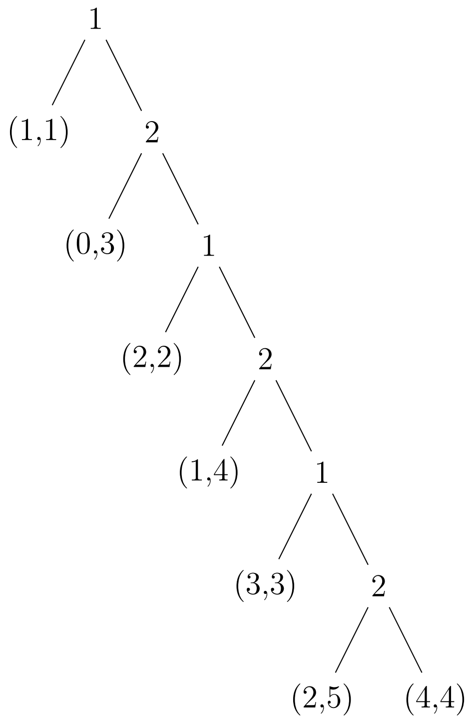
Tutorial 5: solution sketches

1. (a) Going left in the tree indicates stopping, and going right indicates giving.



(b) Recall that a pure strategy for player i is a function that maps each node controlled by player i to an action available at that node. In this way, the strategy tells Player i what to do at each node controlled by it. (More generally, in the case of imperfect information games, a pure strategy for player i is a function that maps each information set controlled by player i to an action available at that information set.) In this game, each player controlled 3 nodes. We can hence describe a pure strategy for each player as just a tuple, e.g., (G, G, S) is the strategy where the play "gives" in the first node it controls, and "stops" in both of the other two nodes (lower down the tree) that it controls.

We can compute an SPNE for this game by using backwards induction algorithm discussed in class, in the context of Kuhn's theorem. In the lowest proper subgame, rooted at the last node controlled by player 2, choosing "Stop" (S) yields a payoff of 5 to player 2, which is strictly higher than the payoff of 4 that player 2 would obtain by choosing "Give" (G). This, in the (unique) pure

SPNE of that subgame, player 2 chooses action S, yielding payoff $(2, 5)$ to the two players.

Knowing this, in the step before, player 1 gets strictly higher payoff of 3 by choosing S, than choosing G and getting payoff 2 (in the SPNE of the subgame below). Hence, in the (unique) SPNE of the subgame rooted at the lowest node for player 1, the action taken by player 1 is S. And so forth, we can work our way back up the game tree, until we reach the root. Thus the SPNE is given in short hand notation by $((S, S, S), (S, S, S))$. In other words, both players choose action "Stop" at every node that they control.

(c) Working backwards in the above argument, we see that at each stage the choice S made by the player is because it gets a *strictly* higher payoff by making that choice than by making the other choice G. There is never the case where either player would get exactly the same payoff by choosing either S or G (assuming the already computed SPNE for the lower subgame). This allows us to establish by induction that each subgame, starting from the lower most subgame and working our way up toward the root, has a unique SPNE. Therefore the entire game has a unique SPNE.

(d) Consider any pure strategy pair $((S, *, *), (S, *, *))$ for the two players, where each player's first move is S, but thereafter their move can be either G or S (it doesn't matter). We claim that ANY such combination of pure strategies for the two players is a Nash Equilbrium in this game.

To see this, note that indeed, since player 1 starts with S, player 2 cannot possibly improve its own payoff by unilaterally deviating from its own strategy, because against such a strategy for player 1 player 2 can't even change its own payoff no matter what strategy it changes to.

On the other hand, since player 2 plays S at the first node it controls, we know that player 1 cannot improve its own payoff by unilaterally changing its own pure strategy, because against such a pure strategy for player 2, if player 1 chooses G instead of S at the root of the tree then its payoff will decrease from 1 to 0. Moreover, if player 1 only changes its actions elsewhere lower in the tree, it will have no effect on its own payoff (because its own first action makes the game stop immediately).

Thus any pair of strategies of the form ((S,*,*),(S,*,*)) is a pure NE for the game. Likewise, in terms of mixed/behavior strategy NEs, note that any behavior strategy profile $((S, -, -), (S, -, -))$

where the first action chosen by both players is action S with probability 1, and where the subsequent choices at the two lower nodes controlled by each player is ANY probability distribution on the two actions S and G, forms a Nash Equilibrium.

(e) This game is indeed very odd. In particular, it doesn't feel that the SPNE or NEs of the game are a good reflection of how humans might actually behave when playing this game.

Consider the same kind of game, but rather than having just 3 nodes belonging to each player, imagine the game was extended to 100 rounds, so to 50 nodes for each player.

I think that if I was confronted with such a game in the "real world", for the first rounds of play I would "take a risk" and Give to the other player, to see if the other player is willing to return the favor and "cooperate with me for a while" so we can both make some money.

It is much harder to argue why, at the very last step of the game, the player whose turn it is to move would do anything other than pick the unique choice (Stop) which maximizes its own payoff. After all, we assume a "rational" player always make choices that maximize its own (expected) payoff.

But that's the troubling aspect: if the other player "knows" that Stop will be chosen at the very last step, then it is also incentivized to choose "Stop" in the prior step, and so on. But this kind of backward reasoning (which is very much related to "iterated illimination of strictly dominated strategies"), would yield both players to choose Stop from the beginning of the game.

If a player could somehow "commit" to the other player that it will play G, for example by yelling out "I promise that I will play (G,G,G)", and if the other player was convinced by this, then the other player's best response to (G, G, G) would give both players a better payoff than just playing the SPNE.

However, there is no mechanism within such a 2-player non-cooperative game for "making firm commitments" about how you will play in the future, since we assume the players choose their moves independently, and we assume that each player is "rational", meaning that its only objective is to maximize its own (expected) payoff.

2. First of all, it is clear that Player 1 will always choose B whenever facing the choice at the leftmost node. Thus, we can and will from now on assume that player 1 will always play B in that leftmost subgame.

Thus with 1/3 probability, the payoff to player 1 will be 3, and the payoff to player 2 will be 2. This is in fact the only proper subgame of the game, as a subgame must consist of a subtree with self-contained information sets, and say starting from player 2s information set doesn't form a subtree (it is a forest). Now let us consider the expected payoff overall, to both players. In effect, let us construct the normal form game corresponding to this extensive form game, after the action B at the leftmost node for player 1 has been fixed.

It is not difficult to calculate the expected payoffs to both players under the remaining combinations of pure strategies (actions) for both players.

Specifically, we get the following payoff table:

	a	b
BC	$((3+5+9)/3, (2+7+2)/3)$	$((3+5+5)/3, (2+7+2)/3)$
BD	$((3+10+6)/3, (2+3+6)/3)$	$((3+4+6)/3, (2+0+6)/3)$

Or equivalently,

	a	b
BC	$(17/3, 11/3)$	$(13/3, 11/3)$
BD	$(19/3, 11/3)$	$(13/3, 8/3)$

To see the above, note that, for example, if Player 1 plays B and C and player 2 plays "a" then the expected utility (payoff) for Player 1 is $(3+5+9)/3 = 17/3$. We can likewise calculate all of the entries of the above table. (Note that in all these entries, it is always assumed that in the leftmost subtree player 1 plays B, because that is the unique optimal action in that subgame. So, without loss of generality, we can assume player 1 has two possible pure strategies: BC and BD, and of course it can also mix (randomize) between these two strategies.)

Now that we have the above normal form, we can easily calculate the Nash equilibria in this game, all of which will be "subgame perfect", because they already incorporate the fact that player 1 plays B in the leftmost subgame.

Note, in particular, that $((BD), (a))$ is a SPNE for the game, by inspection of the above payoff table: neither player can improve its payoff by switching strategies. Likewise $((BC), (b))$ is also an SPNE for the game, since both players can not *strictly* improve their payoff by unilaterally switching their strategy.

It is also not diffcult to check that there are no other, mixed NEs in this 2×2 normal form game. This is because as soon as player 1 puts positive probability on BD, it is preferrable for player 2 to switch its strategy to put probability 1 on pure strategy "a". Likewise, as soon as player 2 puts any positive probability on strategy "a", it is preferable for player 1 to put probability 1 on pure strategy BD.

The above two (pure) Nash Equilibria are both subgame perfect. So, there are exactly two SPNEs, both of which are pure.

Moreover, there are no other Nash Equilibria of any kind in the game. The reason is that, firstly, the only proper "subgame" of this game is the one in the leftmost subtree, rooted at the node controlled by player 1. But since there is a 1/3 probability that the game will end up in that subgame, player 1 MUST play B with probability 1 in that subgame. Otherwise, if it puts positive probability on the action A, then it can always increase its own expected payoff (no matter what the other player does), by playing action B with probability 1 in that subgame. Hence, in all Nash equilibria (not just in all subgame perfect Nash equilibria), player 1 plays the action B with probability 1 in the leftmost subgame. Hence, there are no other NEs, other than the two pure NEs we have mentioned above.