AGTA Tutorial sheet 4 (solutions)

1 Question 1

Given the finite 2 player bimatrix game from below, we will use iterated elimination of *strictly* dominated strategies.

$$\begin{array}{cc} & \begin{array}{cccc} c_1 & c_2 & c_3 & c_4 \end{array} \\ \begin{array}{c} r_1 \\ r_2 \\ r_3 \\ r_4 \end{array} & \begin{pmatrix} (5,2) & (22,4) & (4,9) & (7,6) \\ (16,4) & (18,5) & (1,10) & (10,2) \\ (15,12) & (16,9) & (18,10) & (11,3) \\ (9,15) & (23,9) & (11,5) & (5,13) \end{pmatrix} \end{array}$$

Consider Player 1 = Row player and Player 2 = Column player. We know that a pure strategy can be dominated by either a pure strategy or a *mixed* one. Observe that c_2 is strictly dominated by $(\frac{1}{2}c_1; \frac{1}{2}c_3)$ (easy to check). Therefore, we prune strategy c_2 and obtain a 4×3 residual game.

$$\begin{array}{cc} & \begin{array}{ccc} c_1 & c_3 & c_4 \end{array} \\ \begin{array}{c} r_1 \\ r_2 \\ r_3 \\ r_4 \end{array} & \begin{pmatrix} (5,2) & (4,9) & (7,6) \\ (16,4) & (1,10) & (10,2) \\ (15,12) & (18,10) & (11,3) \\ (9,15) & (11,5) & (5,13) \end{pmatrix} \end{array}$$

Observe that r_1 and r_4 are strictly dominated by the pure strategy r_3. Also, from the new residual game, strategy c_4 is strictly dominated by c_3. The final residual game is:

$$\begin{array}{cc} & \begin{array}{cc} c_1 & c_2 \end{array} \\ \begin{array}{c} r_2 \\ r_3 \end{array} & \begin{pmatrix} (16,4) & (1,10) \\ (15,12) & (18,10) \end{pmatrix} \end{array}$$

Applying the same principles as in question 2 from tutorial 2, we obtain the unique Nash equilibrium $((\frac{1}{4}r_2; \frac{3}{4}r_3); (\frac{17}{18}c_1; \frac{1}{18}c_2))$. Therefore, the final answer: $((0; \frac{1}{4}; \frac{3}{4}; 0); (\frac{17}{18}; 0; \frac{1}{18}; 0))$

2 Question 2

a)

Suppose $x = (x_1, x_2, ...x_n)$ is a Nash equilibrium. Consider the product distribution $p(s_1, s_2, ...s_n) = \prod_{i=1}^{n} x_i(s_i)$. We will show that any such Nash equilibrium is also a *correlated equilibrium* (CE).

Recall that, by definition, for all players i and for all pure strategies $s_i, s_i' \in S_i$,

$$U_i^{s_i'}(p \mid s_i) = \sum_{s_{-i} \in S_{-i}} p(s_{-i} \mid s_i) \cdot u_i(s_{-i}; s_i')$$

is the conditional expected payoff of player i, for playing pure strategy s_i', having received recommendation s_i from the distribution p, assuming other players play according to their recommendation s_j. Here, by definition,

$$p(s_{-i} \mid s_i) = \frac{p(s_1, \ldots, s_n)}{\sum_{t_{-i} \in S_{-i}} p(t_{-i}; s_i)}$$

Recall that we define p to be a correlated equilibrium iff, for all players i and $s_i, s_i' \in S_i$, we have

$$U_i^{s_i}(p \mid s_i) \geq U^{s_i'}(p \mid s_i)$$

(In other words, the conditional expected payoff of using the strategy recommended by the recommender is at least as high as choosing any other pure strategy.)

Recall that from the Claim on page 6 of Lec.3, the fact that $x = (x_1, \ldots, x_n)$ is a Nash equilibrium is equivalent to the following statement: for all pure strategies $s_i' \in S_i$, $U_i(x) \geq U_i(x_{-i}; s_i')$. In other words:

$$\sum_{(s_1,\ldots,s_n) \in S} (\prod_{i=1}^{n} x_i(s_i)) \cdot u_i(s_1, \ldots, s_n) \geq \sum_{s_{-i} \in S_{-i}} (\prod_{j \neq i} x_j(s_j)) \cdot u_i(s_1, \ldots, s_{i-1}, s_i', s_{i+1}, \ldots, s_n)$$

Since $p(s_1, \ldots, s_n) = \prod_{j=1}^{n} x_j(s_j)$, we see that

$$p(s_{-i} \mid s_i) = \frac{\prod_{j=1}^{n} x_j(s_j)}{\sum_{t_{-i} \in S_{-i}} p(t_{-i}; s_i)} = \frac{\prod_{j=1}^{n} x_j(s_j)}{x_i(s_i) \sum_{t_{-i} \in S_{-i}} \prod_{j \neq i} x_j(t_j)} = \frac{\prod_{j=1}^{n} x_j(s_j)}{x_i(s_i)} = \prod_{j \neq i} x_j(s_j)$$

Thus, by definition,

$$U_i^{s_i}(p \mid s_i) = \sum_{s_{-i} \in S_{-i}} (\prod_{j \neq i} x_j(s_j)) \cdot u_i(s_{-i}; s_i).$$

Note that therefore $U_i^{s_i}(p \mid s_i) = U_i(x_{-i}; s_i)$, where $U_i(x_{-i}; s_i)$ is the expected payoff to player i if it were to unilaterally switch to pure strategy s_i, assuming everybody else plays according to the mixed profile x. Moreover, for any pure strategy s_i' for player i, we have

$$U_i^{s_i'}(p \mid s_i) = \sum_{s_{-i} \in S_{-i}} (\prod_{j \neq i} x_j(s_j)) \cdot u_i(s_{-i}; s_i')$$

In other words, $U_i^{s_i'}(p \mid s_i) = U_i(x_{-i}; s_i')$ is the same as the expected payoff to player i if it were to unilaterally switch to pure strategy s_i', assuming everyone else plays according to the mixed profile x.

Our aim is to show that, under the assumption that x is an NE, and that player i was recommended s_i under the joint distribution $p(s_1, \ldots, s_n) = \prod_{i=1}^{n} x_i(s_i)$, we must have $U_i^{s_i}(p \mid s_i) \geq U_i^{s_i'}(p \mid s_i)$, or equivalently, that we must have $U_i(x_{-i}; s_i) \geq U_i(x_{-i}; s_i')$.

But this follows immediately from the "useful corollary to Nash's theorem", because in order for s_i to be recommended to player i, s_i must be in the support of x_i (meaning $x_i(s_i) > 0$), and since we are assuming x is a Nash Equilibrium, this means that the pure strategy s_i itself must be a best response to x_{-i}. Hence $U_i(x_{-i}; x_i) = U_i(x_{-i}; s_i) \geq U_i(x_{-i}; s_i')$, and hence p is a correlated equilibrium.

b)

We can express the fact that p is a Correlated Equilibrium via a system of LP constraints as follows. Firstly, we express that p must define a distribution on the set S of combinations of pure strategies, as follows:

$$p(s_1, ..., s_n) \geq 0, \forall (s_1, ..., s_n) \in S \tag{1}$$

$$\sum_{(s_1, s_2, \ldots s_n) \in S} p(s_1, ..., s_n) = 1 \tag{2}$$

Next, we need to express all constraints of the form: $U_i^{s_i}(p \mid s_i) \geq U^{s_i'}(p \mid s_i)$, for every pair of pure strategies s_i, s_i' such that s_i can possibly be recommended to player i, in other words, such that $\sum_{t_{-i} \in S_{-i}} p(t_{-i}; s_i) > 0$.

If we write these out explictly the constraint $U_i^{s_i}(p \mid s_i) \geq U^{s_i'}(p \mid s_i)$, it looks as follows:

$$\sum_{s_{-i} \in S_{-i}} p(s_{-i} \mid s_i) \cdot u_i(s_{-i}; s_i) \geq \sum_{s_{-i} \in S_{-i}} p(s_{-i} \mid s_i) \cdot u_i(s_{-i}; s_i')$$

Which is equivalent to:

$$\sum_{s_{-i} \in S_{-i}} p(s_{-i} \mid s_i) \cdot (u_i(s_{-i}; s_i) - u_i(s_{-i}; s_i')) \geq 0 \tag{3}$$

But since $p(s_{-i} \mid s_i) = \frac{p(s_{-i}; s_i)}{\sum_{t_{-i} \in S_{-i}} p(t_{-i}; s_i)}$, and since we know the denominator is positive, by mulitplying both sides of the inequality by this denominator, we can rewrite the inequality (3) as:

$$\sum_{s_{-i} \in S_{-i}} p(s_{-i}; s_i) \cdot (u_i(s_{-i}; s_i) - u_i(s_{-i}; s_i')) \geq 0 \tag{4}$$

Note that this is an LP constraint.

Thus the set of Correlated Equilibria of a game can be defined as the set of feasible solutions to a system of linear inequalities. The fact that the set of feasible solutions to a system of linear inequalities is convex is of course well-known, and was discussed in the lectures on LP. (You can prove this yourself easily if you wish, from the definition of convexity.)

c)

Given the game

$$\begin{array}{c c} & \begin{array}{cc} A & B \end{array} \\ \begin{array}{c} a \\ b \end{array} & \begin{pmatrix} (5,2) & (0,0) \\ (0,0) & (2,5) \end{pmatrix} \end{array}$$

Using the methods we have already learned, we see that the only three NE in this game are:

- $[(\frac{5}{7}, \frac{2}{7}), (\frac{2}{7}, \frac{5}{7})]$ with expected payoff both for pl.1 and for player 2 : $\frac{10}{7}$.
- $[(0,1),(0,1)]$ with exp. payoff for player 1 = 2 and exp. payoff for player 2 = 5.
- $[(1,0),(1,0)]$ with exp. payoff for player 1 = 5 and exp. payoff for player 2 = 2.

Now consider the correlated distribution, p, with probabilities: $p(a, A) = 1/2, p(a, B) = 0, p(b, A) = 0, p(b, B) = 1/2$.

We claim that p is a correlated equilibrium, which is not a NE.

To see that p is a CE, we need to show that the constraints

$$U_i^{s_i}(p \mid s_i) \geq U^{s_i'}(p \mid s_i) \tag{5}$$

are satisfied for all pure strategies s_i, s_i' for each player i.

But in the context of this specific game, for example, if we let $i = 1$ and we let $s_i = a$ and $s_i' = b$, we have that inequality (5) is equivalent to:

$$U_1^a(p \mid a) = p(A|a)u_i(a, A) = 5 \geq 0 = p(A \mid a)u_i(b, A) = U_1^b(p \mid a)$$

In the same way, it can be checked that all the inequalities of form (5) hold true. Thus, p is a CE. It is clearly not a NE.

Moreover, the expected payoff to each player under this CE is basically the weighted average of their payoff, weighted by the probabilities of the two possible recommendations $p(a, A)$ and $p(b, B)$. Thus, the expected payoff to each player is $5 + 2/2 = 3.5$, yielding a sum total expected utility of 7 for both players, which is as high as in any Nash equilibrium (and note that this CE is also not "biased" toward either player, unlike the two NEs where the sum total expected payoff is 7).

d)

Open discussion. Relevant arguments might include:

- the CE could give higher expected utilities than a Nash equilibrium; in the game above, for the CE given as example, t he total welfare is as high as any N.E.

- CE is less computationally expensive to compute.
- Nash equilibrium is relevant if one assumes that each player knows which strategies the other players are using. In CE, we do not make such assumption, players do not know in general how others are playing, but they know the prior distribution p of the recommendations. CE does not require any explicit randomization on the part of the players. Each player always chooses a pure strategy with no attempt to randomize. The probabilistic nature of the strategies reflects the uncertainty of other players about his choice.
- NE $\subseteq$ CE, i.e CE is a superset of NE.