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Tutorial 7: solution sketches

1. The Bellman optimality equations are as follows:

$$
\begin{aligned}
x_6 &= 1 \\
x_5 &= \max\{x_1, x_2\} \\
x_4 &= x_4 \\
x_3 &= \max\{x_2, x_4\} \\
x_2 &= 2x_1/5 + x_4/5 + 2x_6/5 \\
x_1 &= x_2/6 + x_4/6 + x_5/6 + x_6/2
\end{aligned}
$$

We wish to find the unique minimal solution $p^* = (p_1^*, \ldots p_6^*)$, which also gives the optimal probalities. It is clear that $p_4^* = 0$, so that at s_3 the node s_2 is always chosen, giving $p_3^* = p_2^*$. As $p_6^* = 1$, it remains to solve p_1^*, p_2^* and p_5^*. From the optimality conditions we see that the equations governing these are as follows:

$$
\begin{aligned}
p_5^* &= \max\{p_1^*, p_2^*\} \\
p_2^* &= 2p_1^*/5 + 2/5 \\
p_1^* &= p_2^*/6 + p_5^*/6 + 1/2
\end{aligned}
$$

There are two cases to consider: (i) $\max\{p_1^*, p_2^*\} = p_2^*$ and (ii) $\max\{p_1^*, p_2^*\} = p_1^*$. In both of these cases we know the value of p_5^*, so we can calculate the rest.

In case (i), the equations reduce to

$$
\begin{aligned}
p_2^* &= 2p_1^*/5 + 2/5 \\
p_1^* &= p_2^*/3 + 1/2
\end{aligned}
$$

These can be solved to get $p_1^* = 19/26$ and $p_2^* = 18/26$. This contradicts our assuption that $p_2^* = \max\{p_1^*, p_2^*\}$.

In case (ii), the equations reduce to

$$
\begin{aligned}
p_2^* &= 2p_1^*/5 + 2/5 \\
p_1^* &= p_1^*/6 + p_6^*/6 + 1/2
\end{aligned}
$$

which gives us $p_1^* = 17/23$ and $p_2^* = 16/23$. This gives us the full solution to the original problem: $p^* = (p_1^*, \ldots p_6^*) = (17/23, 16/23, 16/23, 0, 17/23, 1)$. Player 1s optimal strategy is to choose s_2 when at node s_3, and to choose s_1 when at node s_5.

2. As we are working with a congestion game, we can find a pure Nash Equilibrium by starting at any pure strategy profile, and iteratively improving it until we can't. To get a concrete starting point, let's say all players take the route $s \rightarrow v_3 \rightarrow t$. Then we can do iterative improvements for example[1] as follows:

   (i) Player 1 switches to $s \rightarrow v_2 \rightarrow v_1 \rightarrow t$

   (ii) Player 2 switches to $s \rightarrow v_2 \rightarrow v_1 \rightarrow t$

   (iii) Player 3 switches to $s \rightarrow v_1 \rightarrow t$.

   (iv) Player 2 switches to $s \rightarrow v_1 \rightarrow t$

   At (iv) no further improvements can be made, so we reached the following NE:

   Player 1: $s \rightarrow v_2 \rightarrow v_1 \rightarrow t$
   Player 2: $s \rightarrow v_1 \rightarrow t$
   Player 3: $s \rightarrow v_1 \rightarrow t$

   Note that in the above sequence we weren't done at stage (iii), even though every player had switched once. Other starting points will take through other sequences of steps, and they might end up in a different NE, although it turns out that in this game all pure Nash equilibria send two players via the route $s \rightarrow v_1 \rightarrow t$ and one via $s \rightarrow v_2 \rightarrow v_1 \rightarrow t$, differing only in which player chooses the path $s \rightarrow v_2 \rightarrow v_1 \rightarrow t$.

[1] at many stages there's more than one option on who improves and how