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Tutorial 7: solution sketches

1. The Bellman optimality equations are as follows:

$$
\begin{aligned}
x_6 &= 1 \\
x_5 &= \max\{x_1, x_2\} \\
x_4 &= x_4 \\
x_3 &= \max\{x_2, x_4\} \\
x_2 &= 2x_1/5 + x_4/5 + 2x_6/5 \\
x_1 &= x_2/6 + x_4/6 + x_5/6 + x_6/2
\end{aligned}
$$

Our goal is to compute the unique minimal (least non-negative) solution vector $p^* = (p_1^*, \ldots, p_6^*)$, which gives the optimal probabilities of reaching s_6 starting from each state. It is clear that $p_4^* = 0$, so that at s_3 the node s_2 is always chosen, giving $p_3^* = p_2^*$. Also, since $p_6^* = 1$, it only remains to solve for p_1^*, p_2^*, and p_5^*. From the optimality conditions we see that the equations governing these are as follows:

$$
\begin{aligned}
p_5^* &= \max\{p_1^*, p_2^*\} \\
p_2^* &= 2p_1^*/5 + 2/5 \\
p_1^* &= p_2^*/6 + p_5^*/6 + 1/2
\end{aligned}
$$

We can (as shown on the lecture slides) solve this by computing the unique optimal solution for the following linear programming problem:

Minimize: $x_1 + x_2 + x_5$

Subject to:

$x_5 \geq x_1$

$x_5 \geq x_2$

$x_2 \geq (2/5)x_1 + (2/5)$

$x_1 \geq (1/6)x_2 + (1/6)x_5 + (1/2)$

In this small example, we can also avoid using linear programming, by enumerating all possible cases of the max equation. There are two cases to consider: (i) $\max\{p_1^*, p_2^*\} = p_2^*$ and (ii) $\max\{p_1^*, p_2^*\} = p_1^*$. In both of these cases we know the value of p_5^*, so we can calculate the rest.

In case (i), the equations reduce to

$$p_2^* = 2p_1^*/5 + 2/5$$
$$p_1^* = p_2^*/3 + 1/2$$

These can be solved to get $p_1^* = 19/26$ and $p_2^* = 18/26$. This contradicts our assuption that $p_2^* = \max\{p_1^*, p_2^*\}$.

In case (ii), the equations reduce to

$$p_2^* = 2p_1^*/5 + 2/5$$
$$p_1^* = p_1^*/6 + p_6^*/6 + 1/2$$

which gives us $p_1^* = 17/23$ and $p_2^* = 16/23$. This gives us the full solution to the original problem: $p^* = (p_1^*, \dots p_6^*) = (17/23, 16/23, 16/23, 0, 17/23, 1)$. Player 1s optimal strategy is to choose s_2 when at node s_3, and to choose s_1 when at node s_5.

2. As we are working with a congestion game, we can find a pure Nash Equilibrium by starting at any pure strategy profile, and iteratively improving it until we can't. To get a concrete starting point, let's say all players take the route $s \to v_3 \to t$. Then we can do iterative improvements for example[1] as follows:

   (i) Player 1 switches to $s \to v_2 \to v_1 \to t$
   (ii) Player 2 switches to $s \to v_2 \to v_1 \to t$
   (iii) Player 3 switches to $s \to v_1 \to t$.
   (iv) Player 2 switches to $s \to v_1 \to t$

   At (iv) no further improvements can be made, so we reached the following NE:

   Player 1: $s \to v_2 \to v_1 \to t$
   Player 2: $s \to v_1 \to t$
   Player 3: $s \to v_1 \to t$

   Note that in the above sequence we weren't done at stage (iii), even though every player had switched once. Other starting points will take

[1] at many stages there's more than one option on who improves and how

through other sequences of steps, and they might end up in a different NE, although it turns out that in this game all pure Nash equilibria send two players via the route $s \to v_1 \to t$ and one via $s \to v_2 \to v_1 \to t$, differing only in which player chooses the path $s \to v_2 \to v_1 \to t$.