A Geometry of Oriented Curves*

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Abstract. This technical report presents a geometric approach on the formal description of ordering on curves through space. Curves are usually considered as topological mappings from parameter sets to space and not described in a geometric framework. In contrast, we introduce oriented curves in an axiomatic framework as geometric entities on the same level as points and straight lines. This account does not require any numerical information and therefore enables a qualitative characterization of oriented curves. Oriented curves can for instance be used to represent trajectories of moving objects. They are basically atemporal and therefore allow temporal as well as non-temporal interpretations. Since oriented curves need not to be straight, they provide a generalized notion of direction. This report contains the formal part of this enterprise including the axioms, definitions, theorems and proofs. The general framework and an application is described by Eschenbach, Habel & Kulik (1999).

1 Introduction

The framework of ordering geometry gives a formal account of the spatial relation of betweenness (cf. Hilbert 1902, Huntington & Kline 1917, Huntington 1924, Eschenbach et al. 1998). Traditionally, the embedding of betweenness in non-linear spaces is based on collinearity and straightness. But betweenness can also be used relative to other kinds of linear structures, such as paths that are not straight but bent or generally curved (cf. Habel 1990). In this report we provide a formal geometric account of ordering on linear structures that can be embedded in more complex (spatial) structures. Thereby present the formal background of the idea that every linear structure can be oriented in exactly two ways by introducing the geometric descriptions of oriented curves in the framework of ordering geometry.

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Oriented curves can be considered as a geometric device generalizing polygonal curves since their parts do not have to be straight segments. Therefore, they can represent a variety of entities that are both linear (i.e., contain no cycles and do not branch) and directed (i.e., distinguish between start and end). Applications are given by planning tasks in transportation systems (like rail connections and air routes), traffic directing systems and more generally in qualitative descriptions of trajectories of moving objects. A description of applying oriented curves as geometric representations of simple trajectories of objects moving through space can be found in Eschenbach, Habel & Kulik (1999). The formal description below provides the proofs for the theorems stated this work.

In another paper (Eschenbach et al. 1998), we introduced a geometric description of shape curves a the basis of describing the shape of objects and object parts based on their outline. This description provides the means for distinguishing between straight and bent parts of curves, and between bents and kinks as direction changes in curves. Our intention has been to enrich this framework with the notion of oriented curves based on shape curves. However, the spatial embedding of the curves, i.e., their shape, is irrelevant for the question of ordering and orientation on the curves. Therefore, we refrain from presenting the whole geometric framework and restrict ourselves to describe curves and oriented curves. Nevertheless, one intended specialization for the curves described here are the shape curves of Eschenbach et al. (1998), which provide a general account to characterize essential aspects of shape based on features of the object's boundary.

To extend the classical ordering geometry with oriented curves we employ the axiomatic method. An axiomatic system constitutes a system of constraints and describes the spatial properties of basic terms like 'point' and 'curve' through axioms. The axioms determine the relations between the basic terms. An axiomatic specification of spatial relations has at least two advantages. It provides an exact characterization of the structure of a spatial relation and the systems can be compared concerning their assumptions about the underlying spatial structure. Our description does neither employ any metrical information nor any conception of time measurement. Since all further statements about the spatial properties of curves are derived from the axioms, the theory can also be used as a test-bed for automatic theorem provers.

In contrast to this geometric account, traditional approaches of mathematics and physics assume that linear structures in general are mappings from a prototypical linearly ordered set to the embedding structure. Since we aim at identifying general characteristics of these structures, we develop a description of curves that does not make any particular assumptions about the properties of curves, that is to say, whether they are smoothly bent, have vertices, are rectifiable, etc. The main requirement is that they are linear in the sense that they can be supplied with a total ordering structure.

The plan of this report is the following. We first present the characterization of curves and record their basic properties concerning the relation of incidence between points and curves and the sub-curve relation between curves. In the next step we show how betweenness can be defined on curves. The main idea is that a point *P* is between two other points on a curve if there is a sub-curve that connects the two points and has point P as an inner point Finally, we introduce oriented curves such that every open curve corresponds to two oriented curves that order the points on the curve in opposite manner. Thus, every curve can be oriented in exactly two ways and a ordered pair of points on it supplies a direction on a curve. Oriented curves constitute a more general way than oriented straight lines to describe directions in space.

2 Simple Curves

As the formal framework we assume many sorted predicate logic. The geometric structure introduces three types of entities and two primitive relations. The entities are points (denoted by $P, Q, P', P_1, \ldots$), curves ($c, c_1, \ldots$), and oriented curves ($o, o_1, \ldots$). The primitive relations are the binary relation of *incidence* (denoted by ι) and the ternary relation of *precedence* with respect to oriented curves ($\prec$).

2.1 Points on Curves

Definition 2.1 (Sub-curve)

A curve c' is *part of* another curve c or a *sub-curve* of c (in symbols $c' \sqsubset c$), if all points of c' are incident with c:

$$c' \sqsubset c \quad \Leftrightarrow_{\text{def}} \quad P\ [P \iota c' \quad P \iota c]$$

Definition 2.2 (Endpoint, Inner Point)

An *endpoint* of a curve is on the curve and of any two curve parts that include it one is part of the other. An *inner point* of a curve is on the curve and not an endpoint:

$$\mathrm{ept}(P, c) \quad \Leftrightarrow_{\text{def}} \quad P \iota c \quad c_1, c_2\ [c_1 \sqsubset c \quad c_2 \sqsubset c \quad P \iota c_1 \quad P \iota c_2$$
$$(c_1 \sqsubset c_2 \quad c_2 \sqsubset c_1)]$$

$$\mathrm{ipt}(P, c) \quad \Leftrightarrow_{\text{def}} \quad P \iota c \quad \neg\mathrm{ept}(P, c)$$

Remark. A logically equivalent variant uses the definition of an inner point as basic notion and defines the endpoint on this basis. A point is an inner point of a curve, if there are two sub-curves which include the point and none of the sub-curves is part of the other.

$$\mathrm{ipt}(P, c) \quad \Leftrightarrow \quad c_1, c_2\ [c_1 \sqsubset c \quad c_2 \sqsubset c \quad P \iota c_1 \quad P \iota c_2 \quad \neg c_1 \sqsubset c_2 \quad \neg c_2 \sqsubset c_1]$$
$$\mathrm{ept}(P, c) \quad \Leftrightarrow \quad P \iota c \quad \neg\mathrm{ipt}(P, c)$$

Definition 2.3 (Meeting at a Point, Sum)

Two curves c, c' *meet* at endpoint P, symbolized by $\mathrm{meet}(P, c, c')$, if P is a common point and all their common points are endpoints:

$$\mathrm{meet}(P, c, c') \quad \Leftrightarrow_{\text{def}} \quad P \iota c \quad P \iota c' \quad Q\ [Q \iota c \quad Q \iota c' \quad \mathrm{ept}(Q, c) \quad \mathrm{ept}(Q, c')]$$

Remark. This definition allows two curves to meet at both ends. In particular, if two curves meet at P then P is an endpoint of both curves and the curves cannot be identical (cf. (C9) and (C3)).

A curve c that has exactly the points of curves c_1 and c_2 (see axiom (C9)) is called their *sum* $c_1 \sqcup c_2$:

$$c = c_1 \sqcup c_2 \quad \Leftrightarrow_{\text{def}} \quad Q\,[Q \iota c \Leftrightarrow (Q \iota c_1 \quad Q \iota c_2)]$$

Remark. The operation sum ($\sqcup$) is only partially defined: Two curves need not have a curve as their sum since curves are connected and do not branch. Therefore, e.g., curves that have no point in common do not form a sum. (cf. (C8))

Definition 2.4 (Closed Curve, Open Curve)

If a curve c does not have an endpoint, then we call the curve *closed* (in symbols: $\mathrm{cl}(c)$). Otherwise we call the curve *open* and denote it by $\mathrm{op}(c)$:

$$\mathrm{cl}(c) \quad \Leftrightarrow_{\text{def}} \quad \neg \ P\,[\mathrm{ept}(P, c)]$$

$$\mathrm{op}(c) \quad \Leftrightarrow_{\text{def}} \quad P\,[\mathrm{ept}(P, c)]$$

2.2 Axioms for the Simple Curve Geometry

On this basis we can give the axioms for curves. The curves we define here are strictly linear in that they do not include internal cycles and do not branch. More precisely, according to (C1) every proper sub-curve of a given curve is open, and if three sub-curves of a given curve have one endpoint in common, then one of the three sub-curves is included in one of the others (C2).

(C1) $\quad c \quad c' \qquad [c' \sqsubset c \quad c' \neq c \quad \mathrm{op}(c')]$

(C2) $\quad c \quad c_1 \quad c_2 \quad c_3 \quad [c_1 \sqsubset c \quad c_2 \sqsubset c \quad c_3 \sqsubset c$
$\qquad P\,[\mathrm{ept}(P, c_1) \quad \mathrm{ept}(P, c_2) \quad \mathrm{ept}(P, c_3)]$
$\qquad c_2 \sqsubset c_3 \quad c_3 \sqsubset c_2 \quad c_1 \sqsubset c_2 \quad c_2 \sqsubset c_1 \quad c_1 \sqsubset c_3 \quad c_3 \sqsubset c_1]$

Every curve has at least one inner point, i.e. a point which is not an endpoint of this curve (C3). The axiom (C4) states that every inner point P of a curve divides the curve into two sub-curves meeting at P.

(C3) $\quad c \quad P \qquad [\mathrm{ipt}(P, c)]$

(C4) $\quad c \quad P \qquad [\mathrm{ipt}(P, c) \qquad c_1 \quad c_2\,[\mathrm{meet}(P, c_1, c_2) \quad c = c_1 \sqcup c_2]]$

Curves have at most two endpoints (C5) and, if a curve has one endpoint, then it has another one (C6). On the other hand, if two curves meet and constitute a closed curve, then they meet at all their endpoints (C7).

(C5) $\quad c \quad P \quad Q \quad R \quad [\mathrm{ept}(P, c) \quad \mathrm{ept}(Q, c) \quad \mathrm{ept}(R, c) \qquad (P = Q \quad P = R \quad Q = R)]$

(C6) $\quad c \quad P \qquad [\mathrm{ept}(P, c) \qquad Q\,[\mathrm{ept}(Q, c) \quad P \neq Q]]$

(C7) $\quad c \quad c_1 \quad c_2 \quad P \quad [\mathrm{cl}(c) \quad \mathrm{meet}(P, c_1, c_2) \quad c = c_1 \sqcup c_2$
$\qquad Q\,[\mathrm{ept}(Q, c_1) \quad \mathrm{meet}(Q, c_1, c_2)]]$

If two curves meet at one endpoint, then there is a curve that has exactly the points of the two given curves (C8). Curves differ in the points they are incident with (C9). Therefore, curves can be represented as sets of points, although we do not employ such a representation.

(C8) $\quad c_1 \quad c_2 \qquad [\ P\,[\mathrm{meet}(P, c_1, c_2)] \qquad c\,[c = c_1 \sqcup c_2]]$

(C9) $\quad c \quad c' \qquad [\ P\,[P \iota c \Leftrightarrow P \iota c'] \qquad c = c']$

2.3 Basic Consequences of the Axioms of the Curve Geometry

We note some consequences of these axioms.

Theorem 2.5

The relation sub-curve ($\sqsubset$) is an order relation.

Proof. The reflexivity and the transitivity are obvious, and the antisymmetry follows from (C9). √

The next Corollary says that the sum is monotone. We omit the simple proofs.

Corollary 2.6

(1) $c_1 \; c_2 \; c_3 \; P \quad [c_2 \sqsubset c_3 \quad \mathsf{meet}(P, c_1, c_2) \quad \mathsf{meet}(P, c_1, c_3) \quad c_1 \sqcup c_2 \sqsubset c_1 \sqcup c_3]$

(2) $c_1 \; c_2 \; c_3 \; P \quad [c_1 \sqsubset c_3 \quad c_2 \sqsubset c_3 \quad \mathsf{meet}(P, c_1, c_2) \quad c_1 \sqcup c_2 \sqsubset c_3]$

We record two basic facts about open curves. Every open curve has at least two endpoints, and every sub-curve of an open curve is open.

Theorem 2.7

(1) $c \quad [\mathsf{op}(c) \quad P \;\; Q \, [P \neq Q \quad \mathsf{ept}(P, c) \quad \mathsf{ept}(Q, c)]]$

(2) $c \;\; c' \quad [\mathsf{op}(c) \quad c' \sqsubset c \quad \mathsf{op}(c')]$

Proof of (1). The statement is an immediate consequence of the definition of an open curve and of axiom (C6). √

Remark 2.8. Considering (1) and axiom (C5), we obtain that every open curve has exactly two endpoints.

Proof of (2). If $c' = c$ then the statement is trivial. If $c' \neq c$ then axiom (C1) yields $\mathsf{op}(c')$. √

If one curve is part of another curve then they cannot meet.

Corollary 2.9

$c_1 \;\; c_2 \quad [c_1 \sqsubset c_2 \quad \neg \;\; P \, [\mathsf{meet}(P, c_1, c_2)]]$

Proof. Considering axiom (C3) we know that there is a point Q being incident with c_1 and c_2 without being an endpoint of c_1. Therefore $\mathsf{meet}(P, c_1, c_2)$ does not hold. √

In the next step we record a simple but important property of endpoints: If an endpoint of a given curve lies on a sub-curve then it is also an endpoint of this sub-curve.

Theorem 2.10

$c \;\; c' \;\; P \quad [c' \sqsubset c \quad \mathsf{ept}(P, c) \quad P \iota c' \quad \mathsf{ept}(P, c')]$

Proof. Let c_1, c_2 be two curves with $c_1 \sqsubset c'$, $c_2 \sqsubset c'$, $P \iota c_1$ and $P \iota c_2$. Since $c' \sqsubset c$ holds, $c_1 \sqsubset c$ and $c_2 \sqsubset c$ follow from Theorem 2.5. From $\mathsf{ept}(P, c)$ we get via the definition of ept $c_1 \sqsubset c_2$ or $c_2 \sqsubset c_1$. Applying the definition again, we derive $\mathsf{ept}(P, c')$. √

A simple transformation of Theorem 2.10 yields that inner points of a sub-curve of any curve c are inner points of c.

Corollary 2.11

$c \;\; P \quad [\;\; c' \, [c' \sqsubset c \quad \mathsf{ipt}(P, c')] \quad \mathsf{ipt}(P, c)]$

Similarly, Theorem 2.10 yields that if a point *P* is a meeting point of two curves and lies on a sub-curve of one of the two curves then *P* is also meeting point of the sub-curve and the other curve.

Corollary 2.12

$c_1 \quad c_2 \quad c' \quad P \qquad [c_2 \sqsubset c_1 \quad P \iota c_2 \quad \mathrm{meet}(P, c_1, c') \quad \mathrm{meet}(P, c_2, c')]$

The next theorem states that two distinct points on an open curve uniquely determine the sub-curve connecting these points.

Theorem 2.13

$c \quad c_1 \quad c_2 \qquad [c_1 \sqsubset c \quad c_2 \sqsubset c \quad \mathrm{op}(c) \quad P \quad Q\,[P \neq Q \quad \mathrm{ept}(P, c_1)$
$\mathrm{ept}(P, c_2) \quad \mathrm{ept}(Q, c_1) \quad \mathrm{ept}(Q, c_2)] \quad c_1 = c_2]$

Proof. If c_1, c_2 and c have a common endpoint, then we call this point *P* and c also c_3 and continue with step 2 of the proof. Otherwise, if both *P* and *Q* are inner points of c, then we first construct a sub-curve of c, that has c_1 and c_2 as parts and *P* as an endpoint.

Step 1. If both *P* and *Q* are inner points of c, then c can be divided into sub-curves c_3 and c_4 that have one endpoint in *P* (C4) and another one in the endpoints of c, respectively (Theorem 2.10). According to (C2) c_1 and c_2 are part of c_3 or c_4, or contain one of these. If, say, c_3 is in c_1, then the other endpoint of c_3 is in c_1. But, since it is also an endpoint of c, it must be an endpoint of c_1 (Theorem 2.10) and hence (C5) identical to *Q* contradicting the assumption that *Q* is not an endpoint. Therefore, c_1 and c_2 are part of c_3 or c_4. Moreover, they have to be part of the same sub-curve, either c_3 or c_4, because, if they would lie on different sub-curves, *Q* would lie on c_3 and c_4 and using $\mathrm{meet}(P, c_3, c_4)$ yields that it is an endpoint of both curves and therefore of c, which again contradicts the assumption. Let the partitioning curve that contains both c_1 and c_2 be c_3.

Step 2. *P* is an endpoint of c_1, c_2 and c_3. By the definition of ept, we know that c_1 must be part of c_2 or vice versa. Without loss of generality, we can assume $c_1 \sqsubset c_2$. If c_1 and c_2 are not identical there is a point *R* being incident with c_2 that does not lie on c_1 (C9). *R* cannot be an endpoint of c_2 (C5). Therefore, we can divide c_2 into two sub-curves c_5, c_6 meeting in *R* (C4) and having *P* and *Q*, the endpoints of c_2, as their respective second endpoints (Theorem 2.10). Since c_1 does not contain *R*, it can contain neither c_5 nor c_6. But with *P* and *Q* being endpoints of c_2 and lying on c_1, c_1 itself had to be part of both c_5 and c_6 (definition of ept). Since this contradicts $\mathrm{meet}(R,$Ê$c_5, c_6)$ (C3, definition of meet) a point like *R* cannot exist. Thus $c_1 = c_2$ has been proved (C9) for the case that *P* and *Q* both are inner points of c. √

We record five simple consequences for the meet-relation. (1) If two curves meet and their sum is open, then the only point they have in common is their meeting-point. (2) If two open sub-curves of an open curve meet, then their sum is also open. (3) A meeting point of two curves is not an endpoint of any curve that includes both as sub-curves. (4) If two curves meet and their sum is open, then the endpoints of the two curves that are not the meeting-point are also the endpoints of the sum of these curves. (5) If two curves meet, than the endpoints of the sum are exactly those endpoints of the two curves that are not meeting-points of them.

Proposition 2.14.

(1) $c_1 \quad c_2 \quad P \quad [\mathsf{meet}(P, c_1, c_2) \quad \mathsf{op}(c_1 \sqcup c_2)$
$\quad Q\,[Q \neq P \quad \neg(Q\ \iota\ c_1 \quad Q\ \iota\ c_2)]]$

(2) $c \quad c_1 \quad c_2 \quad P \quad [\mathsf{op}(c) \quad c_1 \sqsubset c \quad c_2 \sqsubset c \quad \mathsf{meet}(P, c_1, c_2) \quad \mathsf{op}(c_1 \sqcup c_2)]$

(3) $c_1 \quad c_2 \quad P \quad [\mathsf{meet}(P, c_1, c_2) \quad c_1 \sqsubset c \quad c_2 \sqsubset c \quad \neg\mathsf{ept}(P, c)]$

(4) $c_1 \quad c_2 \quad P \quad [\mathsf{meet}(P, c_1, c_2) \quad \mathsf{op}(c_1 \sqcup c_2)$
$\quad Q \quad R\,[P \neq Q \quad Q \neq R \quad P \neq R \quad \mathsf{ept}(Q, c_1 \sqcup c_2)$
$\quad \mathsf{ept}(Q, c_1) \quad \mathsf{ept}(R, c_1 \sqcup c_2) \quad \mathsf{ept}(R, c_2)]]$

(5) $c_1 \quad c_2 \quad [\ P\,[\mathsf{meet}(P, c_1, c_2)] \quad Q\,[\mathsf{ept}(Q, c_1 \sqcup c_2) \Leftrightarrow$
$\quad (\neg\mathsf{meet}(Q, c_1, c_2) \quad (\mathsf{ept}(Q, c_1) \quad \mathsf{ept}(Q, c_2)))]]$

Proof of (1). According to the definition of the meet-relation only the endpoints of c_1 and c_2 can be incident with both curves c_1 and c_2. Since $\mathsf{op}(c_1 \sqcup c_2)$ there is an endpoints R of $c_1 \sqcup c_2$. Because of Corollary 2.9 c_1 and c_2 are not in the sub-curve relation and therefore R is not on both c_1 and c_2. If R is on c_1, then it is an endpoint of c_1 (Theorem 2.10). The other endpoint of c_1 is P and therefore (C5) this is the only point common to both sub-curves. If R is on c_2, the argument is the same. √

Proof of (2). If $c_1 \sqcup c_2 = c$ holds then we immediately get $\mathsf{op}(c_1 \sqcup c_2)$. If $c_1 \sqcup c_2 \neq c$ holds then we obtain $\mathsf{op}(c_1 \sqcup c_2)$ by employing axiom (C1) and Corollary 2.6.2. √

Proof of (3). If P is an endpoint of c then c_1 and c_2 fulfill $c_1 \sqsubset c$, $c_2 \sqsubset c$, $P\ \iota\ c_1$ and $P\ \iota\ c_2$ and $c_1 \sqsubset c_2$ or $c_2 \sqsubset c_1$ holds. This contradicts Corollary 2.9 that neither $c_1 \sqsubset c_2$ nor $c_2 \sqsubset c_1$ holds. Therefore, P cannot be an endpoint of c. √

Proof of (4). Since $c_1 \sqcup c_2$ is open, Theorem 2.7.1 shows that there are two points Q and R with $Q \neq R$, $\mathsf{ept}(Q, c_1 \sqcup c_2)$ and $\mathsf{ept}(R, c_1 \sqcup c_2)$. According to (3) $\neg\mathsf{ept}(P, c_1 \sqcup c_2)$ holds, hence we have $P \neq Q$ and $P \neq R$. Since both Q and R are on $c_1 \sqcup c_2$, we can assume without loss of generality $Q\ \iota\ c_1$. Theorem 2.10 then yields $\mathsf{ept}(Q, c_1)$. Now, $R\ \iota\ c_1$ cannot hold, otherwise Theorem 2.10 would yield again $\mathsf{ept}(R, c_1)$. As $\mathsf{ept}(P, c_1)$ holds, this would contradict axiom (C5). Therefore we have $R\ \iota\ c_2$. Applying Theorem 2.10 again we obtain $\mathsf{ept}(R, c_2)$. √

Proof of (5). : Let Q be endpoint of $c_1 \sqcup c_2$. Because of (3) Q is not a meeting-point of c_1 and c_2. But Q is incident with one of them, and, because of Theorem 2.10 endpoint of that sub-curve.

: Let Q be an endpoint of, say, c_1. If Q is also incident with c_2, then $\mathsf{meet}(Q, c_1, c_2)$ holds and nothing has to be proved. If $c_1 \sqcup c_2$ is open, then it has two endpoints (Theorem 2.7.1). Since both c_1 and c_2 have P as an endpoint and P is not one of the endpoints of $c_1 \sqcup c_2$, $c_1 \sqcup c_2$ has one endpoint in common with c_1, and this must be Q. If $c_1 \sqcup c_2$ is closed, then all endpoints of c_1 or c_2 are meeting-points by Axiom (C7) and the condition is fulfilled. √

The next theorem says that for any two points on a curve there is a sub-curve that connects these two points, that is to say these points are the endpoints of the sub-curve.

Theorem 2.15

$P \quad Q \quad c \qquad\qquad [P \neq Q \quad P \iota c \quad Q \iota c \qquad c' \, [c' \sqsubset c \quad \mathrm{ept}(P, c') \quad \mathrm{ept}(Q, c')]]$

Proof. If P and Q are the endpoints of c, the theorem is trivially true. If P is an inner point of c, there are—according to (C4)—two sub-curves c_1, c_2 satisfying

$$\mathrm{meet}(P, c_1, c_2) \quad c = c_1 \sqcup c_2.$$

Q lies on c_1 or on c_2. Without loss of generality we can assume $Q \iota c_2$. If Q is an endpoint of c, then it is endpoint of c_2 as well (Theorem 2.10) and the theorem has been proved. If Q is an inner point of c_2, applying axiom (C4) to Q and c_2 yields two sub-curves c_3 and c_4 fulfilling

$$\mathrm{meet}(Q, c_3, c_4) \quad c_2 = c_3 \sqcup c_4.$$

Since $\mathrm{ept}(P, c_2)$, $c_3 \sqsubset c_2$, $c_4 \sqsubset c_2$ and $Q \iota (c_3 \sqcup c_4)$ hold, Theorem 2.10 yields $\mathrm{ept}(P, c_3)$ or $\mathrm{ept}(P, c_4)$. Let the sub-curve P and Q are endpoint of be designated with c', then c' satisfies all conditions of the theorem. √

For closed curves, we can find two such sub-curves that complement each other.

Theorem 2.16

$c \quad P \quad Q \qquad\qquad [\mathrm{cl}(c) \quad P \iota c \quad Q \iota c \quad P \neq Q$

$c_1 \quad c_2 \, [\mathrm{meet}(P, c_1, c_2) \quad \mathrm{meet}(Q, c_1, c_2) \quad c = c_1 \sqcup c_2]]$

Proof. If $\mathrm{cl}(c)$ then $\neg\mathrm{ept}(P, c)$ and according to (C4) there are curves c_3 and c_4, such that $\mathrm{meet}(P, c_3, c_4) \quad c = c_3 \sqcup c_4$. Since $Q \iota c$ we have $Q \iota c_3$ or $Q \iota c_4$. Without loss of generality we assume $Q \iota c_4$. If $\mathrm{ept}(Q, c_4)$, then we are—because of (C7)—done with $c_1 = c_3$, $c_2 = c_4$. Otherwise (C4) is applicable to c_4, such that there are curves c_5 and c_6 with $\mathrm{meet}(Q, c_5, c_6) \quad c_4 = c_5 \sqcup c_6$. P is on c_5 or c_6. Let it be c_5. Then P and Q are the two endpoints of c_5 (Theorem 2.10). Let R be the endpoint of c_4 that is different from P. R is an endpoint of c_3 (by (C7)) and of c_6 (Theorem 2.10, (C5)). Therefore $\mathrm{meet}(R,$Ê$c_3, c_6)$ (Corollary 2.12) and we are done with $c_1 = c_5$, $c_2 = c_3 \sqcup c_6$. ((C8) and Proposition 2.14.5). √

Every open sub-curve of a closed curve can be complemented by another curve so that their sum constitute the closed curve.

Theorem 2.17

$c \quad c_1 \quad P \quad Q \qquad\qquad [\mathrm{cl}(c) \quad c_1 \sqsubset c \quad \mathrm{ept}(P, c_1) \quad \mathrm{ept}(Q, c_1) \quad P \neq Q$

$c_2 \, [\mathrm{meet}(P, c_1, c_2) \quad \mathrm{meet}(Q, c_1, c_2) \quad c = c_1 \sqcup c_2]]$

Proof. $P \iota c$ is easy to prove. If $\mathrm{cl}(c)$ then $\neg\mathrm{ept}(P, c)$ and according to Theorem 2.16 there are curves c_3 and c_4, such that $\mathrm{meet}(P, c_3, c_4) \quad \mathrm{meet}(Q, c_3, c_4) \quad c = c_3 \sqcup c_4$. According to (C2) c_1 is a sub-curve of c_3 or c_4 or has one of them as sub-curve. Without loss of generality we assume $c_1 \sqsubset c_3 \quad c_3 \sqsubset c_1$. Since $\sqsubset$ is reflexive and both c_1 and c_3 are open, Theorem 2.13 yields $c_1 = c_3$. We are done with $c_2 = c_4$. √

Every proper sub-curve of an open curve that has a common endpoint with the open curve can be complemented by another curve so that their sum constitute the open curve.

Theorem 2.18

$$c \quad c_1 \quad P \qquad [\mathrm{op}(c) \quad c_1 \sqsubset c \quad c_1 \neq c \quad \mathrm{ept}(P, c_1) \quad \mathrm{ept}(P, c)$$
$$Q \quad c_2\, [P \neq Q \quad \mathrm{meet}(Q, c_1, c_2) \quad c = c_1 \sqcup c_2]]$$

Proof. c_1 is a proper sub-curve of c and hence open (axiom (C1)). The other uniquely determined endpoint of c_1 (Remark 2.8) is denoted by Q. Q is not endpoint of c, otherwise Theorem 2.13 yields $c_1 = c$ contradictory to the assumption. Applying axiom (C4) we know that there are two curves c' and c_2 that satisfy $\mathrm{meet}(Q, c', c_2)$ and $c = c' \sqcup c_2$. Since $P \iota c' \sqcup c_2$ holds we can assume without loss of generality $P \iota c'$. Considering $\mathrm{ept}(P, c)$ Theorem 2.10 yields $\mathrm{ept}(P, c')$ and because of $\mathrm{ept}(Q, c')$, $c' = c_1$ follows from Theorem 2.13. √

Theorem 2.18 can be intensified in the following way:

Corollary 2.19

$$c \quad c_1 \quad P \qquad [c_1 \sqsubset c \quad c_1 \neq c \quad \mathrm{op}(c) \quad \mathrm{ept}(P, c_1) \quad \mathrm{ept}(P, c)$$
$$Q \quad R \quad c_2\, [\mathrm{meet}(Q, c_1, c_2) \quad c = c_1 \sqcup c_2$$
$$P \neq Q \quad Q \neq R \quad P \neq R \quad \mathrm{ept}(R, c_2) \quad \mathrm{ept}(R, c)]]$$

Proof. Considering Theorem 2.18 we have to prove only that there is a point R satisfying the above requirement. Regarding Remark 2.8 we know that there is another uniquely determined endpoint of c. We call this point R. Therefore $\mathrm{ept}(R, c)$ as well as $P \neq R$ holds. According to Proposition 2.14.3 Q is not an endpoint of c. Hence we conclude $Q \neq R$. Using Proposition 2.14.1 we get $\neg(R \iota c_1) \quad R \iota c_2$. Assuming $R \iota c_1$ Theorem 2.10 yields $\mathrm{ept}(R, c_1)$ and applying axiom (C9) we obtain $c_1 = c$ contradictory to the assumption $c_1 \neq c$. Hence we get $R \iota c_2$. Applying Theorem 2.10 again $\mathrm{ept}(R, c_2)$ follows. √

If two sub-curves of one curve have a common endpoint and include a sub-curve starting at this endpoint, then one of the two sub-curves is included in the other.

Theorem 2.20

$$c_1 \quad c_2 \qquad [\quad c_3 \quad P\, [c_3 \sqsubset c_1 \quad c_3 \sqsubset c_2 \quad \mathrm{ept}(P, c_1) \quad \mathrm{ept}(P, c_2) \quad \mathrm{ept}(P, c_3)]$$
$$c\, [c_1 \sqsubset c \quad c_2 \sqsubset c] \quad c_1 \sqsubset c_2 \quad c_2 \sqsubset c_1]$$

Proof. If $c_3 = c_1$ or $c_3 = c_2$ holds then the theorem is proved. Therefore we assume in the following $c_3 \neq c_1$ and $c_3 \neq c_2$. Hence, we can apply Theorem 2.18 to c_3 and c_1 as well as to c_3 and c_2, respectively. It follows that there is point Q distinct from P and curves c'_1 and c'_2 satisfying $\mathrm{meet}(Q, c_3, c'_1)$ and $c_1 = c_3 \sqcup c'_1$ as well as $\mathrm{meet}(Q, c_3, c'_2)$ and $c_2 = c_3 \sqcup c'_2$. Since $c_3 \sqsubset c$, $c'_1 \sqsubset c$ and $c'_2 \sqsubset c$ is fulfilled, axiom (C2) yields $c'_1 \sqsubset c'_2$ or $c'_2 \sqsubset c'_1$. The monotony of the sum gives in agreement with Corollary 2.6 $c_3 \sqcup c'_1 \sqsubset c_3 \sqcup c'_2$ or $c_3 \sqcup c'_2 \sqsubset c_3 \sqcup c'_1$, that is to say $c_1 \sqsubset c_2$ or $c_2 \sqsubset c_1$. √

If two sub-curves of an open curve have a common endpoint and another point in common, then one of the two sub-curves is included in the other. This weaker statement is a direct consequence of the above theorem.

Corollary 2.21

$$c \quad c_1 \quad c_2 \quad P \quad Q \qquad [\mathrm{op}(c') \quad c_1 \sqsubset c \quad c_2 \sqsubset c \quad \mathrm{ept}(P, c_1) \quad \mathrm{ept}(P, c_2)$$
$$P \neq Q \quad Q \iota c_1 \quad Q \iota c_2 \quad c_1 \sqsubset c_2 \quad c_2 \sqsubset c_1]$$

Proof. Since $Q \iota c_1$, $P \iota c_1$, $Q \iota c_2$, $P \iota c_2$ hold, there are due to Theorem 2.15 curves c_3 and c_4 with $c_3 \sqsubset c_1$, $c_4 \sqsubset c_2$, $\mathsf{ept}(P, c_3)$, $\mathsf{ept}(Q, c_3)$, $\mathsf{ept}(P, c_4)$, $\mathsf{ept}(Q, c_4)$. Because of Theorem 2.5 $c_3 \sqsubset c$ and $c_4 \sqsubset c$. And Theorem 2.13 yields $c_3 = c_4$. Theorem 2.20 proves the statement. √

If two sub-curves of a given open curve have a common endpoint then the sub-curves meet or one is included in the other.

Theorem 2.22

$$c \quad c_1 \quad c_2 \quad P \qquad [\mathsf{ept}(P, c_1) \quad \mathsf{ept}(P, c_2) \quad c_1 \sqsubset c \quad c_2 \sqsubset c \quad \mathsf{op}(c)$$
$$\mathsf{meet}(P, c_1, c_2) \quad c_1 \sqsubset c_2 \quad c_2 \sqsubset c_1]$$

Proof. If $\mathsf{meet}(P, c_1, c_2)$ holds then the theorem is proved. Therefore, we assume in the following $\neg\mathsf{meet}(P, c_1, c_2)$. Considering the definition of the meet-relation we get from $\mathsf{ept}(P, c_1)$ and $\mathsf{ept}(P, c_2)$ that there is a point Q satisfying $Q \iota c_1 \quad Q \iota c_2 \quad \neg(\mathsf{ept}(Q, c_1) \quad \mathsf{ept}(Q, c_2))$. According to Theorem 2.15 there are two sub-curves c', c'' satisfying $c' \sqsubset c_1 \quad \mathsf{ept}(P, c') \quad \mathsf{ept}(Q, c')$ and $c'' \sqsubset c_2 \quad \mathsf{ept}(P, c'') \quad \mathsf{ept}(Q, c'')$. Since c is open and $c' \sqsubset c$ as well as $c'' \sqsubset c$ holds, Theorem 2.13 yields $c' = c''$. The application of Theorem 2.20 to c', c_1 and c_2 shows $c_1 \sqsubset c_2$ or $c_2 \sqsubset c_1$ and completes the proof. √

An immediate consequence of Corollary 2.21 and Theorem 2.22 is the following statement: If two sub-curves of an open curve meet at a point and this point is an endpoint for another sub-curve then this sub-curve meets one of the former sub-curves at this point.

Proposition 2.23

$$c \quad c_1 \quad c_2 \quad c_3 \quad P \qquad [c_1 \sqsubset c \quad c_2 \sqsubset c \quad c_3 \sqsubset c \quad \mathsf{meet}(P, c_1, c_2) \quad \mathsf{ept}(P, c_3) \quad \mathsf{op}(c)$$
$$\mathsf{meet}(P, c_1, c_3) \quad \mathsf{meet}(P, c_2, c_3)]$$

Proof. If the only points that are incident with both c_1 and c_3 are endpoints of c_1, then c_1 is not a sub-curve of c_3 and c_3 is not a sub-curve of c_1 (Axiom (C3) and Corollary 2.11). Theorem 2.22 then yields $\mathsf{meet}(P, c_1, c_3)$.

If on the other hand there is an inner point Q of c_1 being incident with c_3 then Corollary 2.21 yields $c_1 \sqsubset c_3 \quad c_3 \sqsubset c_1$. If $c_3 \sqsubset c_1$ is satisfied then, by Corollary 2.12, we get $\mathsf{meet}(P, c_2, c_3)$. If, on the other hand, $c_1 \sqsubset c_3$, then Theorem 2.22 yields $\mathsf{meet}(P,$Ê$c_2, c_3) \quad c_2 \sqsubset c_3 \quad c_3 \sqsubset c_2$ and we have to check whether the conditions $c_2 \sqsubset c_3$ and $c_3 \sqsubset c_2$ can hold. If $c_3 \sqsubset c_2$ holds then we have $c_1 \sqsubset c_2$, which contradicts $\mathsf{meet}(P,$Ê$c_1, c_2)$. If $c_2 \sqsubset c_3$ holds then we have found two sub-curves of c_3, namely c_1 and c_2, that fulfill $\mathsf{meet}(P, c_1, c_2)$ and this contradicts the assumption $\mathsf{ept}(P, c_3)$ (Proposition 2.14.3). Therefore $\mathsf{meet}(P, c_2, c_3)$ holds. √

We conclude this section with a theorem about closed curves. If two sub-curves have two common endpoints then they are identical or their sum is the whole curve.

Theorem 2.24

$$c \quad c_1 \quad c_2 \qquad [\ P \quad Q\ [\mathsf{ept}(P, c_1) \quad \mathsf{ept}(Q, c_1) \quad \mathsf{ept}(P, c_2) \quad \mathsf{ept}(Q, c_2)$$
$$P \neq Q] \quad \mathsf{cl}(c) \quad c_1 \sqsubset c \quad c_2 \sqsubset c \quad c_1 = c_2 \quad c = c_1 \sqcup c_2]$$

Proof. From Theorem 2.16 we know that there is a partitioning of c in sub-curves c_1 and c_2 such that $\mathsf{meet}(P, c_1, c_2) \quad \mathsf{meet}(Q, c_1, c_2) \quad c = c_1 \sqcup c_2$. We just have to prove that any c_3 with $\mathsf{ept}(P, c_3) \quad \mathsf{ept}(Q, c_3) \quad c_3 \sqsubset c$ is identical to one of c_1 and c_2.

From (C2) we get that $c_2 \sqsubset c_3 \quad c_3 \sqsubset c_2 \quad c_1 \sqsubset c_2 \quad c_2 \sqsubset c_1 \quad c_1 \sqsubset c_3 \quad c_3 \sqsubset c_1$. Since $\mathsf{meet}(Q, c_1, c_2)$ we get from Corollary 2.9 that $c_1 \sqsubset c_2$ and $c_2 \sqsubset c_1$ do not hold. Since the three sub-curves are open and they have the same endpoints, we get by Theorem 2.13 that if any of them is a sub-curve of the other, then the two curves are identical. Thus, we have $c_2 = c_3 \quad c_1 = c_3$. √

3 Orderings on Simple Curves

3.1 The Definition of Betweenness

The main idea for the definition of betweenness on curves is the following: A point Q is between two other points if there is a sub-curve that connects the two points and Q is an inner point of this sub-curve.[1]

Definition 3.1 (Betweenness on Curves)

$$\beta_1(c, P, Q, R) \ \Leftrightarrow_{\text{def}} \ P \neq R \quad c' \,[c' \sqsubset c \quad \mathsf{ept}(P, c') \quad \mathsf{ept}(R, c') \quad \mathsf{ipt}(Q, c')]$$

An alternative proposal consists of the idea that a point is between two other points with respect to the curve if there are two sub-curves that connect the three points and the point in question is the meeting-point of the two sub-curves.

Definition 3.2 (Betweenness on Curves)

$$\beta_2(c, P, Q, R) \ \Leftrightarrow_{\text{def}} \ P \neq Q \quad P \neq R \quad Q \neq R \quad c_1 \quad c_2 \,[\mathsf{meet}(Q, c_1, c_2) \quad c_1 \sqsubset c \quad c_2 \sqsubset c \quad \mathsf{ept}(P, c_1) \quad \mathsf{ept}(R, c_2)]$$

In the framework proposed here, these definitions are equivalent:

Theorem 3.3

$$c \quad P \quad Q \quad R \qquad [\beta_1(c, P, Q, R) \Leftrightarrow \beta_2(c, P, Q, R)]$$

Proof. On account of axiom (C4) $\beta_2(c, P, Q, R)$ follows from $\beta_1(c, P, Q, R)$. The application of axiom (C8), Proposition 2.14.3 and 5 prove $\beta_1(c, P, Q, R)$ on the basis of $\beta_2(c, P, Q, R)$. √

In the following we use both variants. A third alternative is suggested by axiom (C4). It says that a point is between two points with respect to a curve, if it is the meeting point of two sub-curves that partition the curve such that the other two points are on different sub-curves:

Definition 3.4 (Betweenness on Curves)

$$\beta_3(c, P, Q, R) \ \Leftrightarrow_{\text{def}} \ P \neq Q \quad P \neq R \quad Q \neq R \quad c_1 \quad c_2 \,[\mathsf{meet}(Q, c_1, c_2) \quad (c_1 \sqcup c_2 = c) \quad P \iota c_1 \quad R \iota c_2]$$

[1] This definition of betweenness for points on curves is based on Theorem 2.15 that for two given points there is always a sub-curve of this curve such that the two points are the endpoints for the sub-curve.

Theorem 3.5

$$c \quad P \quad Q \quad R \qquad [\beta_2(c, P, Q, R) \Leftrightarrow \beta_3(c, P, Q, R)]$$

Proof. $\beta_2(c, P, Q, R)$ follows from $\beta_3(c, P, Q, R)$ and axiom (C4). The application of axiom (C4) and (C8) prove $\beta_3(c, P, Q, R)$ on the basis of $\beta_2(c, P, Q, R)$. √

Since we need not distinguish between the variants of defining betweenness, we will use the symbol β in the following.

A simple consequence is that every endpoint of an open curve is not between any other point of this curve.

Theorem 3.6

$$c \quad P \qquad [\mathsf{op}(c) \quad (\mathsf{ept}(P, c) \Leftrightarrow P \iota c \quad \neg \quad Q \quad R\, [\beta(c, Q, P, R)])]$$

Proof. We show: if $\mathsf{op}(c)$ then $\neg\,\mathsf{ept}(P, c) \Leftrightarrow \neg(P \iota c) \quad Q \quad R\, [\beta(c, Q, P, R)]$.

: If $\neg(P \iota c)$ then $\neg\,\mathsf{ept}(P, c)$ follows trivially. If $P \iota c$ and if there are two points Q and R fulfilling $\beta(c, Q, P, R)$ the point P cannot be an endpoint because of Definition 3.1 and Corollary 2.11.

: If $\neg(P \iota c)$ holds, there is nothing to prove. Therefore we can assume $P \iota c$. Since P is not an endpoint of c there are in accordance with axiom (C4) two curves c_1 and c_2 satisfying $\mathsf{meet}(P, c_1, c_2)$ and $c = c_1 \sqcup c_2$. c is open, thus there are exactly two endpoints Q and R of c (cf. Remark 2.8). Proposition 2.14.3 shows $P \neq Q$ and $P \neq R$. Hence, Q, R, c_1 and c_2 fulfill the conditions of Definition 3.2. √

In contrast to open curves, we find that betweenness for closed curves is trivial, since on closed curves every triple of points fulfills that one point is between the other two points with respect to the curve. Thus, the development of oriented curves in the next chapter is based on open curves only.

Theorem 3.7

$$c \quad P\, Q\, R \qquad [\mathsf{cl}(c) \quad P \neq Q \quad Q \neq R \quad P \neq R \quad P \iota c \quad Q \iota c \quad R \iota c \quad \beta(c, P, Q, R)]$$

Proof. Considering Theorem 2.16 we get that there are curves c_1 and c_2 fulfilling $\mathsf{meet}(P, c_1, c_2) \quad \mathsf{meet}(R, c_1, c_2) \quad c = c_1 \sqcup c_2$. It follows from $Q \iota c$ that $Q \iota c_1$ or $Q \iota c_2$. Either c_1 or c_2 comply with Definition 3.1. This proves the theorem. √

3.2 **The properties of β**

In this section we show that all basic properties of orderings on linear structures are satisfied (cf. Huntington, 1924). We summarize the fundamental properties of orderings on linear structures in the following theorem: Let c denote a curve and P, Q and R points: (1) If Q is between P and R wrt. c, then P, Q and R are incident with c and are pairwise distinct. (2) If Q is between P and R wrt. c, then Q is between R and P wrt. c. (3) If c is open and Q is between P and R wrt. c, then P is not between Q and R wrt. c. (4) If P, Q and R are distinct and on c then one of the points is between the others wrt. c. Finally, (5) if Q is between P and R wrt. c and Q' another point distinct from Q and lying on c then Q is either between P and Q' or between Q' and R wrt. c.

Theorem 3.8

(1) $c \quad P\ Q\ R \quad [\beta(c, P, Q, R)$
$P \imath c \quad Q \imath c \quad R \imath c \quad P \neq Q \quad Q \neq R \quad P \neq R]$

(2) $c \quad P\ Q\ R \quad [\beta(c, P, Q, R) \quad \beta(c, R, Q, P)]$

(3) $c \quad P\ Q\ R \quad [\mathrm{op}(c) \quad \beta(c, P, Q, R) \quad \neg\beta(c, Q, P, R)]$

(4) $c \quad P\ Q\ R \quad [P \imath c \quad Q \imath c \quad R \imath c \quad P \neq Q \quad Q \neq R \quad P \neq R$
$\beta(c, P, Q, R) \quad \beta(c, Q, P, R) \quad \beta(c, P, R, Q)]$

(5) $c \quad P\ Q\ R\ Q' \quad [\beta(c, P, Q, R) \quad Q' \imath c \quad Q \neq Q'$
$(\beta(c, P, Q, Q') \quad \beta(c, Q', Q, R))]$

Proof of (1). If $\beta(c, P, Q, R)$, then $P \neq R$ (Definition 3.1) and there is a curve c' satisfying $c' \sqsubset c \quad \mathrm{ept}(P, c') \quad \mathrm{ept}(R, c') \quad \mathrm{ipt}(Q, c')$. It ensues from $\neg\mathrm{ept}(Q, c')$ that PÊ$\neq Q$ and $Q \neq R$. From $\mathrm{ept}(P, c')$ and $\mathrm{ept}(R, c')$ we derive $P \imath c'$ and $R \imath c'$. Considering $Q \imath c'$ we have on account of $c' \sqsubset c$ finally $P \imath c$, $Q \imath c$ and $R \imath c$. √

Proof of (2). The proof is trivial, since we can interchange the order of the conjuncts in Definition 3.1. √

Proof of (3). Since $\beta(c, P, Q, R)$ there are (Definition 3.2) two curves c_1 and c_2 with

$$(c_1 \sqcup c_2 \sqsubset c) \quad \mathrm{ept}(P, c_1) \quad \mathrm{ept}(R, c_2) \quad \mathrm{meet}(Q, c_1, c_2).$$

Let c_3 be the sum of c_1 and c_2. Curve c_3 is open because of Theorem 2.7.2. Applying Proposition 2.14.4 we obtain $\mathrm{ept}(P, c_3)$ and $\mathrm{ept}(R, c_3)$. Let us assume that $\beta(c, Q, P, R)$ also holds. Then there are also two curves c'_1 and c'_3 with

$$(c'_1 \sqcup c'_3 \sqsubset c) \quad \mathrm{ept}(Q, c'_1) \quad \mathrm{ept}(R, c'_3) \quad \mathrm{meet}(P, c'_1, c'_3).$$

Let c'_2 be the sum of c'_1 and c'_3 which is open according to Theorem 2.7.2. Applying Proposition 2.14.4 again we obtain $\mathrm{ept}(Q, c'_2)$ and $\mathrm{ept}(R, c'_2)$. Therefore we can apply Theorem 2.13 and get

$$\begin{array}{lllll} \mathrm{ept}(P, c_1) & \mathrm{ept}(P, c'_1) & \mathrm{ept}(Q, c_1) & \mathrm{ept}(Q, c'_1) & c_1 = c'_1 \\ \mathrm{ept}(R, c_2) & \mathrm{ept}(R, c'_2) & \mathrm{ept}(Q, c_2) & \mathrm{ept}(Q, c'_2) & c_2 = c'_2 \\ \mathrm{ept}(P, c_3) & \mathrm{ept}(P, c'_3) & \mathrm{ept}(R, c_3) & \mathrm{ept}(R, c'_3) & c_3 = c'_3 \end{array}$$

Summarizing, we obtain $c_3 = c_1 \sqcup c_2$ and $c_2 = c_1 \sqcup c_3$, and therefore $c_3 \sqsubset c_2$ and $c_2 \sqsubset c_3$. This leads to the conclusion $c_2 = c_3$. According to axiom (C5) it follows that $P = Q$ or $Q = R$ or $P = R$ contradictory to the assumption $P \neq Q$, $Q \neq R$, $P \neq R$. Thus we have $\neg\beta(c, Q, P, R)$. √

Proof of (4). Considering Theorem 3.7 we can assume that c is open. We will prove that given the assumptions of the theorem and additionally supposing that neither $\neg\beta(c, P, R, Q)$ nor $\neg\beta(c, Q, P, R)$ holds, $\beta(c, P, Q, R)$ follows. According to Theorem 2.15 there are three curves c_1, c_2 and c_3 satisfying

$$\begin{array}{lll} c_1 \sqsubset c & \mathrm{ept}(P, c_1) & \mathrm{ept}(Q, c_1), \\ c_2 \sqsubset c & \mathrm{ept}(Q, c_2) & \mathrm{ept}(R, c_2), \\ c_3 \sqsubset c & \mathrm{ept}(P, c_3) & \mathrm{ept}(R, c_3). \end{array}$$

Since we assume $\mathrm{op}(c)$ Theorem 2.22 yields due to $\mathrm{ept}(Q, c_1)$ and $\mathrm{ept}(Q, c_2)$

$$\mathrm{meet}(Q, c_1, c_2) \quad c_1 \sqsubset c_2 \quad c_2 \sqsubset c_1.$$

Assuming $c_1 \sqsubset c_2$, then in particular $P \iota c_2$ holds. $\neg \mathsf{ept}(P, c_2)$ follows since c_2 has at most two endpoints (C5). But then we would have found a curve—this is c_2—with $\mathsf{ept}(Q, c_2)$, $\mathsf{ept}(R, c_2)$ and $\mathsf{ipt}(P, c_2)$. Therefore, Definition 3.1 is fulfilled for P, Q, R and c_2 contrary to the assumption $\neg\beta(c, Q, P, R)$. Analogously, $c_2 \sqsubset c_1$ cannot be valid, otherwise $\neg\beta(c, P, R, Q)$ would be violated. Hence $\mathsf{meet}(Q, c_1, c_2)$ follows and on account of Proposition 2.14.3 $\neg \mathsf{ept}(P, c_1 \sqcup c_2)$ holds. Consequently, Definition 3.2 is fulfilled. √

Proof of (5). We distinguish two cases. First let c be closed. We know from $\beta(c, P, Q, R)$ that P, Q and R are on c and mutually distinct. Since $P \neq R$ holds, $Q' \neq P$ or $Q' \neq R$ follows. In the first case we get $\beta(c, P, Q, Q')$ from $Q' \iota c$, $Q \neq Q'$ and Theorem 3.7, in the second case we get $\beta(c, Q', Q, R)$.

Now, let curve c be open. Since $Q \neq Q'$ holds, Theorem 2.15 says that there is curve $c_3 \sqsubset c$ with $\mathsf{ept}(Q', c_3)$ and $\mathsf{ept}(Q, c_3)$. On the other hand $\beta(c, P, Q, R)$ is satisfied. According to Definition 3.2, there are two curves c_1, c_2 with

$$(c_1 \sqcup c_2 \sqsubset c) \quad \mathsf{ept}(P, c_1) \quad \mathsf{ept}(R, c_2) \quad \mathsf{meet}(Q, c_1, c_2).$$

Hence, all assumptions of Proposition 2.23 are fulfilled and we get $\mathsf{meet}(Q, c_1, c_3)$ or $\mathsf{meet}(Q, c_2, c_3)$. In the first case Definition 3.2 is satisfied due to $c_1 \sqcup c_3 \sqsubset c$, $\mathsf{ept}(P, c_1)$ and $\mathsf{ept}(Q', c_3)$, and we obtain $\beta(c, P, Q, Q')$. In the second case we get $\beta(c, Q', Q, R)$ from $\mathsf{ept}(Q', c_3)$, $\mathsf{ept}(R, c_2)$ and $c_2 \sqcup c_3 \sqsubset c$. √

If P, Q and R are points on an open curve c then Q is not between P and R wrt. c, iff P is between R and Q wrt. c or R is between Q and P wrt. c or at least two of the points are identical.

Corollary 3.9

$$c \quad P\, Q\, R \qquad [\mathsf{op}(c) \quad P \iota c \quad Q \iota c \quad R \iota c$$
$$(\neg\beta(c, P, Q, R) \Leftrightarrow \beta(c, R, P, Q) \quad \beta(c, Q, R, P)$$
$$R = Q \quad R = P \quad P = Q)]$$

Proof. : is proved by Theorem 3.8.4 and Theorem 3.8.2. The other direction is derived from Theorem 3.8.3, Theorem 3.8.4 and Theorem 3.8.1. √

4 Oriented Curves

The axioms for oriented curves and the precedence structure are closely related to the axioms of ordering for oriented straight lines in Eschenbach & Kulik (1997). Oriented curves constitute a more general basis than oriented straight lines to describe directions in space. The description of oriented curves is given on the basis of a primitive ternary relation of precedence that distinguishes the order of points on an oriented curve and is compatible with the relation of betweenness as defined above.

A point is between two other points on an oriented curve iff one of them precedes it and the other one is preceded by it:

Definition 4.1 (Betweenness on an Oriented Curve)

$$\beta(o, P, Q, R) \quad \Leftrightarrow_{\text{def}} \quad (\prec(o, P, Q) \quad \prec(o, Q, R)) \quad (\prec(o, R, Q) \quad \prec(o, Q, P))$$

A *starting point* of an oriented curve precedes any other point on it, and a *finishing point* of an oriented curve is preceded by any other point on it.

Definition 4.2 (Starting Point, Finishing Point)

$\mathsf{stpt}(P, o) \quad \Leftrightarrow_{\mathrm{def}} \quad P \iota o \quad Q\, [P \neq Q \quad Q \iota o \quad \prec(o, P, Q)]$

$\mathsf{fpt}(P, o) \quad \Leftrightarrow_{\mathrm{def}} \quad P \iota o \quad Q\, [P \neq Q \quad Q \iota o \quad \prec(o, Q, P)]$

Remark. It is possible to omit the requirement $P \iota o$ in both definitions, since it is provable from the following Axiom (O1).

4.1 Axioms of Oriented Curves and Precedence

Points that are ordered by an oriented curve are incident with the curve (O1). Since the basic spatial structure of the oriented curves shall correspond to the structure of open curves, we require that every oriented curve coincides with an open curve in all points (O2) and betweenness on an oriented curve is compatible with betweenness on the underlying curve (O3). Additionally, every oriented curve has a starting point (O4). For every open curve and two points on it there is an oriented curve that coincides with the curve at all points and orders the two points (O5). Finally, oriented curves are identical if they totally agree in the ordering of points (O6).

(O1) $\quad o \quad P \quad Q \quad [\prec(o, P, Q) \quad P \iota o \quad Q \iota o]$

(O2) $\quad o \quad c \quad [\mathsf{op}(c) \quad P\, [P \iota o \Leftrightarrow P \iota c]]$

(O3) $\quad P \quad Q \quad R \quad o \quad [\beta(o, P, Q, R) \Leftrightarrow \quad c\, [\quad P\, [P \iota o \Leftrightarrow P \iota c] \quad \beta(c, P, Q, R)]]$

(O4) $\quad o \quad P \quad [\mathsf{stpt}(P, o)]$

(O5) $\quad P \quad Q \quad c \quad [\mathsf{op}(c) \quad P \neq Q \quad P \iota c \quad Q \iota c$
$\qquad o[\quad R\, [R \iota o \Leftrightarrow R \iota c] \quad \prec(o, P, Q)]$

(O6) $\quad o_1 \quad o_2 \quad [\quad P \quad Q\, [\prec(o_1, P, Q) \Leftrightarrow \prec(o_2, P, Q)] \quad o_1 = o_2]$

We will use the function uc to denote the underlying curve of an oriented curve.

Definition 4.3 (Underlying Curve)

$c = \mathsf{uc}(o) \quad \Leftrightarrow_{\mathrm{def}} \quad P\, [P \iota o \Leftrightarrow P \iota c]$

Remark. According to Axiom (O2) there is at least one open curve c for every oriented curve o. Considering Axiom (C9) we obtain that c is uniquely determined.

4.2 Theorems

Precedence on oriented curves is irreflexive.

Theorem 4.4

$o \quad P \quad [\neg \prec(o, P, P)]$

Proof. If $\prec(o, P, P)$ then by $\beta(o, P, P, P)$, contradicting Theorem 3.8.1 with Definition 4.1. √

Precedence on oriented curves is asymmetric.

Theorem 4.5

$$o \quad P \quad Q \qquad [\prec(o, P, Q) \qquad \neg\prec(o, Q, P)]$$

Proof. If $\prec(o, P, Q)$ and $\prec(o, Q, P)$ then by Definition 4.1 $\beta(o, P, Q, P)$, contradicting Theorem 3.8.1 with (O3). √

The starting point and the finishing point of an oriented curve are endpoints of the underlying curve.

Theorem 4.6

(1) $\quad o \quad P \qquad [\mathsf{stpt}(P, o) \qquad \mathsf{ept}(P, \mathsf{uc}(o))]$

(2) $\quad o \quad P \qquad [\mathsf{fpt}(P, o) \qquad \mathsf{ept}(P, \mathsf{uc}(o))]$

Proof. We just give the proof for (1) since the proof for (2) proceeds in the same way. More precisely we prove for a point P and an oriented line o:

$$\mathsf{ipt}(P, \mathsf{uc}(o)) \qquad \neg\,\mathsf{stpt}(P, o).$$

According to Axiom (O2) $\mathsf{uc}(o)$ is open. Let P_1 and P_2 denote its endpoints. In agreement with Definition 3.1 we have $\beta(\mathsf{uc}(o), P_1, P, P_2)$, by Axiom (O3) we get $\beta(o, P_1, P, P_2)$. According to Definition 4.1 this means $\prec(o, P_1, P) \quad \prec(o, P, P_2)$ or $\prec(o, P_2, P) \quad \prec(o, P, P_1)$. Theorem 4.5 finally yields $\prec(o, P_1, P)$ or $\prec(o, P_2, P)$. This contradicts $\mathsf{stpt}(P, o)$. √

The other way round, every endpoint of the underlying curve of an oriented curve is either a starting point or finishing point of the oriented curve.

Theorem 4.7

$$o \quad P \qquad [\mathsf{ept}(P, \mathsf{uc}(o)) \qquad \mathsf{stpt}(P, o) \quad \mathsf{fpt}(P, o)]]$$

Proof. We assume $\mathsf{ept}(P, \mathsf{uc}(o))$ and $\neg\,\mathsf{stpt}(P, o)$ in order to show $\mathsf{fpt}(P, o)$. According to Axiom (O4), o has a starting point, which we call Q. Q is also an endpoint of $\mathsf{uc}(o)$ (Theorem 4.6). Let R be any point on o, different from P and Q. We just have to show that $\prec(o, R, P)$ holds. Since P and Q are endpoints of $\mathsf{uc}(o)$ we get by Theorem 3.6 $\neg\beta(\mathsf{uc}(o), Q, P, R)$ and $\neg\beta(\mathsf{uc}(o), P, Q, R)$. Via Theorem 3.8.4 and Axiom (O3) we get $\beta(o, P, R, Q)$. This means (Definition 4.1) $(\prec(o, P, R) \quad \prec(o, R, Q)) \quad (\prec(o, Q, R) \quad \prec(o, R, P))$. The first case is excluded because of $\mathsf{stpt}(Q, o)$. Thus, $\prec(o, R, P)$ holds and P is a finishing point of o. √

Every underlying curve of an oriented curve has exactly two endpoints. Since the proof shows that an endpoint of the underlying curve which is not a starting point of the oriented curve must be a finishing point of the oriented curve, one endpoint of the underlying curve has to be the starting point (Axiom (O4)) and the other has to be the finishing point of the oriented curve. Therefore we obtain:

Corollary 4.8

$$o \quad P \qquad [\mathsf{fpt}(P, o)]$$

These prerequisites are sufficient to prove that every oriented curve orders all points on it.

Theorem 4.9

$$o \quad P \quad Q \qquad [P \iota\, o \quad Q \iota\, o \qquad (\prec(o, P, Q) \quad P = Q \quad \prec(o, Q, P))]$$

Proof. Let us assume $P \iota o \quad Q \iota o$ and $P \neq Q$. If one of P or Q is starting point or finishing point of o then we also have $\prec(o, P, Q) \quad \prec(o, Q, P)$.

If P and Q are neither a starting point nor a finishing point thenÑin accordance with Theorem 4.7—P and Q have to be inner points of the open curve $\mathrm{uc}(o)$. Let P_1 and P_2 be the endpoints of $\mathrm{uc}(o)$ (Theorem 2.7.1), such that $\mathrm{stpt}(P_1, o)$ and $\mathrm{fpt}(P_2, o)$. Theorem 3.6 yields $\beta(\mathrm{uc}(o), P_1, P, P_2)$. Since $Q \iota \mathrm{uc}(o)$ we obtain using Theorem 3.8.5:

$$\beta(\mathrm{uc}(o), P_1, P, Q) \quad \beta(\mathrm{uc}(o), Q, P, P_2).$$

Applying Axiom (O3) and Definition 4.1 we get

$$(\prec(o, P_1, P) \quad \prec(o, P, Q)) \quad (\prec(o, Q, P) \quad \prec(o, P, P_1))$$
$$(\prec(o, Q, P) \quad \prec(o, P, P_2)) \quad (\prec(o, P_2, P) \quad \prec(o, P, Q)).$$

Since P_1 is a starting point of o and P_2 is a finishing point of o we conclude using Theorem 4.5 $\prec(o, P, Q) \quad \prec(o, Q, P)$. √

Every oriented curve has at most one starting point and one finishing point.

Corollary 4.10

(1) $\quad o \quad P \quad Q \qquad [\mathrm{stpt}(P, o) \quad \mathrm{stpt}(Q, o) \qquad P = Q]$

(2) $\quad o \quad P \quad Q \qquad [\mathrm{fpt}(P, o) \quad \mathrm{fpt}(Q, o) \qquad P = Q]$

Every oriented curve orders some points.

Theorem 4.11

$$o \quad P \quad Q \qquad [\prec(o, P, Q) \quad P \neq Q]$$

Proof. The open curve $\mathrm{uc}(o)$ has because of Theorem 2.7.1 two points on it which are ordered according to Theorem 4.9. √

Incidence on oriented curves can be defined in terms of precedence.

Theorem 4.12

$$o \quad P \qquad [P \iota o \Leftrightarrow \quad Q\, [\prec(o, P, Q) \quad \prec(o, Q, P)]]$$

Proof. : is given by (O1).

: By (O2) o coincides totally with an open curve, by Theorem 2.7.1 it has at least one point Q different from P and by Theorem 4.9 P is related to this point by precedence on o. √

If P precedes Q with respect to o, then any point R on o precedes Q or is preceded by P.

Theorem 4.13

$$o \quad P \quad Q \qquad [\prec(o, P, Q) \qquad R\, [R \iota o \qquad (\prec(o, R, Q) \quad \prec(o, P, R))]]$$

Proof. If $R = P$ or $R = Q$, then there is nothing to prove. Otherwise, we get from Theorem 4.9 that $(\prec(o, R, Q) \quad \prec(o, Q, R))$ and $(\prec(o, R, P) \quad \prec(o, P, R))$. We just have to exclude the case that both $\prec(o, Q, R)$ and $\prec(o, R, P)$ hold. Assuming this we conclude from $\prec(o, P, Q)$ and by Definition 4.1 that $\beta(o, R, P, Q)$ and $\beta(o, P, Q, R)$. Applying Axiom (O3) and Theorem 3.8.2 the last conjunction contradicts Theorem 3.8.3. √

Precedence on an oriented curve is a transitive relation.

Theorem 4.14

$$o \quad P \quad Q \quad R \quad [\prec(o, P, Q) \quad \prec(o, Q, R) \quad \prec(o, P, R)]$$

Proof. From $\prec(o, P, Q) \quad \prec(o, Q, R)$ we know by Definition 4.1 $\beta(o, P, Q, R)$ and by (O3) and Theorem 3.8.1 $P \neq R$. From Theorem 4.9 we know $\prec(o, P, R) \quad \prec(o, R, P)$. If $\prec(o, R, P)$ then $\beta(o, R, P, Q)$ by Definition 4.1, which contradicts Theorem 3.8.3 together with Theorem 3.8.2. √

Theorem 4.15

$$o \quad P \quad Q \quad R \quad [\beta(o, P, Q, R) \quad (\prec(o, P, Q) \Leftrightarrow \prec(o, Q, R))]$$

Proof. Definition 4.1, (O3) and Theorem 4.5. √

Corollary 4.16

$$P \quad Q \quad R \quad o \quad [\beta(o, P, Q, R) \quad (\prec(o, P, Q) \Leftrightarrow \prec(o, P, R))]$$

Proof. Theorem 4.15 and Theorem 4.14. √

Theorem 4.17

(1) $\quad o \quad P \quad Q \quad S \quad [\prec(o, P, Q) \quad P \neq S \quad S \iota o \quad (\prec(o, P, S) \Leftrightarrow \neg\beta(o, S, P, Q))]$

(2) $\quad o \quad P \quad Q \quad R \quad S \quad [\prec(o, P, Q) \quad R \neq S \quad S \iota o \quad \beta(o, R, P, Q)$
$\qquad (\prec(o, R, S) \Leftrightarrow \neg\beta(o, S, R, Q))]$

(3) $\quad o \quad P \quad Q \quad R \quad S \quad [\prec(o, P, Q) \quad P \neq R \quad \neg\beta(o, R, P, Q)$
$\qquad (\prec(o, R, S) \Leftrightarrow \beta(o, P, R, S))]$

Proof of (1). If $\prec(o, P, Q) \quad P \neq S \quad S \iota o$ then $\prec(o, P, S) \Leftrightarrow \neg\prec(o, S, P)$ by Theorem 4.9 and Theorem 4.5. $\prec(o, S, P) \Leftrightarrow \beta(o, S, P, Q)$ can be shown by Theorem 4.15 and Definition 4.1. √

Proof of (2). If $\prec(o, P, Q) \quad \beta(o, R, P, Q)$ then by Corollary 4.16 $\prec(o, R, Q)$. Hence we can apply (1)—by substituting P with R in (1). √

Proof of (3). If $\prec(o, P, Q) \quad P \neq R \quad \neg\beta(o, R, P, Q)$ we get from (O3) and Theorem 4.5 $\neg\prec(o, R, P)$. Hence we apply (1)—substitute P with R and Q with P—and get

$$\prec(o, R, S) \Leftrightarrow \beta(o, S, R, P).$$

Definition 4.1, Axiom (O3) and Theorem 3.8.2 yield the statement. √

Corollary 4.18 and Theorem 4.19 show how the ordering of any pair of points R and S on an oriented line o can be determined on the basis of a given pair of points P and Q using betweenness and incidence only. Thus, the underlying curve and one pair of points are sufficient for the ordering of the points on the oriented curve.

Corollary 4.18

$$o \quad P \quad Q \quad [\prec(o, P, Q) \quad R \quad S\, [\prec(o, R, S) \Leftrightarrow$$
$$((S \iota o \quad (\beta(o, R, P, Q) \quad P = R) \quad R \neq S \quad \neg\beta(o, S, R, Q))$$
$$(\beta(o, P, R, S) \quad \neg\beta(o, Q, P, R)))]]$$

Theorem 4.19

$$o \quad P \quad Q \qquad [\prec(o, P, Q) \qquad R \quad S\,[\prec(o, R, S) \Leftrightarrow$$
$$(\beta(o, R, P, Q) \quad (\beta(o, R, S, Q) \quad \beta(o, R, Q, S) \quad Q = S))$$
$$(\beta(o, P, R, S) \quad (\beta(o, P, R, Q) \quad \beta(o, P, Q, R) \quad Q = R))$$
$$(P = R \quad (\beta(o, P, S, Q) \quad \beta(o, P, Q, S) \quad Q = S))]]$$

Proof. Let o be an oriented curve and P, Q points such that $\prec(o, P, Q)$. On this basis and according to Corollary 3.9, Theorem 3.8.1 and Theorem 4.4 the right-hand side of the equivalence in Corollary 4.18 is equivalent to

$$\beta(o, R, P, Q) \quad (\beta(o, R, Q, S) \quad \beta(o, R, S, Q) \quad S = Q)$$
$$P = R \quad (\beta(o, P, Q, S) \quad \beta(o, P, S, Q) \quad S = Q)$$
$$\beta(o, P, R, S) \quad (\beta(o, P, R, Q) \quad \beta(o, P, Q, R) \quad Q = R)$$

which proves the theorem. √

With these preparations we are able to show that oriented lines consisting of the same points and ordering one pair of points in the same way, are identical.

Theorem 4.20

$$o_1 \quad o_2 \qquad [\quad P\,[P \imath o_1 \Leftrightarrow P \imath o_2] \qquad P \quad Q\,[\prec(o_1, P, Q) \quad \prec(o_2, P, Q)]$$
$$o_1 = o_2]$$

Proof. Let o_1, o_2 be oriented curves that coincide in all their points and P, Q points such that $\prec(o, P, Q)$. Because of (O6) it suffices to show that for any points R and S $\prec(o_2, R, S)$ follows from $\prec(o_1, R, S)$. Since o_1 and o_2 coincide in all points and because of (O2) and (C9) we know that there is a curve c that coincides in all points with both of them. Corollary 4.18 can—based on (O3)—be transformed into

$$\prec(o_i, P, Q) \qquad R \quad S\,[\prec(o_i, R, S) \Leftrightarrow$$
$$((S \imath c \quad (\beta(c, R, P, Q) \quad P = R) \quad R \neq S \quad \neg\beta(c, S, R, Q))$$
$$(\beta(c, P, R, S) \quad \neg\beta(c, Q, P, R)))]$$

with $i = 1, 2$, which proves the theorem. √

The proof shows that a curve and a ordered pair of points uniquely determine an oriented curve:

Corollary 4.21

$$o_1 \quad o_2 \qquad [\mathsf{uc}(o_1) = \mathsf{uc}(o_2) \qquad P \quad Q\,[\prec(o_1, P, Q) \quad \prec(o_2, P, Q)]$$
$$o_1 = o_2]$$

For every oriented curve there is an oppositely oriented curve with the same underlying curve.

Theorem 4.22

$$o \quad o' \qquad [\mathsf{uc}(o) = \mathsf{uc}(o') \qquad P \quad Q\,[\prec(o, P, Q) \qquad \prec(o', Q, P)]]$$

Proof. Let P' and Q' be two points with $\prec(o, P', Q')$. According to Axiom (O5) for P', Q' and $\mathsf{uc}(o)$ there is an oriented curve o'

$$R\,[R \imath o' \Leftrightarrow R \imath \mathsf{uc}(o)] \qquad \prec(o', Q', P').$$

We note that for every point R follows: $R \imath o' \Leftrightarrow R \imath o$. If there is a pair of points R' and S' with $\prec(o, R', S')$ and $\prec(o', R', S')$ then Theorem 4.20 yields $\prec(o', P, Q)$ for all P and Q satisfying $\prec(o, P, Q)$ contrary to the assumption $\prec(o', Q', P')$. Hence we conclude $\neg\prec(o', P, Q)$ for all P and Q fulfilling $\prec(o, P, Q)$. Theorem 4.9 concludes the proof. √

Remark. Combining Theorem 4.22 with Corollary 4.21 we obtain that for every oriented line there is exactly one uniquely determined oriented line with the same underlying curve that orders the points in the opposite way.

Theorem 4.23

(1) $o \quad P \quad Q \quad R \qquad [\prec(o, P, Q) \qquad (\prec(o, R, P) \Leftrightarrow \beta(o, R, P, Q))]$

(2) $o \quad P \quad Q \quad R \qquad [\prec(o, P, Q)$
$\qquad (\prec(o, R, Q) \Leftrightarrow (\beta(o, R, P, Q) \qquad \beta(o, P, R, Q) \qquad P = R))]$

(3) $o \quad P \quad Q \quad R \qquad [\prec(o, P, Q)$
$\qquad (\prec(o, P, R) \Leftrightarrow (\beta(o, P, R, Q) \qquad \beta(o, P, Q, R) \qquad Q = R))]$

(4) $o \quad P \quad Q \quad R \qquad [\prec(o, P, Q) \qquad (\prec(o, Q, R) \Leftrightarrow \beta(o, P, Q, R))]$

(5) $o \quad P \quad Q \quad R \quad S \quad [\prec(o, P, Q) \quad P \neq R \quad P \neq S \quad Q \neq S \quad Q \neq R$
$\qquad (\prec(o, R, S) \Leftrightarrow$
$\qquad\qquad (\beta(o, R, P, Q) \quad (\beta(o, R, S, Q) \quad \beta(o, R, Q, S)))$
$\qquad\qquad (\beta(o, P, R, S) \quad (\beta(o, P, R, Q) \quad \beta(o, P, Q, R))))]$

Proof. Application of Theorem 4.19, Theorem 4.4 and Theorem 3.8.1. √

If R is the starting point of o, then P precedes Q wrt. o, iff P is identical with R and Q is on o but different from R or P is between R and Q on o.

Theorem 4.24

$$o \quad R\ [\mathrm{stpt}(R, o) \qquad P \quad Q\ [\prec(o, P, Q) \Leftrightarrow$$
$$(P = R \quad Q \neq R \quad Q \iota o) \quad \beta(o, R, P, Q)]]$$

Proof. Let R be starting point of o. If $\prec(o, P, Q)$ then by (O1) $Q \iota o$ and by Theorem 4.4 $Q \neq R$.. If Q is on o and different from R then by Theorem 4.9 $\prec(o, R, Q)$. If $P \neq R$, then $\prec(o, R, P)$ follows from the definition of starting point. Theorem 4.23.4 yields $\prec(o, P, Q) \Leftrightarrow \beta(o, R, P, Q)$. √

5 Conclusion

The results demonstrate that oriented curves can be considered as generalized directions: Every curve can be oriented in exactly two ways and a pair ordered points (or a starting or a finishing point) supplies a direction on a curve since the order for every other pair of points is already determined.

In order to prove the consistency of the axioms (C1) – (C9) and (O1) – (O6) respectively, a model is required for these axioms. Instead of giving a proof, we just mention that polygonal curves fulfill the axioms (O1) – (O6) and their traces fulfill axioms (C1) – (C9).

The basic restrictions on the relation between curves, oriented curves and points are given in an axiomatic framework. The results are therefore applicable to any set of curves that fulfill the requirements expressed here, independently of their mathematical or spatial definition or specification. The axioms for curves specify general restrictions of linear structures that are not straight, such that the relation of betweenness on curves can be defined on the basis of their (connected) parts. The betweenness structure on curves is independent of the two orientations that can be assigned to a linear structure and define two oriented curves.

An important difference between time and space is that the positions in space can be traversed in many possible ways while the positions in the linear structure of time cannot. Curves and oriented curves in space reflect this possibility on the geometric level since different curves can connect a collection of positions in different orders. Curves and oriented curves can therefore be interpreted as reified ordering relations in a non-linear space. This provides the option to use oriented curves for representing trajectories of objects moving through space without the additional representation of time, as presented by Eschenbach, Habel, Kulik (1999).

6 References

Eschenbach, C., C. Habel, L. Kulik, A. Leßmöllmann (1998). Shape nouns and shape concepts: A geometry for ‘corner’. In C. Freksa, C. Habel & K.F. Wender (eds.), *Spatial Cognition* (pp. 177–201). Berlin: Springer.

Eschenbach, C., C. Habel & L. Kulik (to appear 1999). Representing simple trajectories as oriented curves. To appear in *Proceedings of the 12th Florida AI Research Symposium*, Orlando, Florida.

Eschenbach, C. & L. Kulik (1997). An axiomatic approach to the spatial relations underlying ‘left’–‘right’ and ‘in front of’–‘behind’. In G. Brewka, C. Habel & B. Nebel (eds.), *KI-97: Advances in Artificial Intelligence* (pp. 207–218). Berlin: Springer.

Habel, Christopher (1990). Propositional and depictorial representations of spatial knowledge: The case of path concepts. In R. Studer (ed.), *Natural Language and Logic* (pp. 94-117). Berlin: Springer.

Hilbert, David (1902). *Grundlagen der Geometrie*. (8th ed. 1956, with revisions and additions by Paul Bernays. Stuttgart: Teubner.)

Huntington, Edward V. (1924). A new set of postulates for betweenness, with proof of complete independence. *Transactions of the American Mathematical Society 26*, 257-282.

Huntington, Edward V. & J.R. Kline (1917). Sets of independent postulates for betweenness. *Transactions of the American Mathematical Society 18*, 301-325.