}$:

$$\begin{tikzcd}
& f(f(f(x))) \arrow[dl] \arrow[dr] & \\
g(f(x)) & & f(g(x))
\end{tikzcd}$$

$\langle g(f(x)), f(g(x)) \rangle$ is the critical pair.

- This CP is not joinable, so R is not locally confluent.
- Adding the rule $f(g(x)) \Rightarrow g(f(x))$ to R makes the pair joinable.
- The enlarged R is terminating (how?), but
- (After renaming) new CP: $\langle g(g(z)), f(g(f(z))) \rangle$ arises (how?);
- LC test: it is joinable, $f(g(f(z))) \to g(f(f(z)) \to g(g(z))$.

Knuth-Bendix (KB) Completion Algorithm

Start with a set R of terminating rewrite rules

While there are non-conflatable critical pairs in R:

1. Take a critical pair $\langle s_1, s_2 \rangle$ in R
2. Normalise s_1 to s'_1 and s_2 to s'_2 (and we know $s'_1 \neq s'_2$)
3. if $R \cup \{s'_1 \Rightarrow s'_2\}$ is terminating then
   $R := R \cup \{s'_1 \Rightarrow s'_2\}$
   else if $R \cup \{s'_2 \Rightarrow s'_1\}$ is terminating then
   $R := R \cup \{s'_2 \Rightarrow s'_1\}$
   else Fail

- If KB succeeds then we have a locally confluent and terminating (and hence confluent) rewrite set (KB may run forever!)
- Depends on the termination check: define a measure and use that to test for termination.

Summary

- Rewriting (Bundy Ch. 9)
  - Local confluence
  - Local confluence + Termination = Confluence
  - Canonical Normal Forms
  - Critical Pairs and Knuth-Bendix Completion